Great Theoretical Ideas In Computer Science

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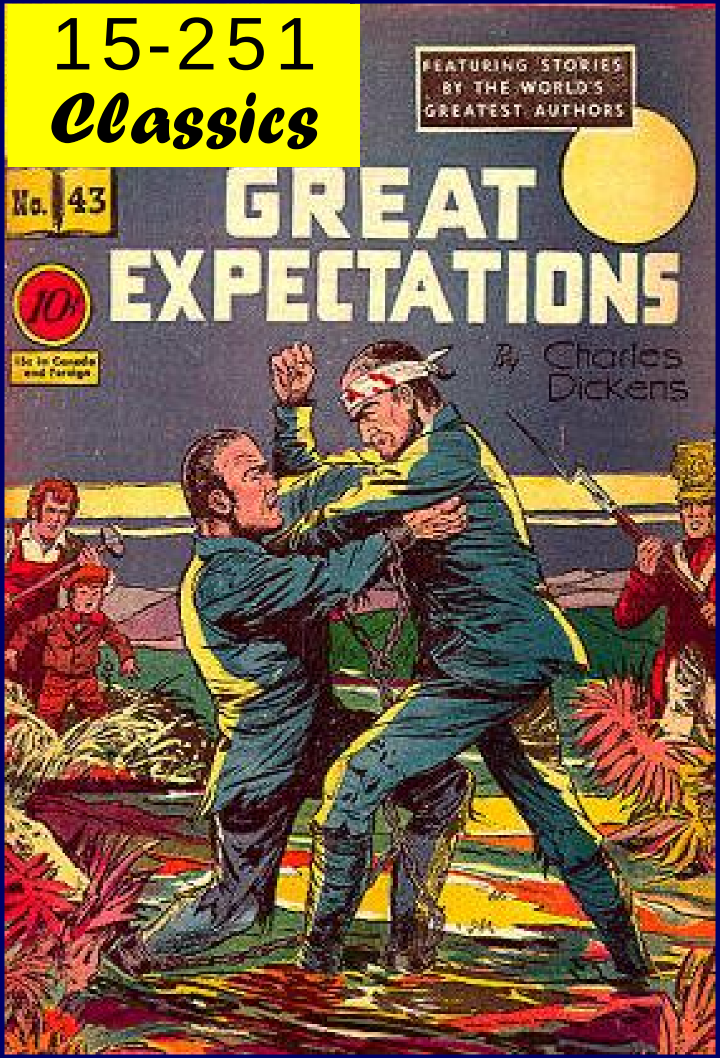
CS 15-251 Fall 2006

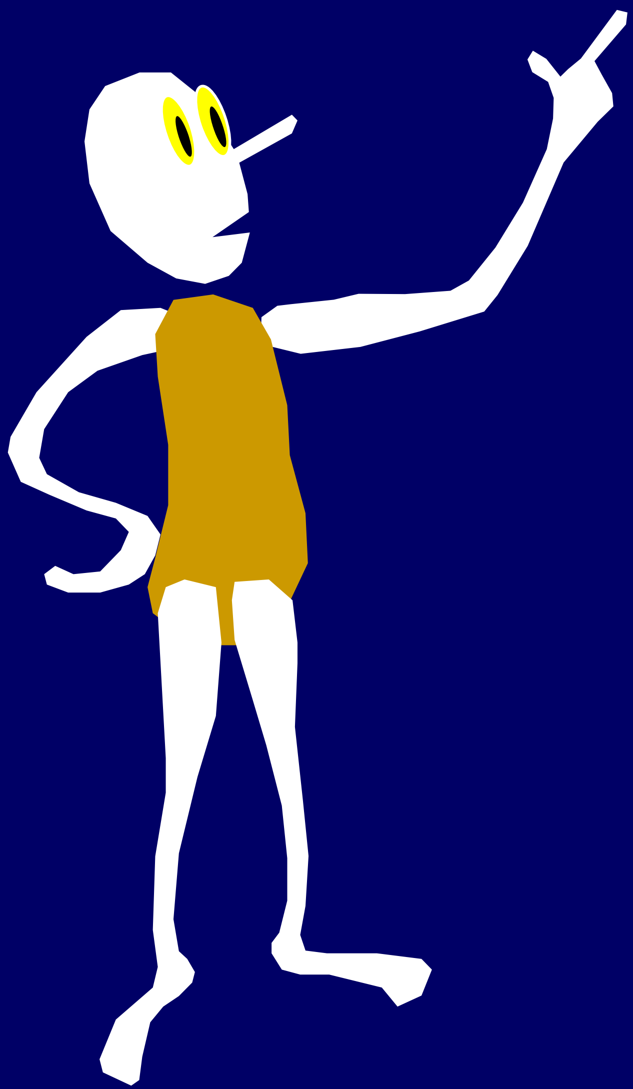
Lecture 10 Sept. 28, 2006

Carnegie Mellon University



Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem <u>really really messy</u>…

Language Of Probability



The formal language of probability is a very important tool in describing and analyzing probability distributions.

Finite Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

The weights must satisfy:

$$\sum_{x \in S} p(x) = 1$$

Finite Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

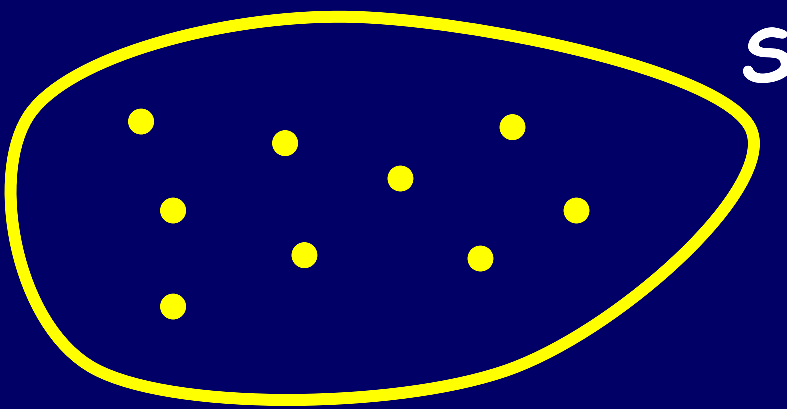
For notational convenience we will define D(x) = p(x).

S is often called the sample space and elements x in S are called samples.

Sample space

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

Sample space



Probability

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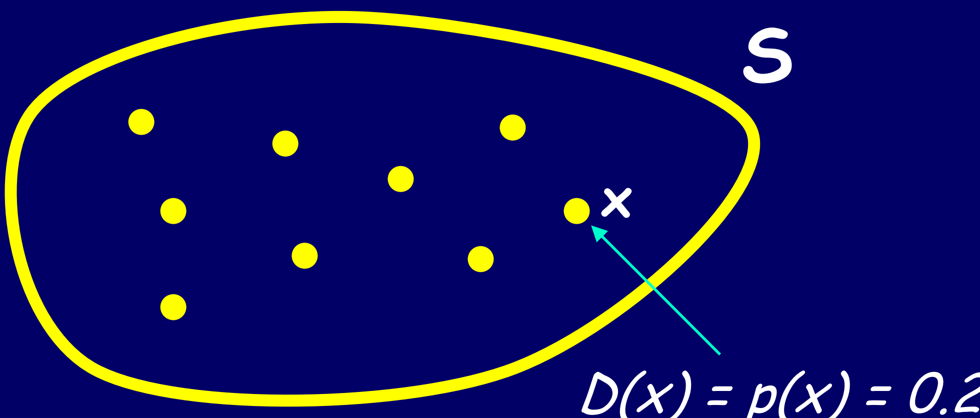

weight or probability of x

Probability Distribution

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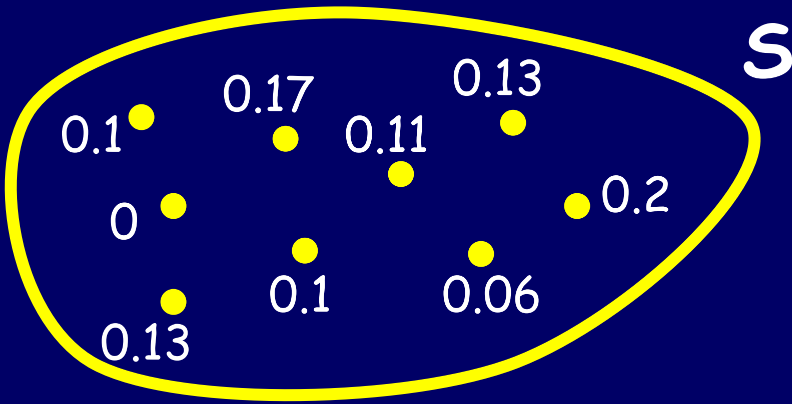

weights must sum to 1

Events

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

Any subset E of S is called an event.
The probability of event E is

$$Pr_D[E] = \sum_{x \in E} p(x)$$

Events

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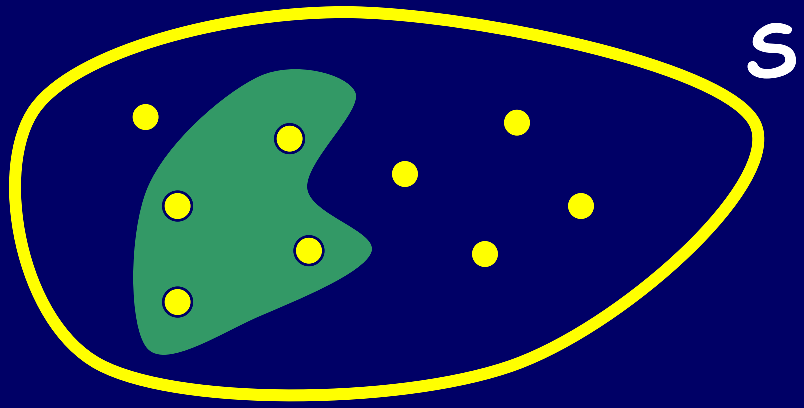
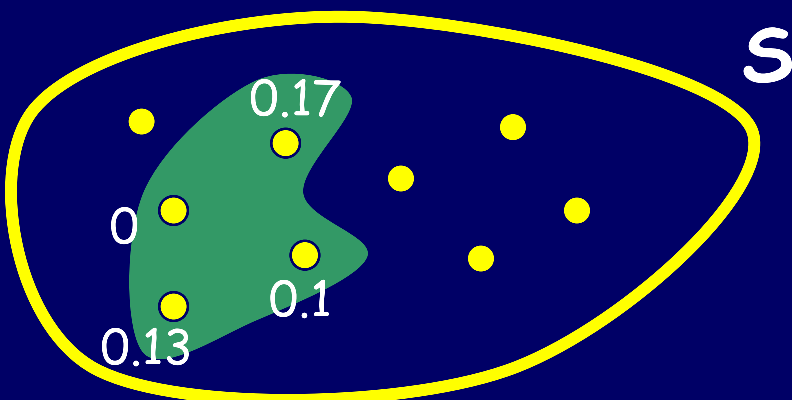


Event E

Events

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).



$Pr_D[E] = 0.4$

Uniform Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

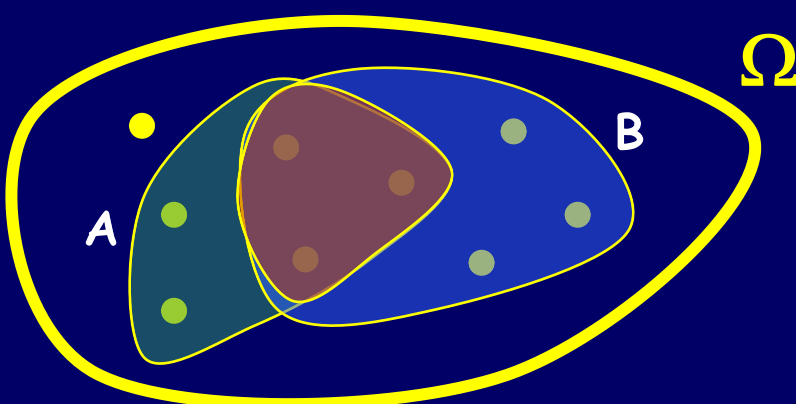
If each element has equal probability, the distribution is said to be uniform.

$$Pr_D[E] = \sum_{x \in E} p(x) = \frac{|E|}{|S|}$$

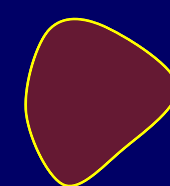
More Language Of Probability: Conditioning

The probability of event A given event B is written Pr[A | B]

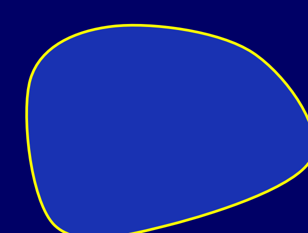
and is defined to be = $\dfrac{\Pr[A \cap B]}{\Pr[B]}$

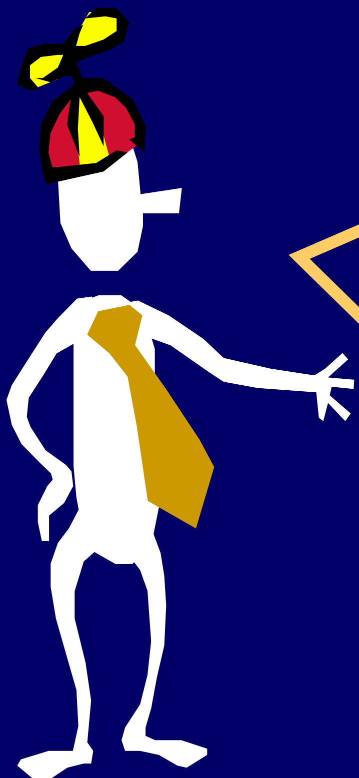


proportion of $A \cap B$



to B





Suppose we roll a ***white*** die and ***black*** die.

