Great Theoretical Ideas In Computer Science

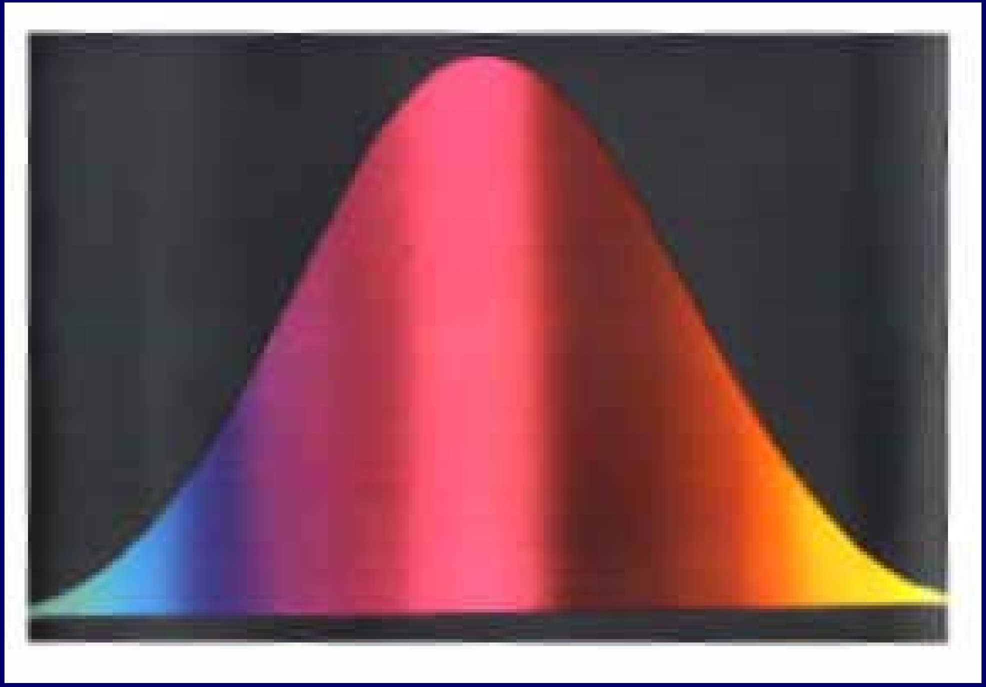
John Lafferty

CS 15-251 Fall 2006

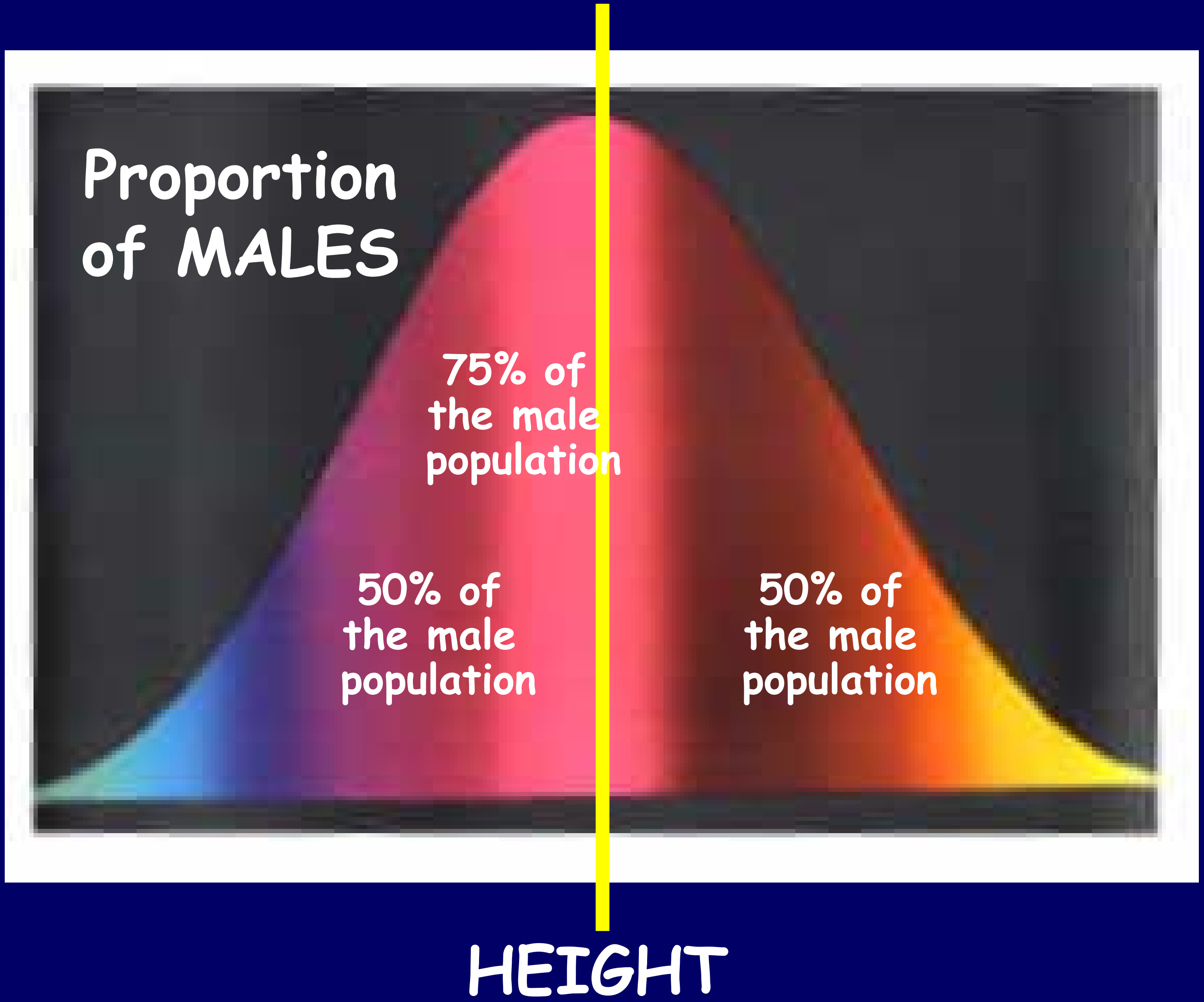
Lecture 9 September 26 2006

Carnegie Mellon University

Probability Theory: Counting in Terms of Proportions