What is the probability that the white is 1 given that the total is 7?

event A = {white die = 1}

event B = {total = 7}

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
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event A = {white die = 1}

event B = {total = 7}

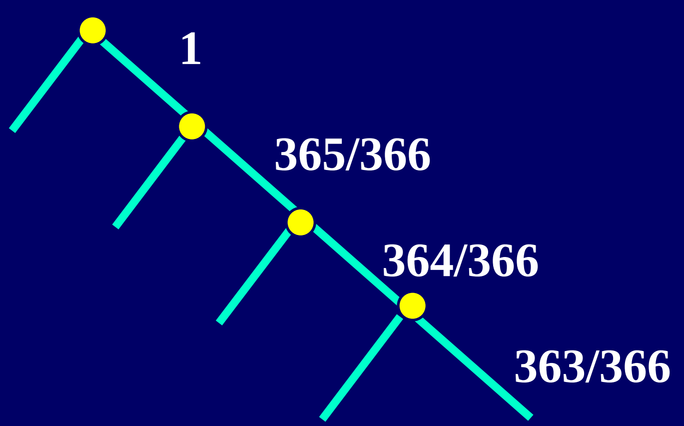
$$\frac{|A \cap B|}{|B|} = \Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{1/36}{1/6}$$

Can do this because Ω is uniformly distributed.

This way does not care about the distribution.

Another way to calculate Birthday probability Pr(no collision)

Pr(1st person doesn't collide) = 1.
Pr(2nd doesn't | no collisions yet) = 365/366.
Pr(3rd doesn't | no collisions yet) = 364/366.
Pr(4th doesn't | no collisions yet) = 363/366.
…
Pr(23rd doesn't| no collisions yet) = 344/366.



Independence!

A and B are <u>independent events</u> if

if $B = S$

$$Pr[\ A \mid B\ ] = Pr[\ A\ ]$$

$$\Leftrightarrow$$

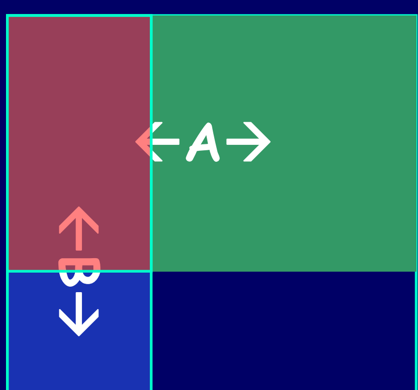
$$Pr[\ A \cap B\ ] = Pr[\ A\ ]\,Pr[\ B\ ]$$

$$\Leftrightarrow$$

$$Pr[\ B \mid A\ ] = Pr[\ B\ ]$$

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = Pr(A)$$

$Pr(B) = 1$



$$Pr(B^c \mid A) = 1 - Pr(B|A) = Pr(B^c)$$

What about Pr[A | not(B)] ?

Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

E.g., {A_1, A_2, A_3} are independent events if:

$$\Pr[A_1 \mid A_2 \cap A_3] = \Pr[A_1]$$
$$\Pr[A_2 \mid A_1 \cap A_3] = \Pr[A_2]$$
$$\Pr[A_3 \mid A_1 \cap A_2] = \Pr[A_3]$$

$$\Pr[A_1 \mid A_2] = \Pr[A_1]$$
$$\Pr[A_2 \mid A_1] = \Pr[A_2]$$
$$\Pr[A_3 \mid A_1] = \Pr[A_3]$$

$$\Pr[A_1 \mid A_3] = \Pr[A_1]$$
$$\Pr[A_2 \mid A_3] = \Pr[A_2]$$
$$\Pr[A_3 \mid A_2] = \Pr[A_3]$$

Independence!

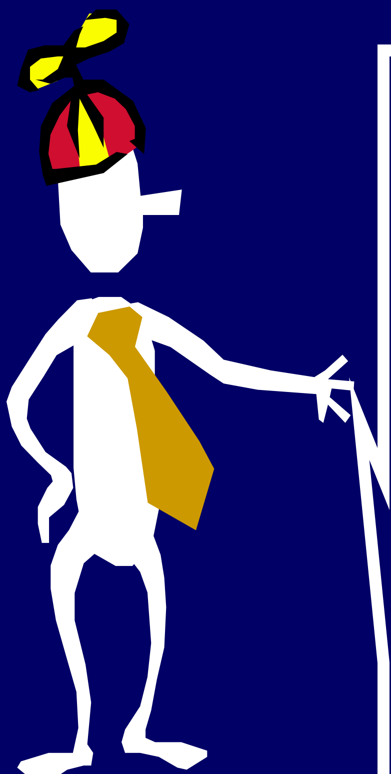
$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

$$\Pr[A|X] = \Pr[A]$$

for all A = some A_i

for all X = a conjunction of any of the others (e.g., A_2 and A_6 and A_7)

Silver and Gold



One bag has two silver coins, another has two gold coins, and the third has one of each.

One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

What is the probability that the other coin is gold?

Let G_1 be the event that the first coin is gold.

$\Pr[G_1] = 1/2$

Let G_2 be the event that the second coin is gold.

$$\Pr[G_2 \mid G_1] = \Pr[G_1 \text{ and } G_2] / \Pr[G_1]$$

$$= (1/3) / (1/2)$$

$$= 2/3$$

Note: G_1 and G_2 are not independent.

Monty Hall problem

- Announcer hides prize behind
  one of 3 doors at random.
- You select some door.
- Announcer opens one of others with no prize.
- You can decide to keep or switch.

What to do?

Monty Hall problem

- Sample space Ω =

{ prize behind door 1,
prize behind door 2,
prize behind door 3 }.

Each has probability 1/3.

Staying

we win if we choose the correct door

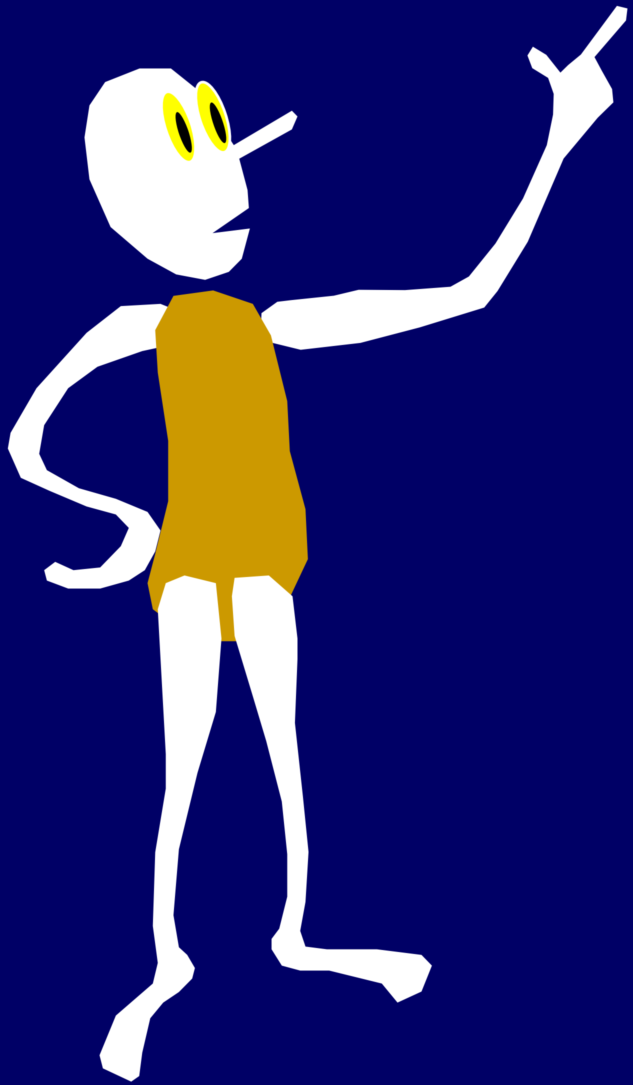
Pr[choosing correct door] = 1/3.

Switching

we win if we choose the <u>incorrect</u> door

Pr[choosing incorrect door] = 2/3.

why was this tricky?



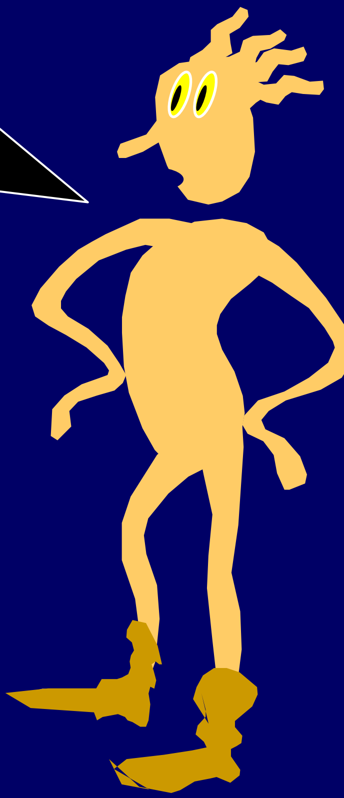
We are inclined to think:

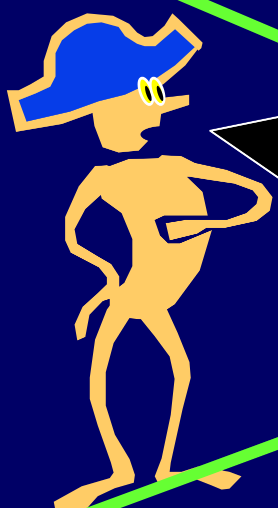
"After one door is opened, others are equally likely…"

But his action is not independent of yours!

Now, about that formidable tool that will allow us to solve problems that seem <u>really really messy</u>…

If I randomly put 100 letters into 100 addressed envelopes, <u>on average</u> how many letters will end up in their correct envelopes?



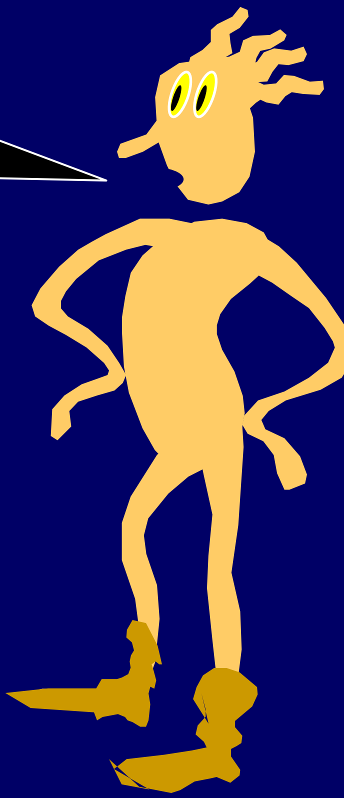


Hmm…

$\sum_k$ k·Pr(*exactly* k letters end up in correct envelopes)

$= \sum_k$ k· (…aargh!!…)

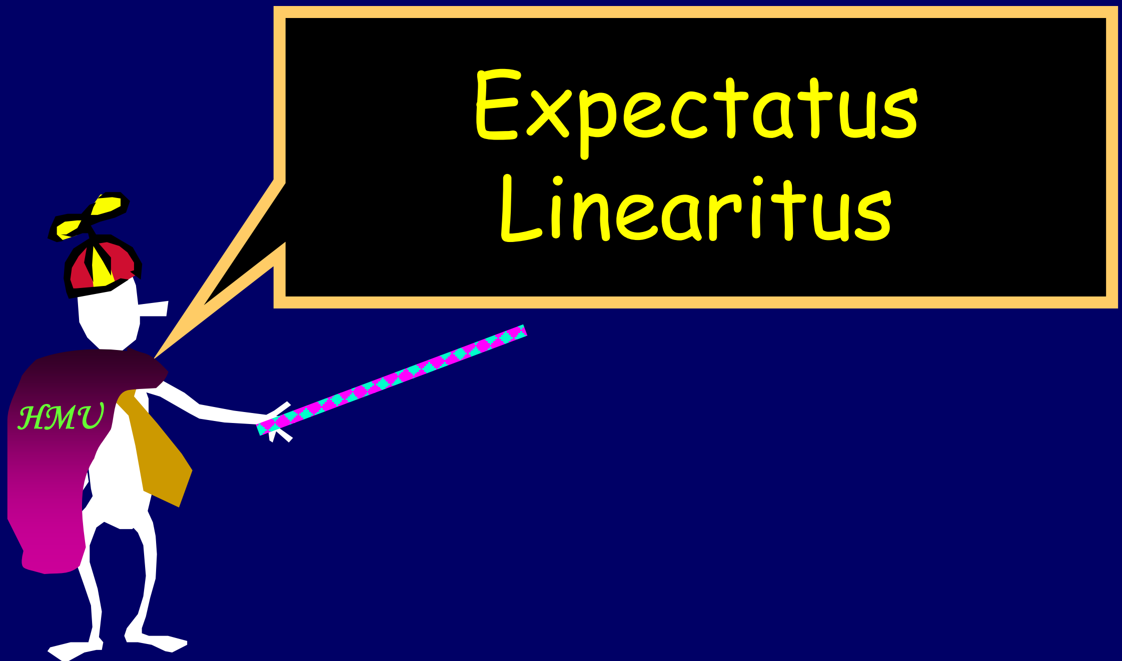
On average, in class of size m, how many pairs of people will have the same birthday?



~~$\sum_k k\cdot$ Pr(exactly k collisions)~~

~~$= \sum_k k\cdot$ (...aargh!!!!...)~~

The new tool is called "Linearity of Expectation"



Random Variable

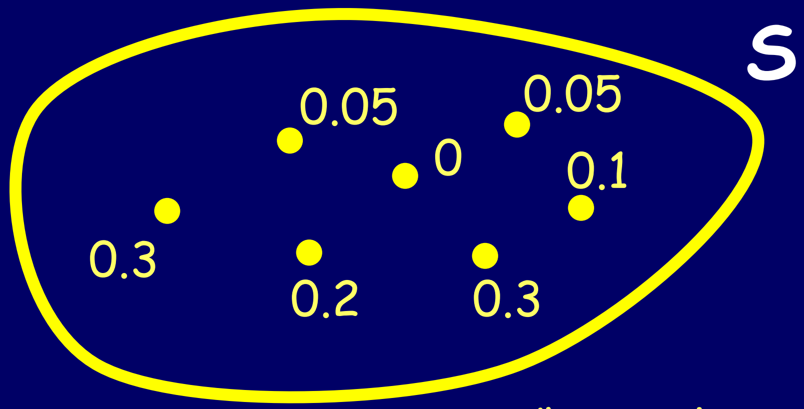
To use this new tool, we will also need to understand the concept of a <u>Random Variable</u>

Today's lecture: not too much material, but need to understand it well.

Probability Distribution

A (finite) <u>probability distribution</u> D

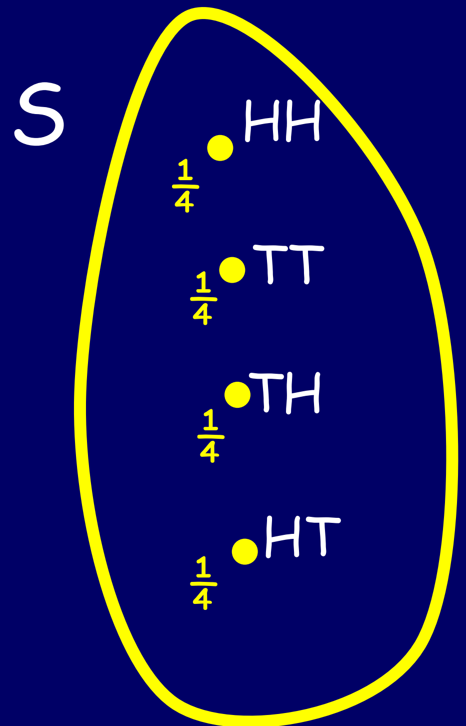
- a finite set S of elements (samples)
- each $x \in S$ has weight or <u>probability</u> $p(x) \in [0,1]$



weights must sum to 1

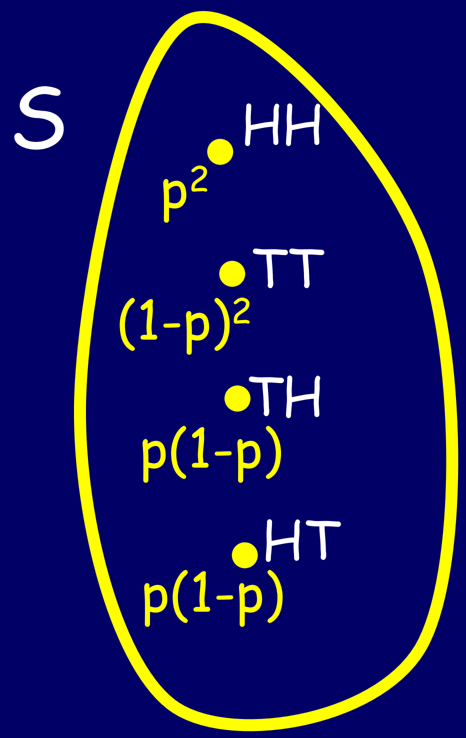
"Sample space"

Flip penny and nickel (unbiased)

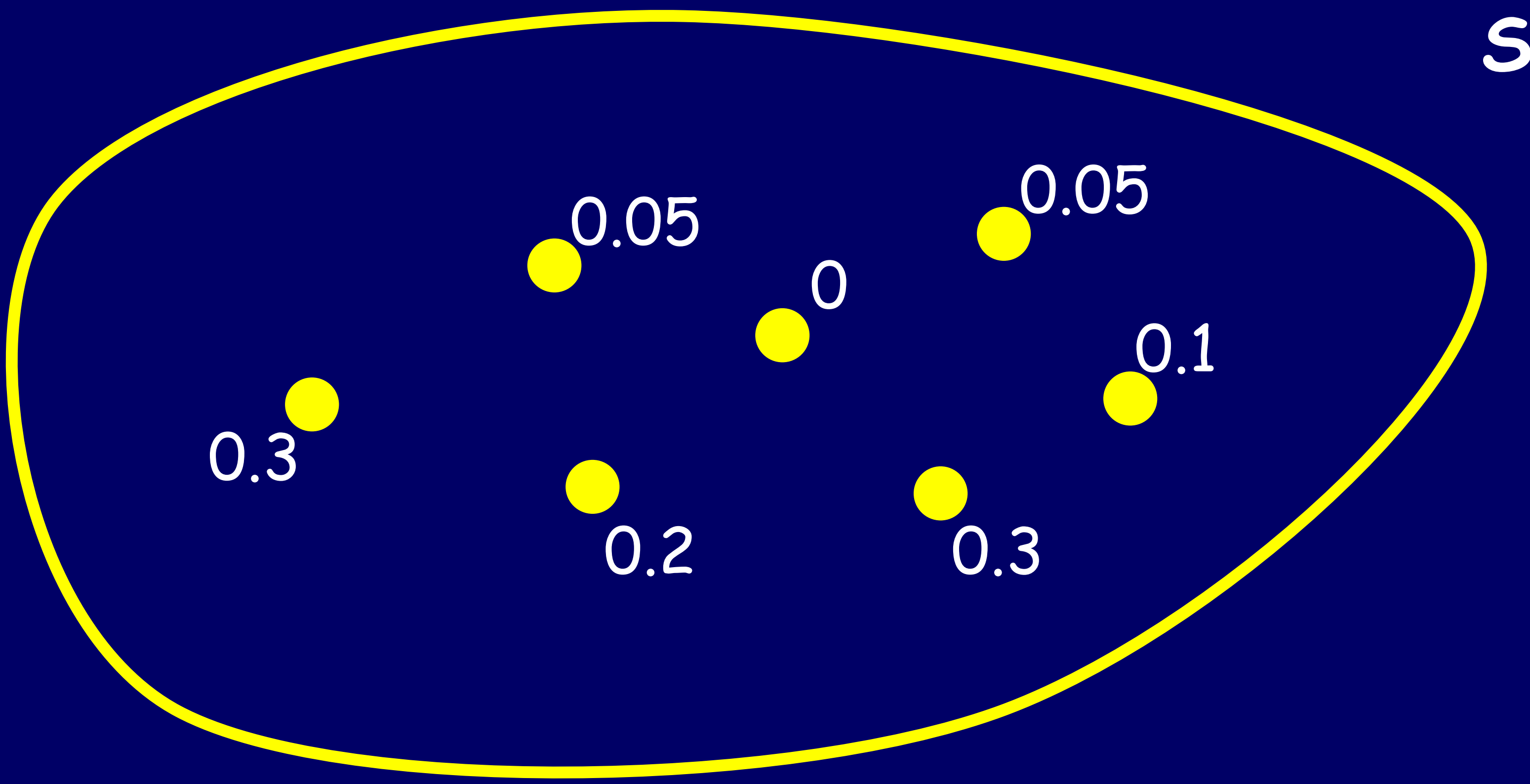


Flip penny and nickel (biased)

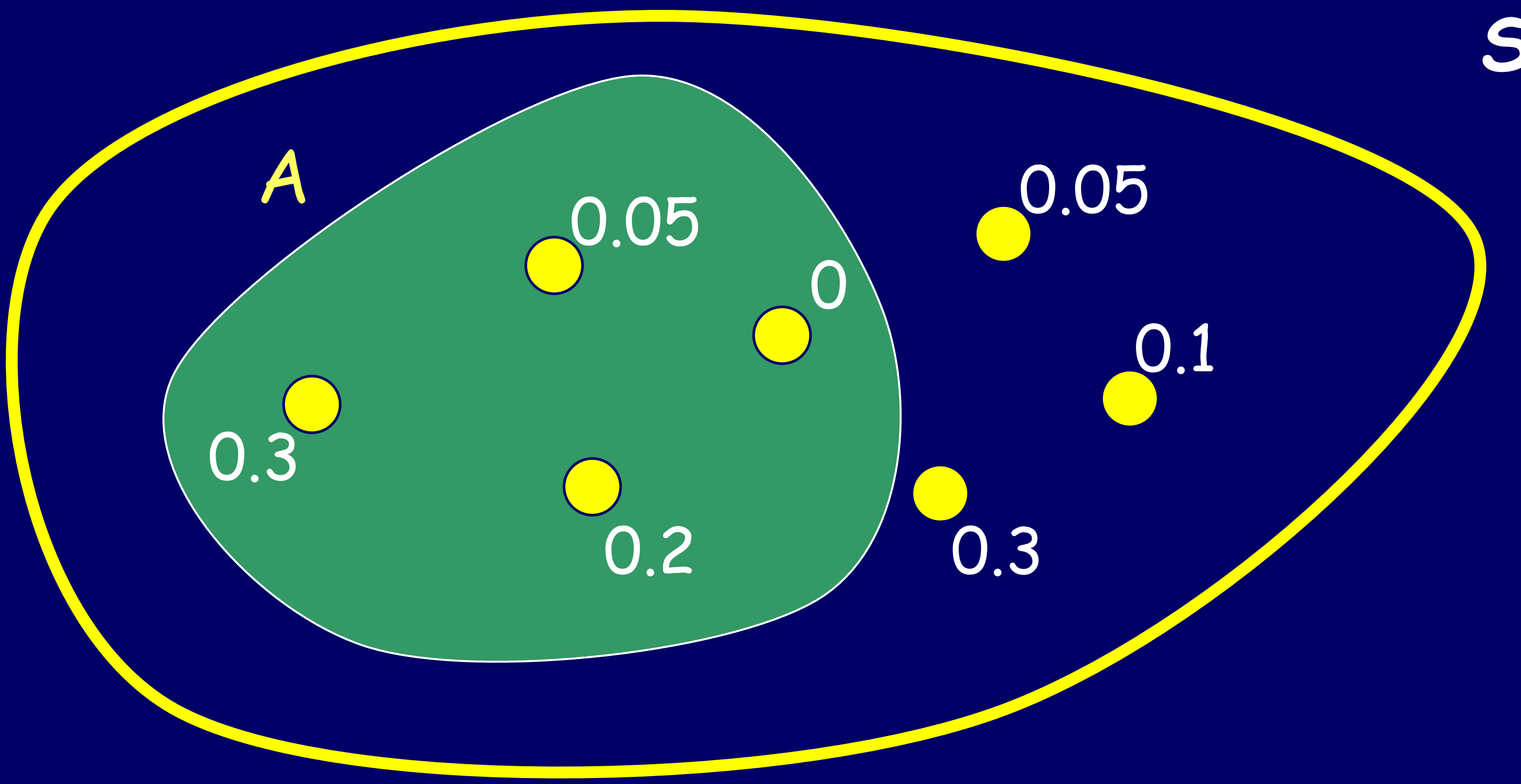
heads probability = p



Probability Distribution



An event is a subset



$\Pr[A] = \sum_{x \in A} p(x) = 0.55$

Running Example

I throw a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
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(6,1),	(6,2),	(6,3),	(6,4),	(6,5),	(6,6) }

$\Pr(x) = 1/36$
$\forall\, x \in S$

E = event that sum ≤ 3

$\Pr[E] = |E|/|S|$ = proportion of E in S = 3/36

New concept:
<u>Random Variables</u>

Random Variable: A measurement for an experimental outcome

Random Variable: a (real-valued) function on S

Toss a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
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Examples:

X = value of white die.

$X(3,4) = 3$, $X(1,6) = 1$ etc.

Y = sum of values of the two dice.

$Y(3,4) = 7$, $Y(1,6) = 7$ etc.

W = (value of white die)$^{\text{value of black die}}$

$W(3,4) = 3^4$ $Y(1,6) = 1^6$

Z = (1 if two dice are equal, 0 otherwise)

$Z(4,4) = 1$, $Z(1,6) = 0$ etc.

E.g., tossing a fair coin n times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S

$\Rightarrow D(x) = (\frac{1}{2})^n$ for all $x \in S$

Random Variables (say $n = 10$)

X = # of heads

$X(HHHTTHTHTT) = 5$

Y = (1 if #heads = #tails, 0 otherwise)

$Y(HHHTTHTHTT) = 1$, $Y(THHHHTTTTT) = 0$

Notational conventions

Use letters like A, B, E for events.

Use letters like X, Y, f, g for R.V.'s.

R.V. = random variable

Two views of random variables

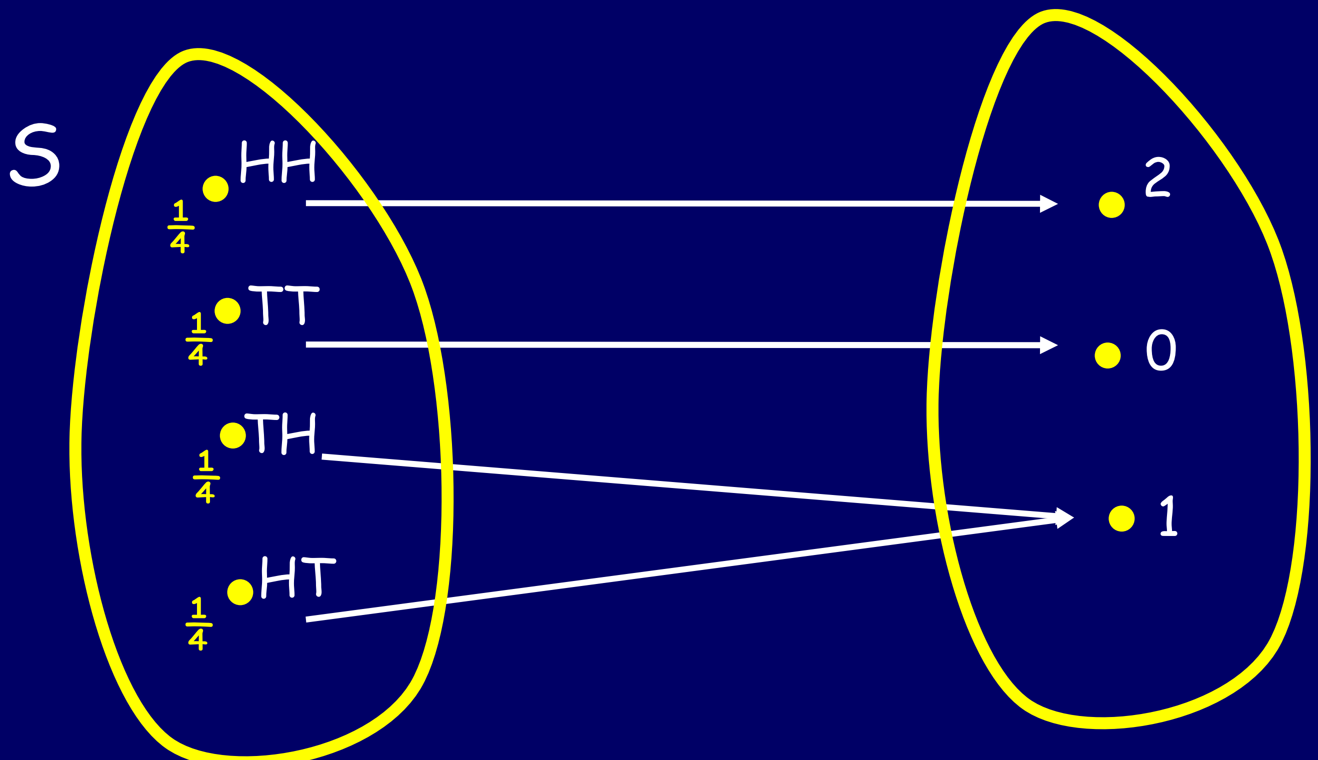
Think of a R.V. as

- a function from S to the reals $\mathbb{R}$
- or think of the induced distribution on $\mathbb{R}$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}

counts the number of heads



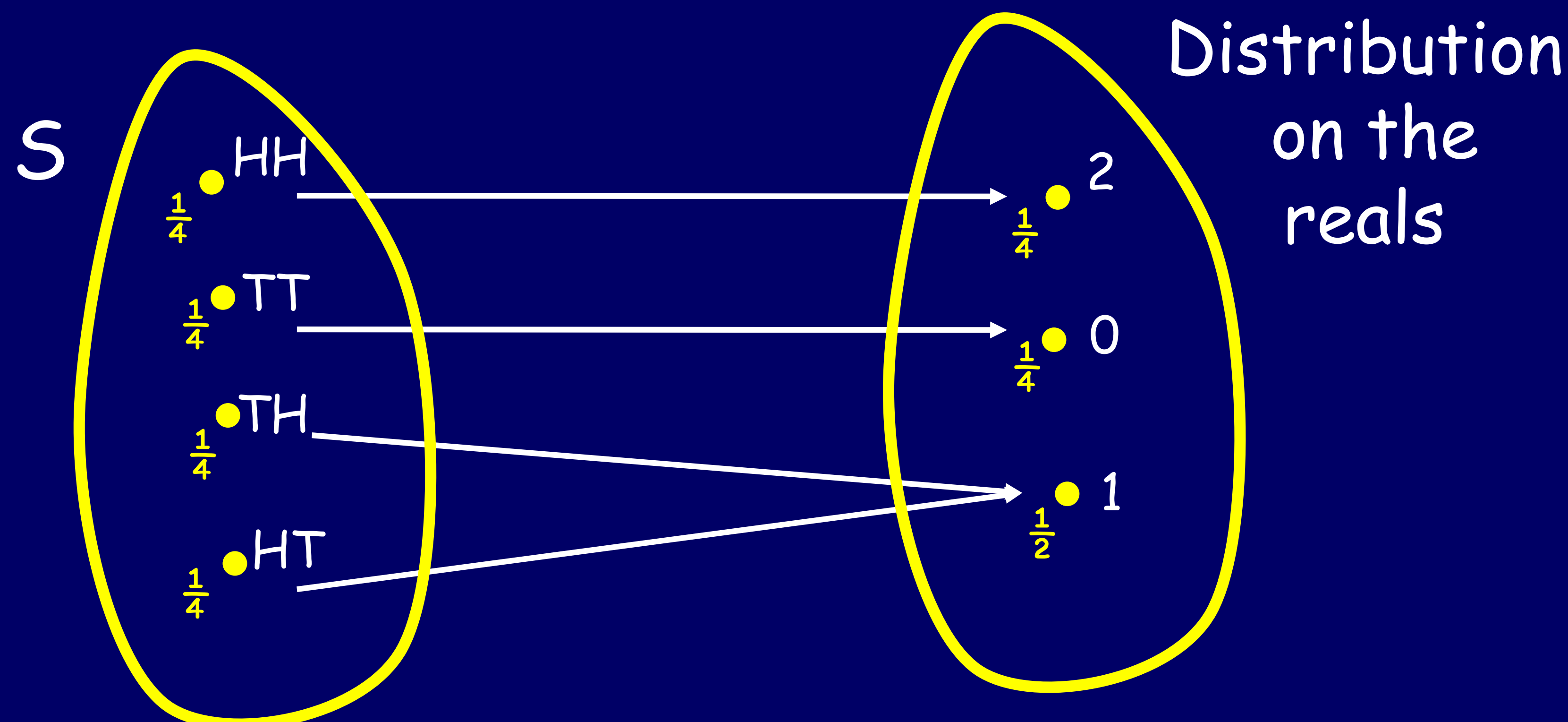
Two views of random variables

Think of a R.V. as

- $\boxed{\text{a function from S to the reals } \mathbb{R}}$
- or think of the induced distribution on $\mathbb{R}$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



Two views of random variables

Think of a R.V. as

- a function from S to the reals
- or think of the induced distribution on reals

Two dice

I throw a white die and a black die.

Sample space S =

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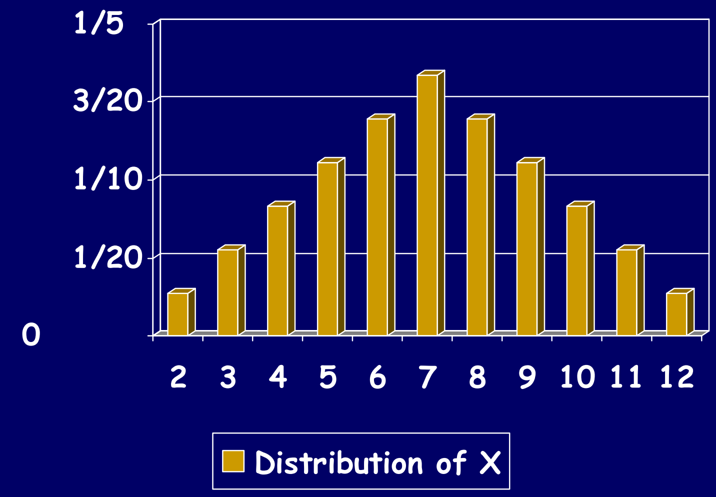
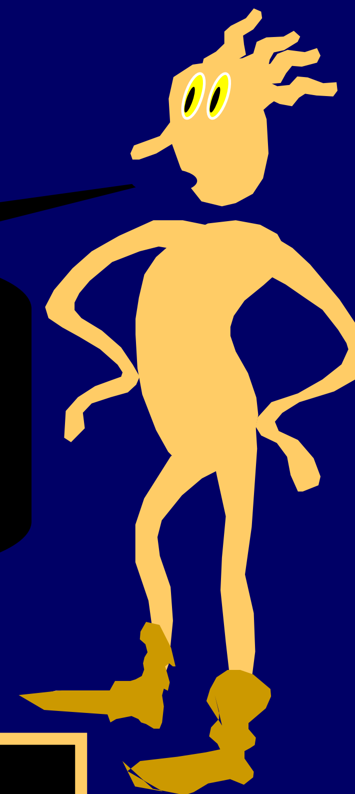


X = sum of both dice

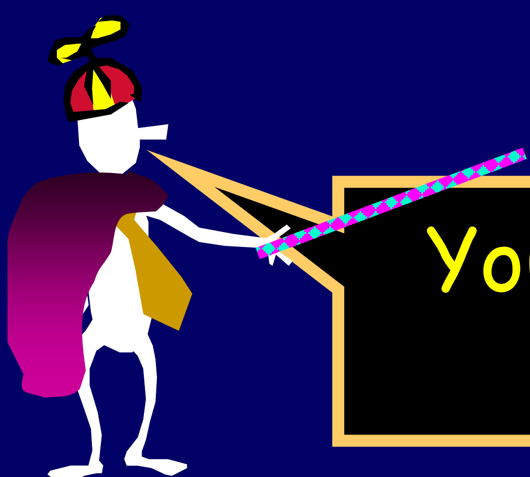
function with X(1,1) = 2, X(1,2)=X(2,1)=3, …, X(6,6)=12

It's a floor wax *and* a dessert topping

It's a function on the sample space S.



It's a variable with a probability distribution on its values.



You should be comfortable with both views.

From Random Variables to Events

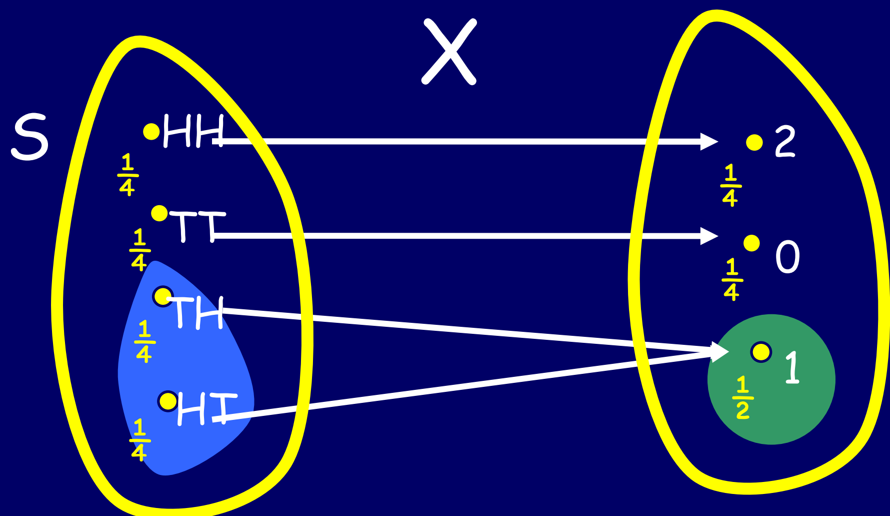
For any random variable X and value a, we can define the <u>event</u> A that $X=a$.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\}).$$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads

$$\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})$$



Distribution on X

$\Pr(X = 1)$

$= \Pr(\{x \in S \mid X(x) = 1\})$

$= \Pr(\{TH, HT\}) = \frac{1}{2}$.

Two dice

I throw a white die and a black die. X = sum

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
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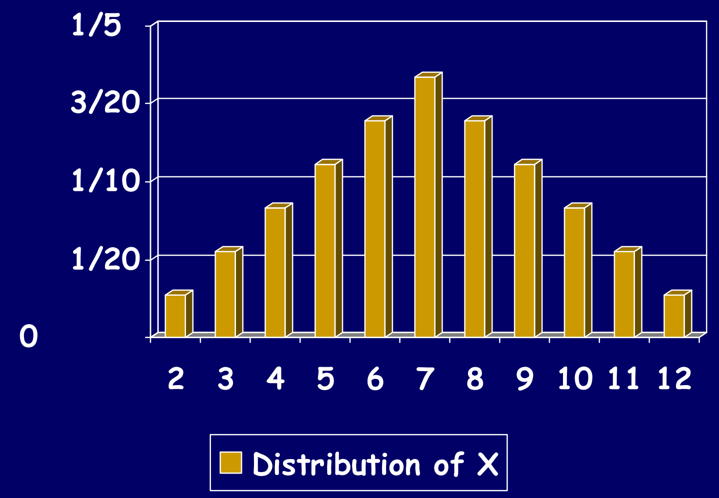


$\Pr(X = 6)$

$= \Pr(\{x \in S : X(x) = 6\})$

$= 5/36.$

$$\Pr(X = a) = \Pr(\{x \in S: X(x) = a\})$$

From Random Variables to Events

For any random variable X and value a,
we can define the <u>event</u> A that X=a.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S: X(x)=a\}).$$

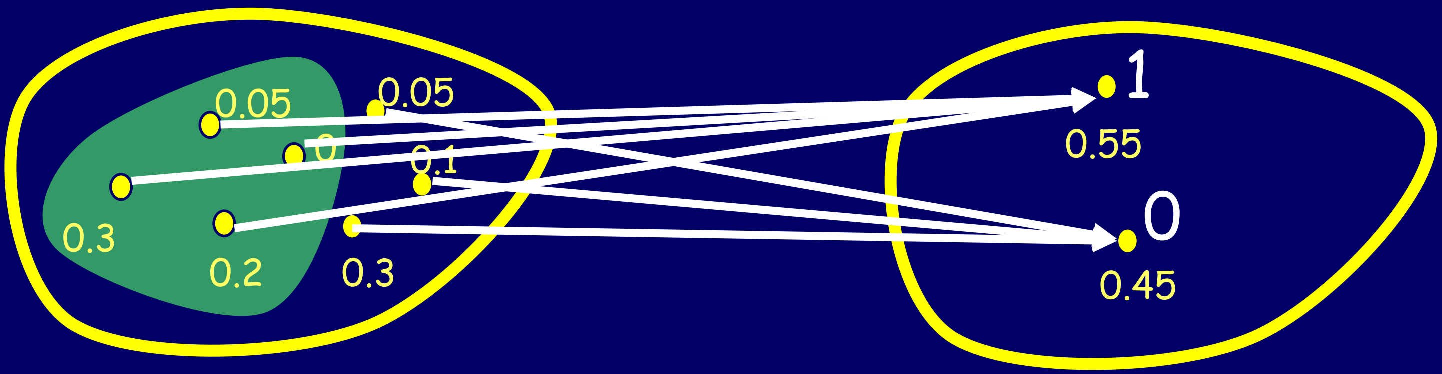
X has a distribution on its values

X is a function on the sample space S

From Events to Random Variables

For any event A, can define the indicator random variable for A:

$$
\begin{aligned}
X_A(x) &= 1 \quad \text{if } x \in A \\
&= 0 \quad \text{if } x \notin A
\end{aligned}
$$



Definition: expectation

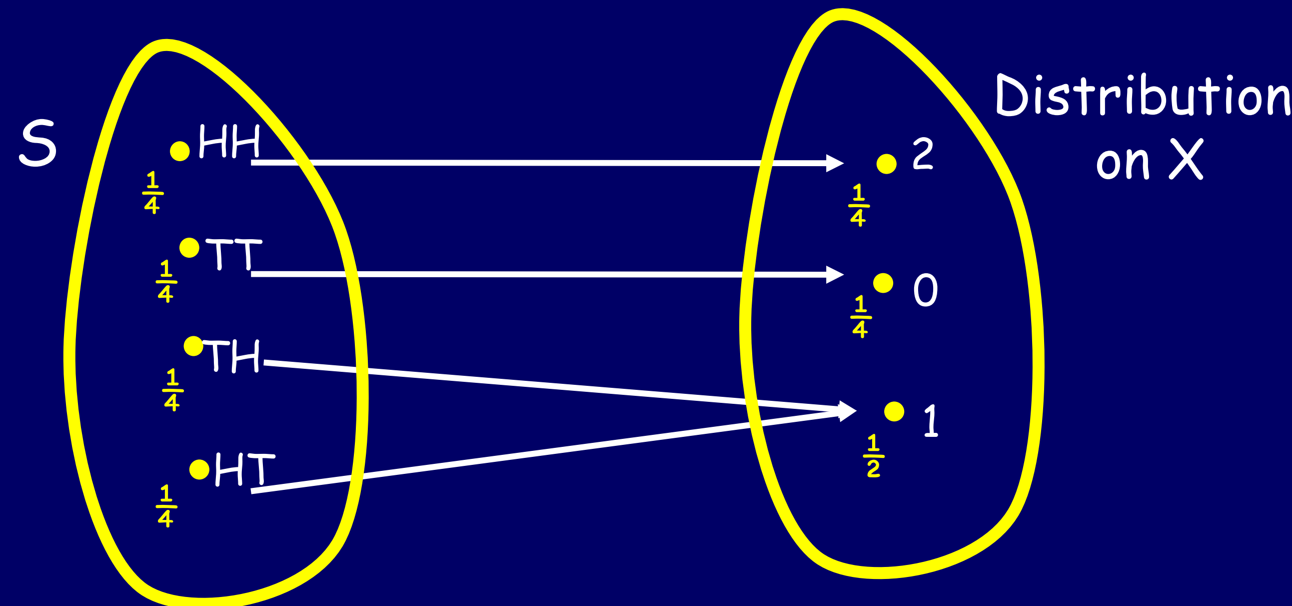
The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x) X(x) = \sum_k k \Pr(X = k)$$

X is a function on the sample space S

X has a distribution on its values

Thinking about expectation



$\boxed{\sum_{x \in S} X(x)\, Pr(x)}$ $\boxed{\sum_k k\, Pr(X = k)}$

$E[X] = \frac{1}{4}*0 + \frac{1}{4}*1 + \frac{1}{4}*1 + \frac{1}{4}*2 = 1.$

$E[X] = \frac{1}{4}*0 + \frac{1}{2}*1 + \frac{1}{4}*2 = 1.$

A quick calculation…

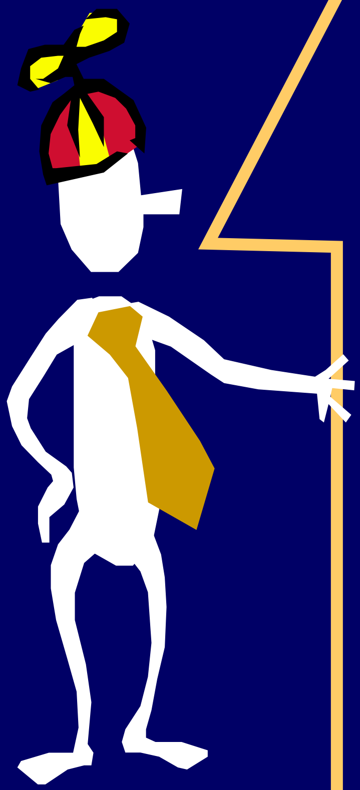
What if I flip a coin 3 times? Now what is the expected number of heads?

$$E[X] = (1/8) \times 0 + (3/8) \times 1 + (3/8) \times 2 + (1/8) \times 3 = 1.5$$

But Pr[X = 1.5] = 0…

Moral: don't always expect the expected.
Pr[X = E[X]] may be 0 !

Type checking



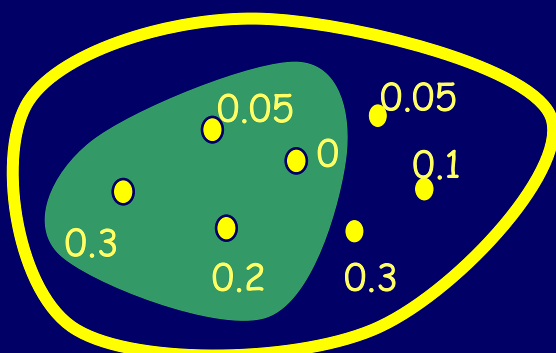
A Random Variable is the type of thing you might want to know an expected value of.

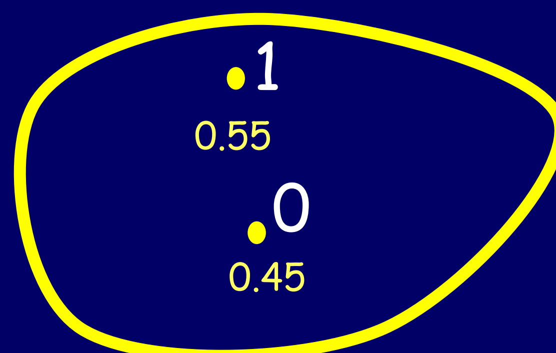
If you are computing an expectation, the thing whose expectation you are computing is a random variable.

Indicator R.V.s: $E[X_A] = \Pr(A)$

For event A, the indicator random variable for A:

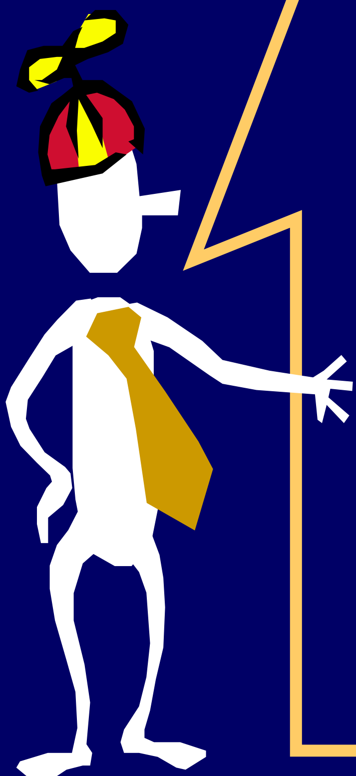
$$X_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \notin A$$





$$E[X_A] = 1 \times \Pr(X_A = 1) = \Pr(A)$$

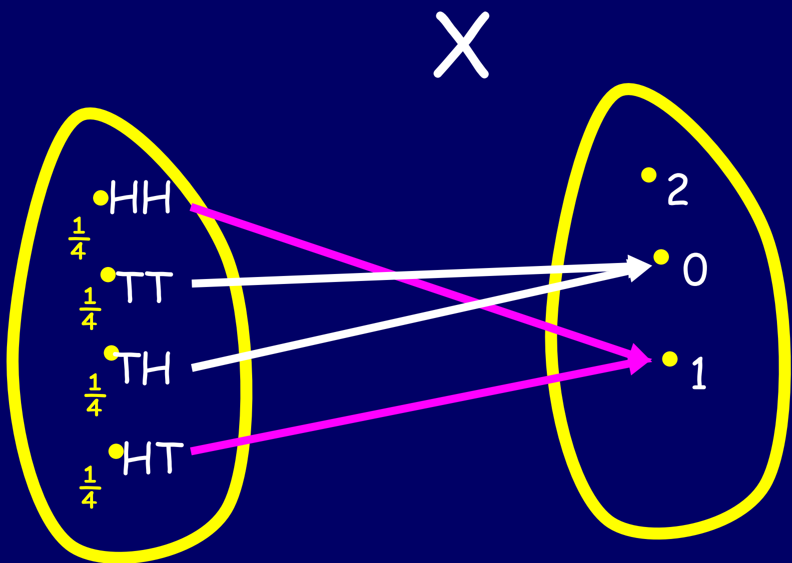
Adding Random Variables

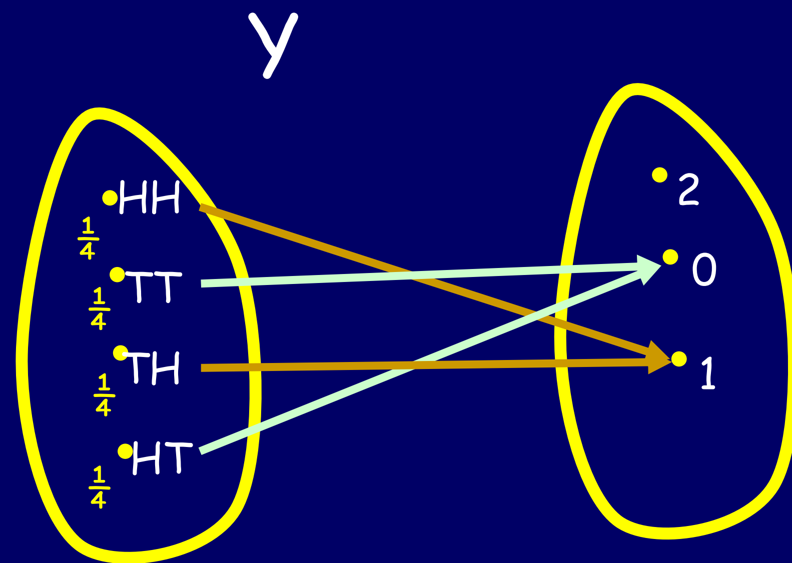


If X and Y are random variables (on the same set S), then $Z = X + Y$ is also a random variable.

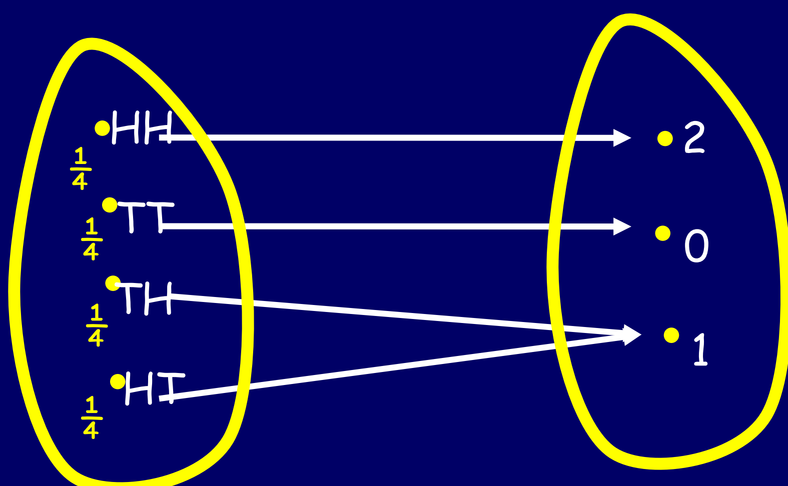
$$Z(x) \equiv X(x) + Y(x)$$

E.g., rolling two dice.
X = 1st die, Y= 2nd die,
Z = sum of two dice.

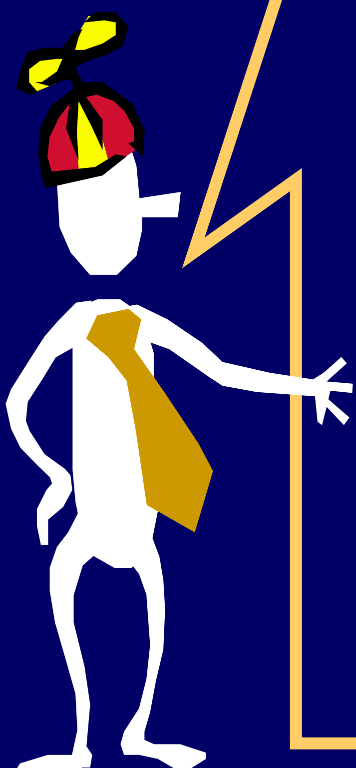

X
HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1


Y
HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1

$Z = X + Y$


HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1

Adding Random Variables



Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let Z = X+Y. Z measures the combined arm lengths.

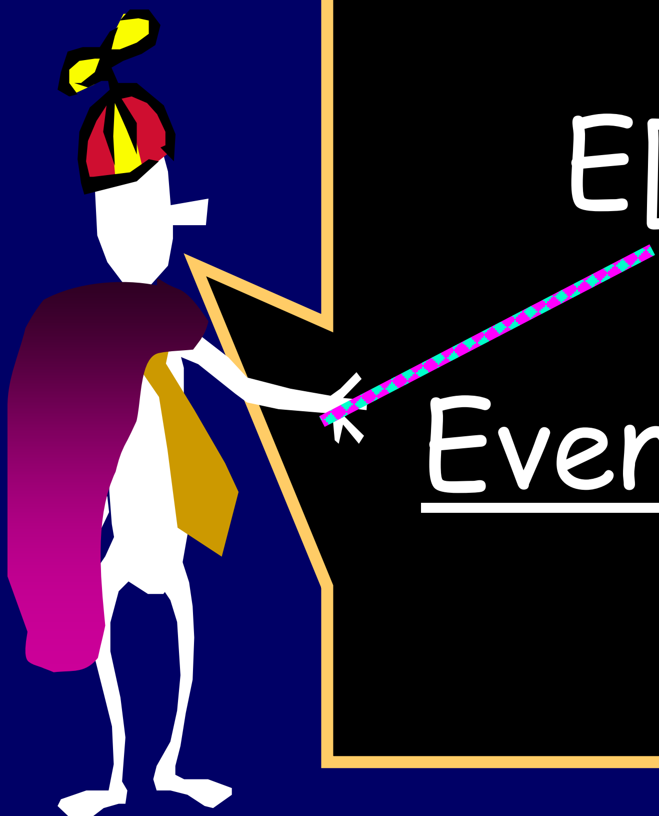
Formally, S = {people in world}, D = uniform distribution on S.

Independence

Two random variables X and Y are <u>independent</u> if for every a,b, the events X=a and Y=b are independent.

How about the case of
X=1st die, Y=2nd die?
X = left arm, Y=right arm?

Linearity of Expectation



If $Z = X+Y$, then

$$E[Z] = E[X] + E[Y]$$

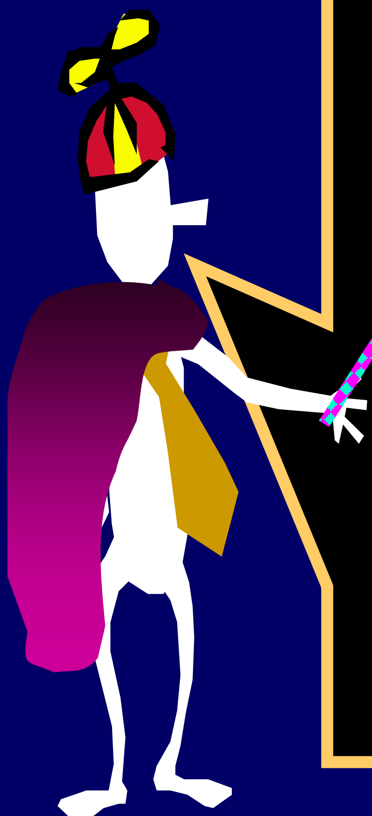
Even if X and Y are not independent.

Linearity of Expectation

If $Z = X+Y$, then

$$E[Z] = E[X] + E[Y]$$

Proof:



$$E[X] = \sum_{x\in S} \Pr(x)\, X(x)$$

$$E[Y] = \sum_{x\in S} \Pr(x)\, Y(x)$$

$$E[Z] = \sum_{x\in S} \Pr(x)\, Z(x)$$

but $Z(x) = X(x) + Y(x)$

Linearity of Expectation

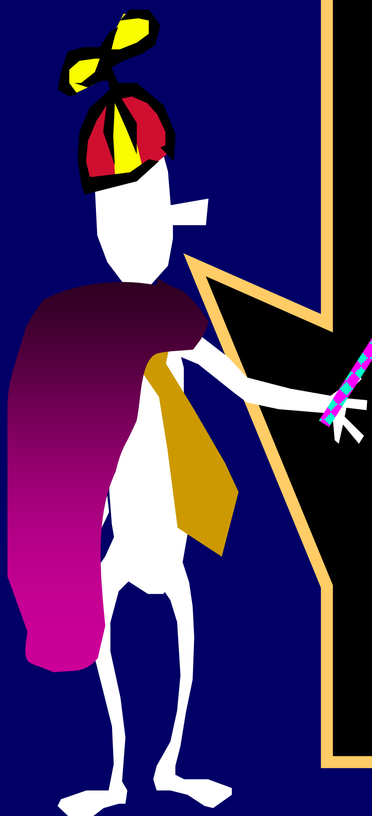
E.g., 2 fair flips:

X = 1 if 1st coin heads,

Y = 1 if 2nd coin heads

Z = X+Y = total # heads.

What is $E[X]$? $E[Y]$? $E[Z]$?



1,1,2
HH

1,0,1
HT

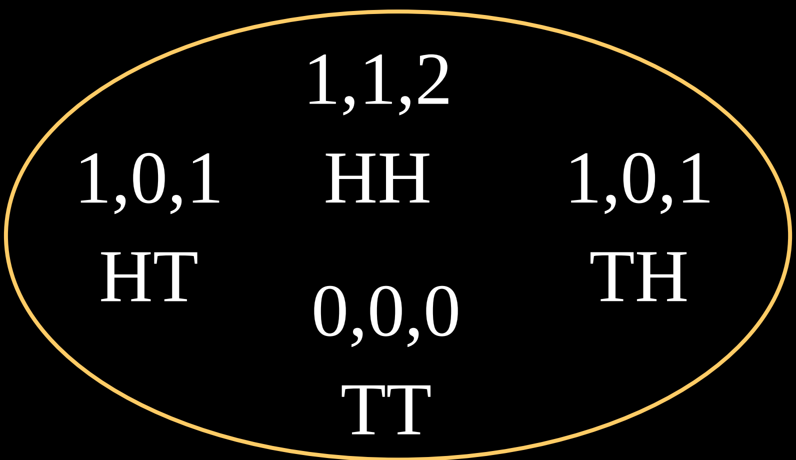
0,1,1
TH

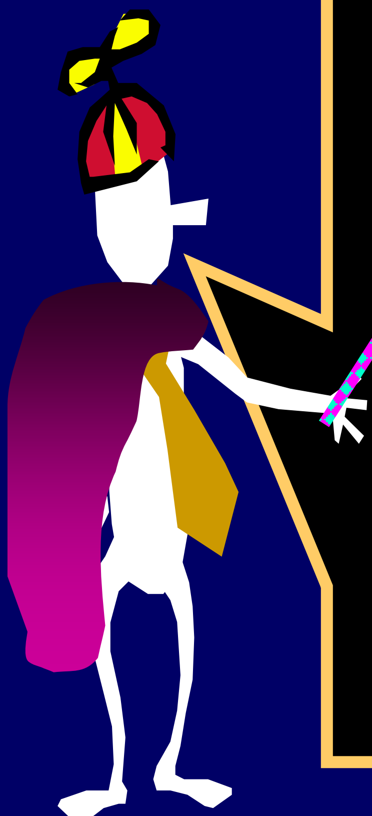
0,0,0
TT

Linearity of Expectation

E.g., 2 fair flips:
X = at least one coin heads,
Y = both coins are heads, $Z = X+Y$

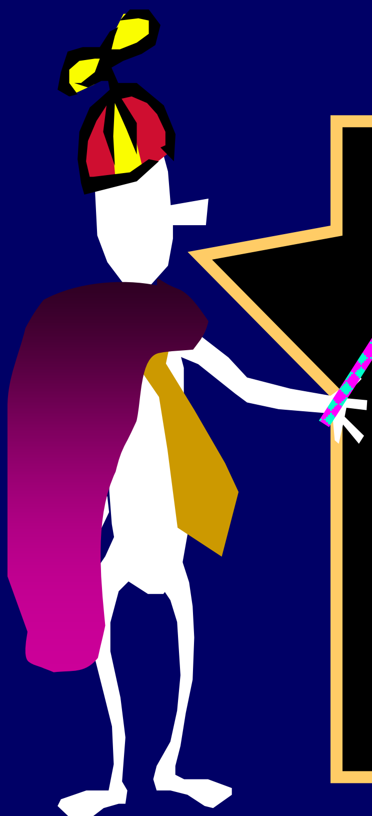
Are X and Y independent?
What is E[X] ? E[Y] ? E[Z] ?





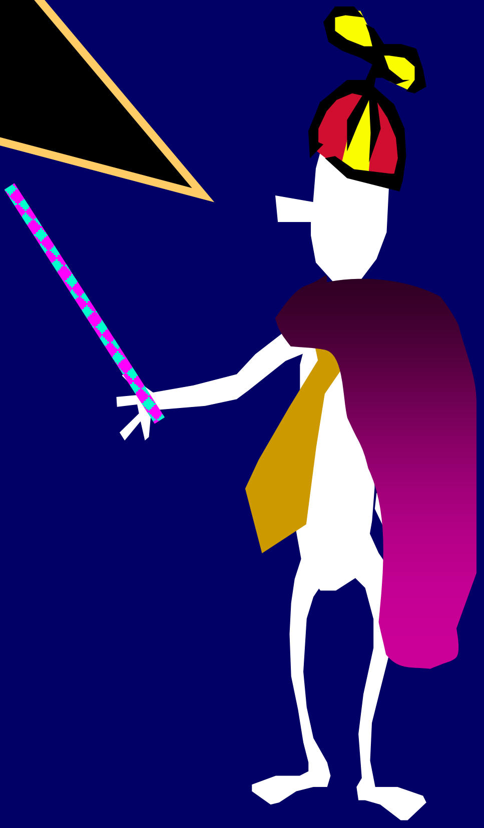
By induction

$$E[X_1 + X_2 + \ldots + X_n] =$$
$$E[X_1] + E[X_2] + \ldots. + E[X_n]$$

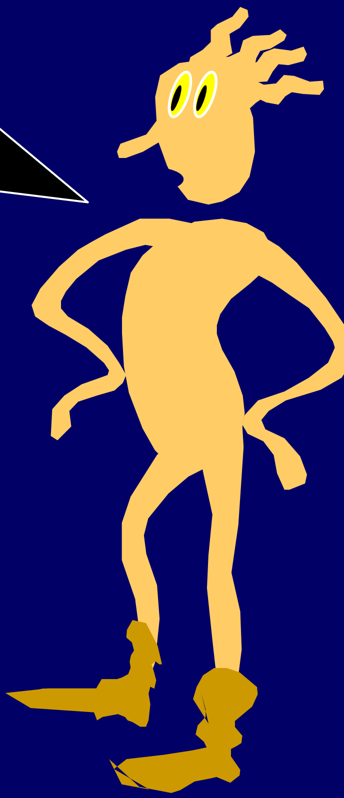


the expectation
of the sum
=
the sum of the
expectations

It is finally time to show off our probability prowess…



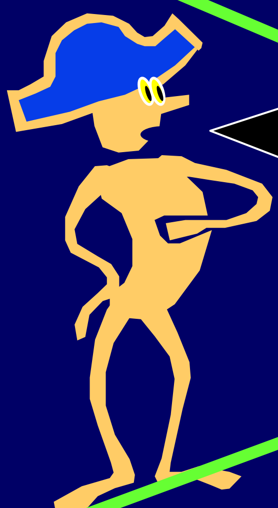
If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?



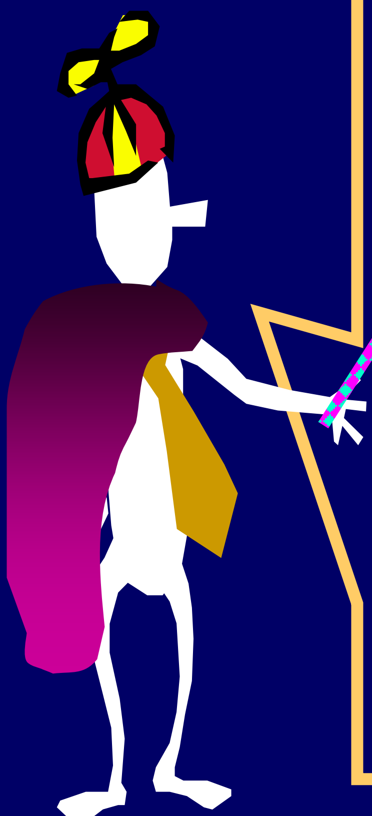
Hmm…

$\sum_k k \cdot \Pr($*exactly* k letters end up in correct envelopes)

$= \sum_k k \cdot$ (…aargh!!…)



Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in its correct envelope.

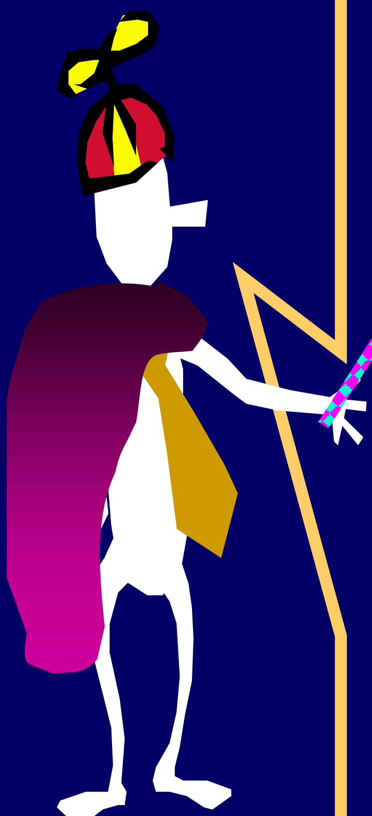
Let X_i be the indicator R.V. for A_i.

$$X_i = \begin{cases} 1 \text{ if } A_i \text{ occurs} \\ 0 \text{ otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$.

We are asking for $E[Z]$.

Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in the correct envelope.
Let X_i be the indicator R.V. for A_i.
Let $Z = X_1 + \ldots + X_n$. We are asking for E[Z].

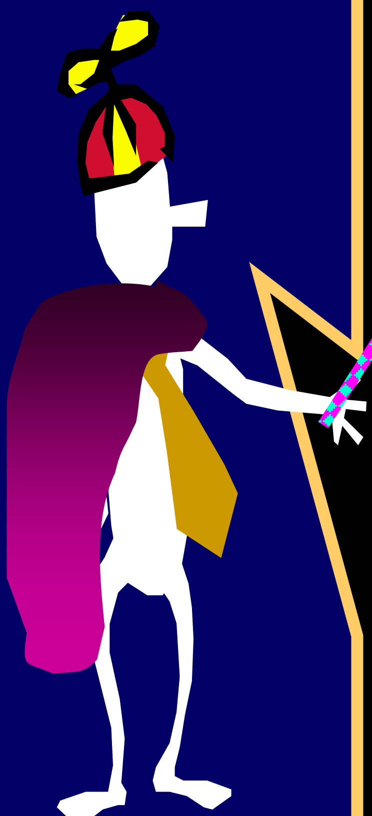
What is $E[X_i]$?

$$E[X_i] = \Pr(A_i) = 1/100.$$

What is E[Z]?

$$\begin{aligned} E[Z] &= E[X_1 + \ldots + X_{100}] \\ &= E[X_1] + \ldots + E[X_{100}] \\ &= 1/100 + \ldots + 1/100 = 1. \end{aligned}$$

Use Linearity of Expectation



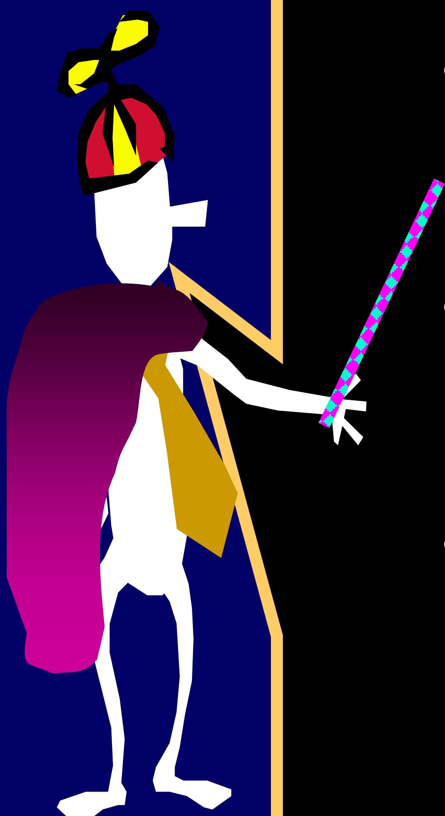
So, in expectation, 1 card will be in the same position as it started.

Pretty neat: it doesn't depend on how many cards!

Question: were the X_i independent?

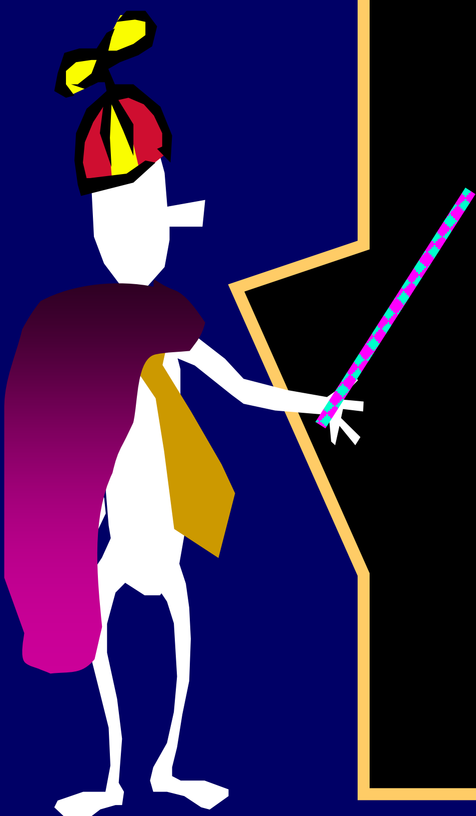
No! E.g., think of n=2.

Use Linearity of Expectation



General approach:

- View thing you care about as expected value of some RV.
- Write this RV as sum of simpler RVs (typically indicator RVs).
- Solve for their expectations and add them up!



Example

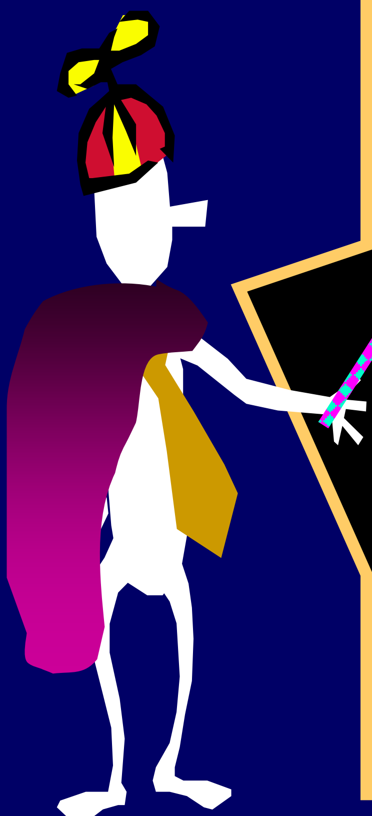
We flip n coins of bias p. What is the expected number of heads?

We <u>could</u> do this by summing

$$\sum_k k \Pr(X = k)$$

$$= \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

But now we know a better way



Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

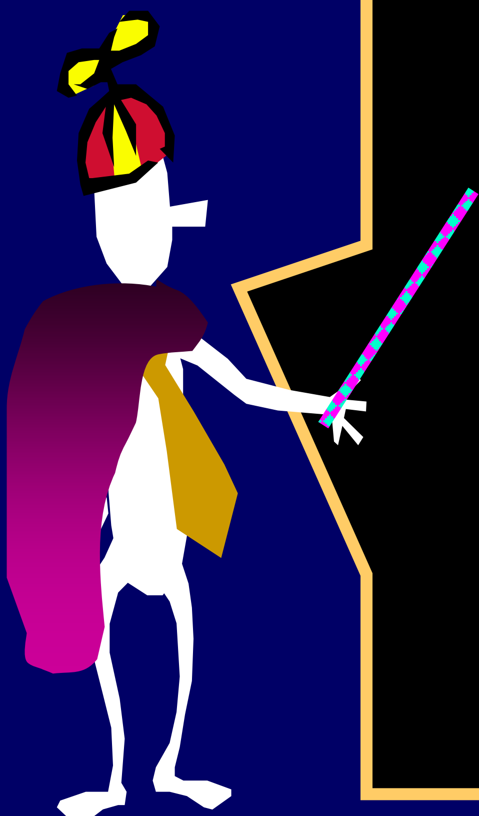
$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[\ X\ ] = E[\ \Sigma_i X_i\ ] = ?$$

Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

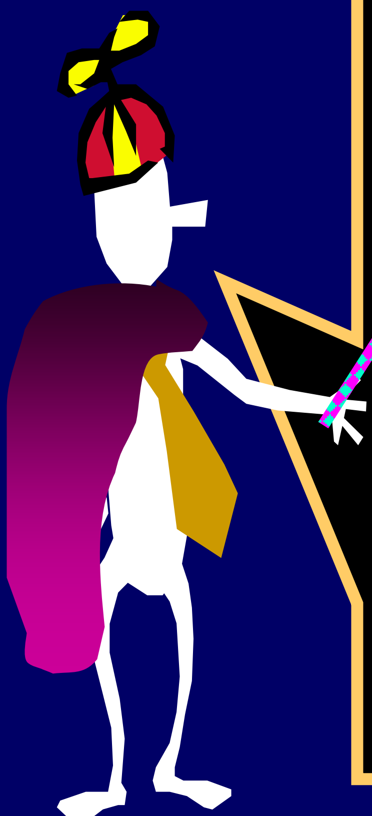


$$E[\,X\,] = E[\,\Sigma_i X_i\,] = np$$

What about Products?

If $Z = XY$, then
$E[Z] = E[X] \times E[Y]$?

No!



X=indicator for "1st flip is heads"
Y=indicator for "1st flip is tails".

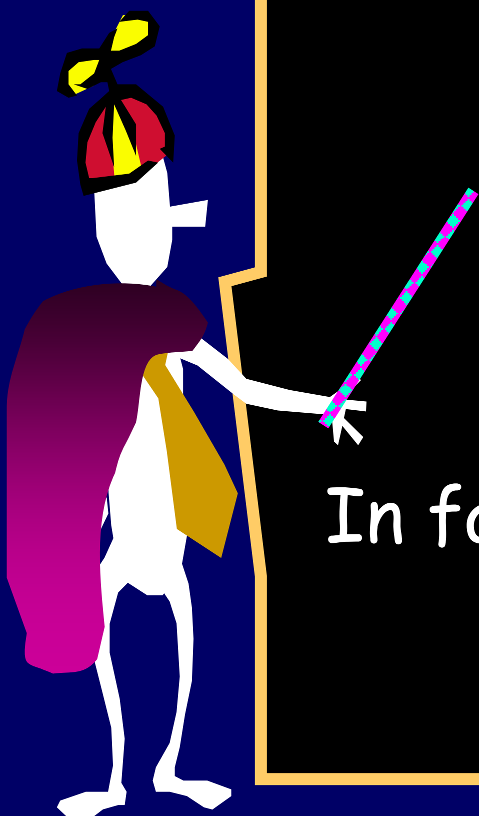
$E[XY]=0$.

But it is true if RVs are independent

Proof:

$$E[X] = \textstyle\sum_a a \times \Pr(X=a)$$
$$E[Y] = \textstyle\sum_b b \times \Pr(Y=b)$$

$$\begin{aligned}
E[XY] &= \textstyle\sum_c c \times \Pr(XY = c) \\
&= \textstyle\sum_c \sum_{a,b:ab=c} c \times \Pr(X=a \cap Y=b) \\
&= \textstyle\sum_{a,b} ab \times \Pr(X=a \cap Y=b) \\
&= \textstyle\sum_{a,b} ab \times \Pr(X=a)\Pr(Y=b) \\
&= E[X]\, E[Y]
\end{aligned}$$



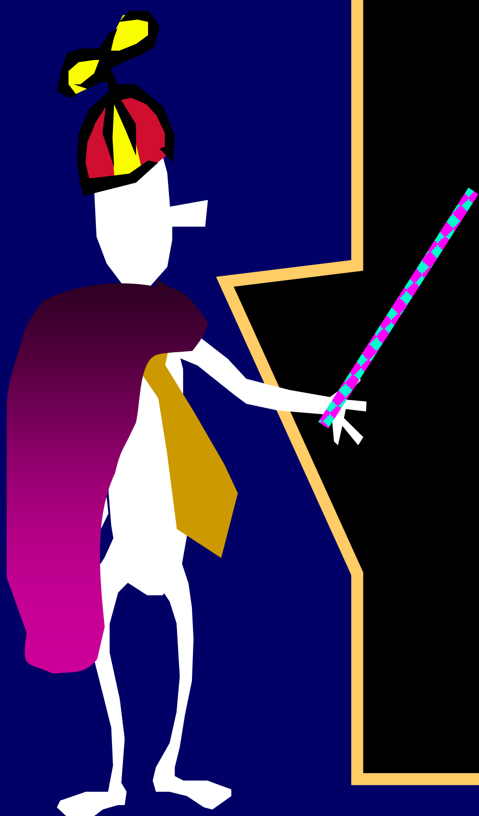
E.g., 2 fair flips.
X = indicator for 1st coin being heads,
Y = indicator for 2nd coin being heads.
XY = indicator for "both are heads".

$$E[X] = \tfrac{1}{2},\ E[Y] = \tfrac{1}{2},\ E[XY] = \tfrac{1}{4}.$$

$$E[X*X] = E[X]^2?$$

No: $E[X^2] = \tfrac{1}{2},\ E[X]^2 = \tfrac{1}{4}.$

In fact, $E[X^2] - E[X]^2$ is called the *variance* of X.

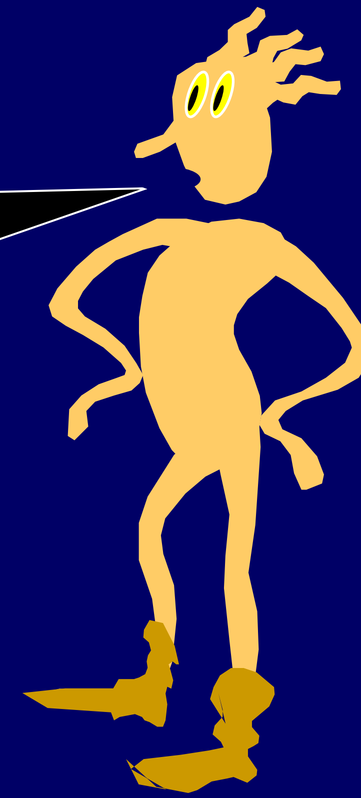


Most of the time, though, power will come from using sums.

Mostly because Linearity of Expectations holds even if RVs are not independent.

Another problem

On average, in class of size m, how many pairs of people will have the same birthday?



$\sum_k k \cdot \Pr(\text{exactly } k \text{ collisions})$

$= \sum_k k \cdot (\ldots\text{aargh!!!!}\ldots)$

Use linearity of expectation.

Suppose we have m people each with a uniformly chosen birthday from 1 to 366.

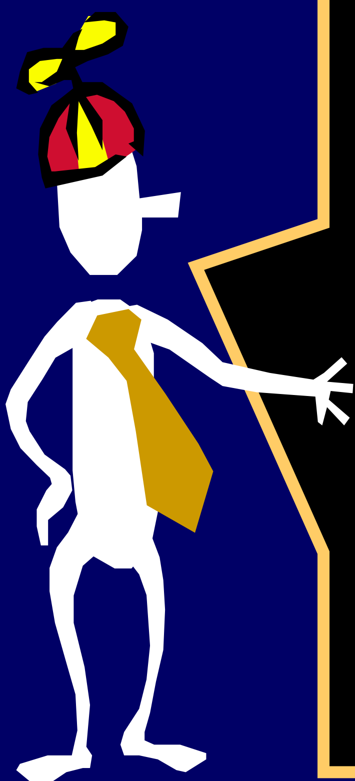
X = number of pairs of people with the same birthday.

E[X] = ?

X = number of pairs of people with the same birthday.

$E[X] = ?$

Use $m(m-1)/2$ indicator variables, one for each pair of people.



$X_{jk} = 1$ if person j and person k have the same birthday; else 0.

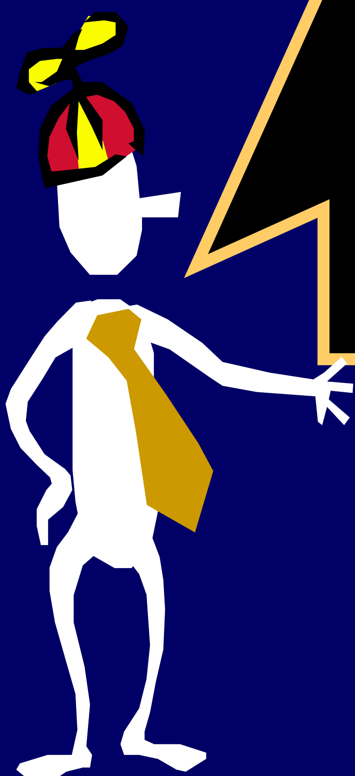
$$E[X_{jk}] = (1/366)\ 1 + (1 - 1/366)\ 0$$
$$= 1/366$$

X = number of pairs of people with the same birthday.

$E[X] = E[\,\Sigma_{j \le k \le m} X_{jk}\,]$

There are many dependencies among the indicator variables. E.g., X_{12} and X_{13} and X_{23} are dependent.

But we don't care!



X = number of pairs of people with the same birthday.

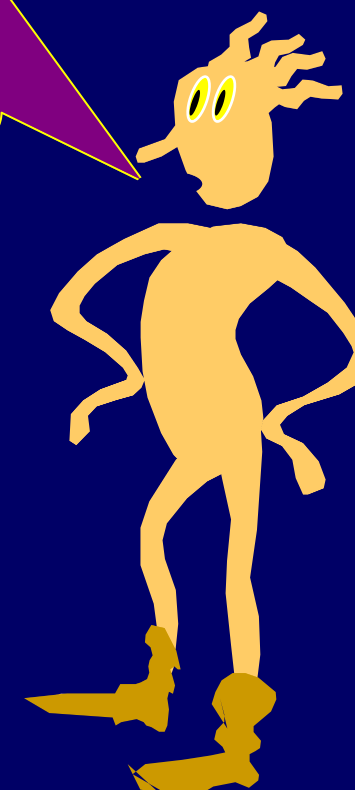
$$E[X] = E[\, \Sigma_{j \le k \le m} X_{jk} \,]$$

$$= \Sigma_{j \le k \le m} E[\, X_{jk} \,]$$

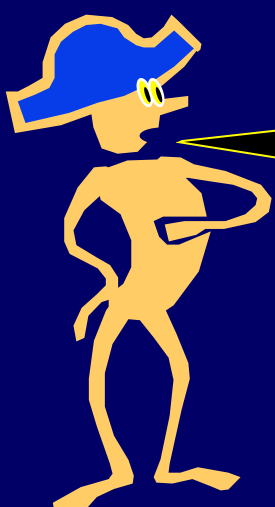
$$= m(m-1)/2 \times 1/366$$

Step right up…

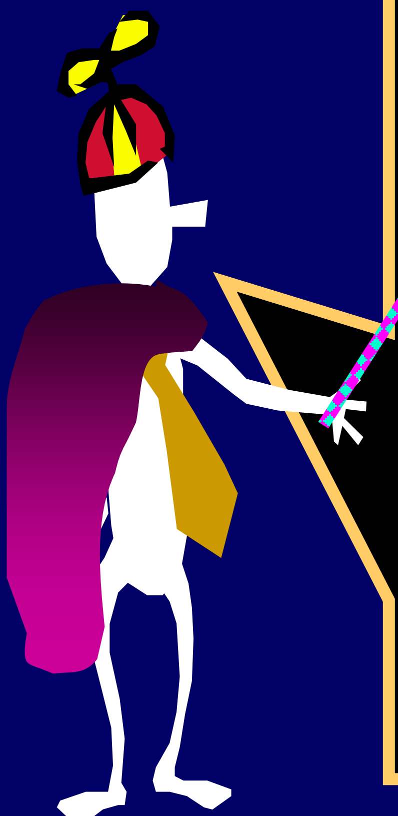
You pick a number $n \in [1..6]$. You roll 3 dice. If any match n, you win \$1. Else you pay me \$1. Want to play?



Hmm…
let's see



Analysis



A_i = event that i-th die matches.

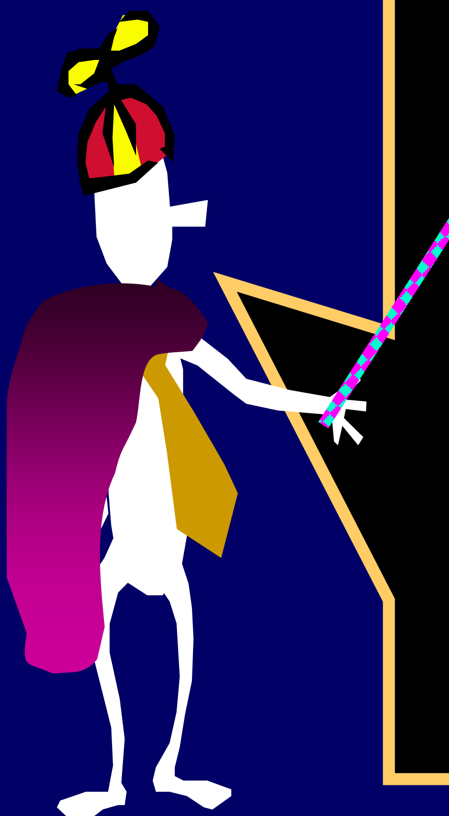
X_i = indicator RV for A_i.

Expected number of dice that match:

$$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = \tfrac{1}{2}.$$

But this is not the same as Pr(at least one die matches).

Analysis



Pr(at least one die matches)

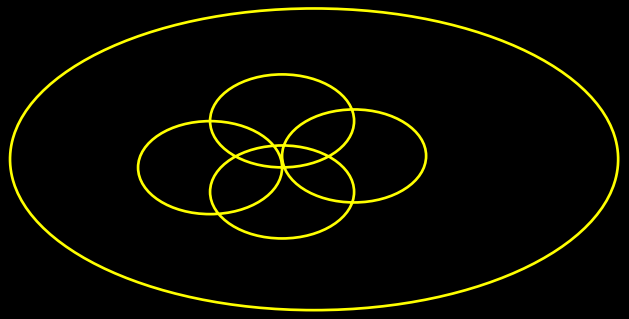
$= 1 - \Pr(\text{none match})$

$= 1 - (5/6)^3 = 0.416.$

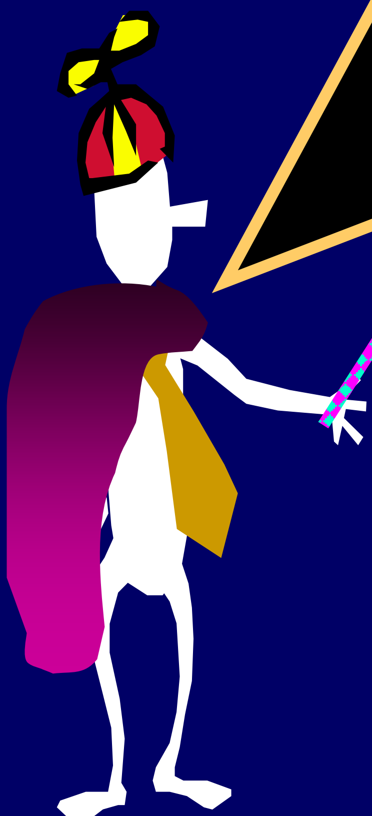
What's going on?

Say we have a collection of events A_1, A_2, ...

How does E[# events that occur] compare to Pr(at least one occurs)?



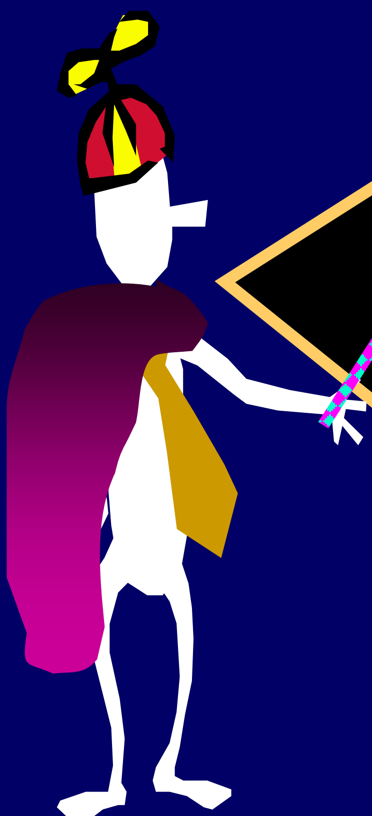
What's going on?



$$E[\#\text{ events that occur}]$$
$$= \sum_k \Pr(k \text{ events occur}) \times k$$
$$= \sum_{(k>0)} \Pr(k \text{ events occur}) \times k$$

$$\Pr(\text{at least one event occurs})$$
$$= \sum_{(k>0)} \Pr(k \text{ events occur})$$

What's going on?



Moral #1: be careful you are modeling problem correctly.

Moral #2: watch out for carnival games.