A Probability Distribution



The Descendants Of Adam

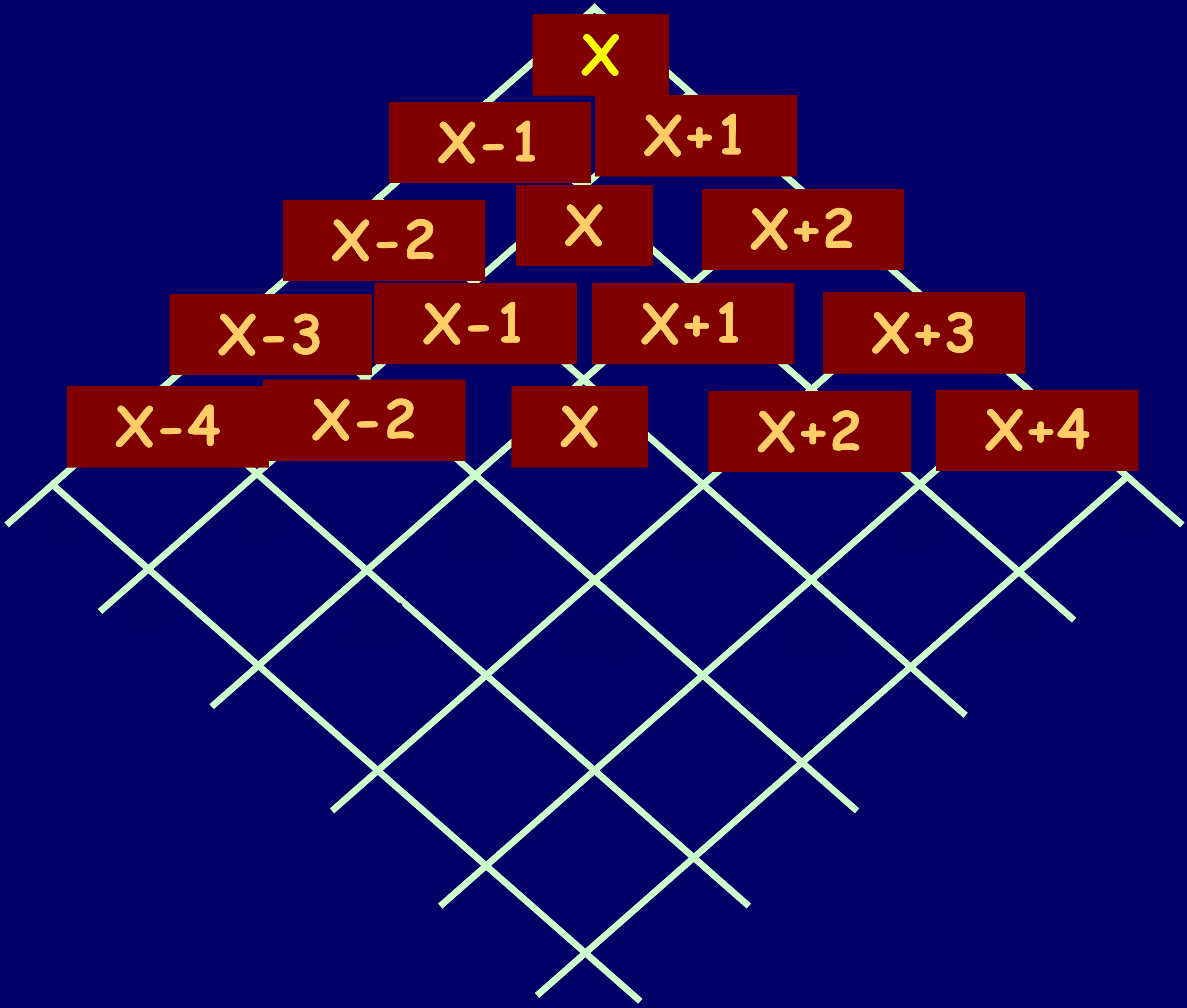
Adam was X inches tall.

He had two sons

One was X+1 inches tall

One was X-1 inches tall

Each of his sons had two sons …


X
X-1
X+1
X-2
X
X+2
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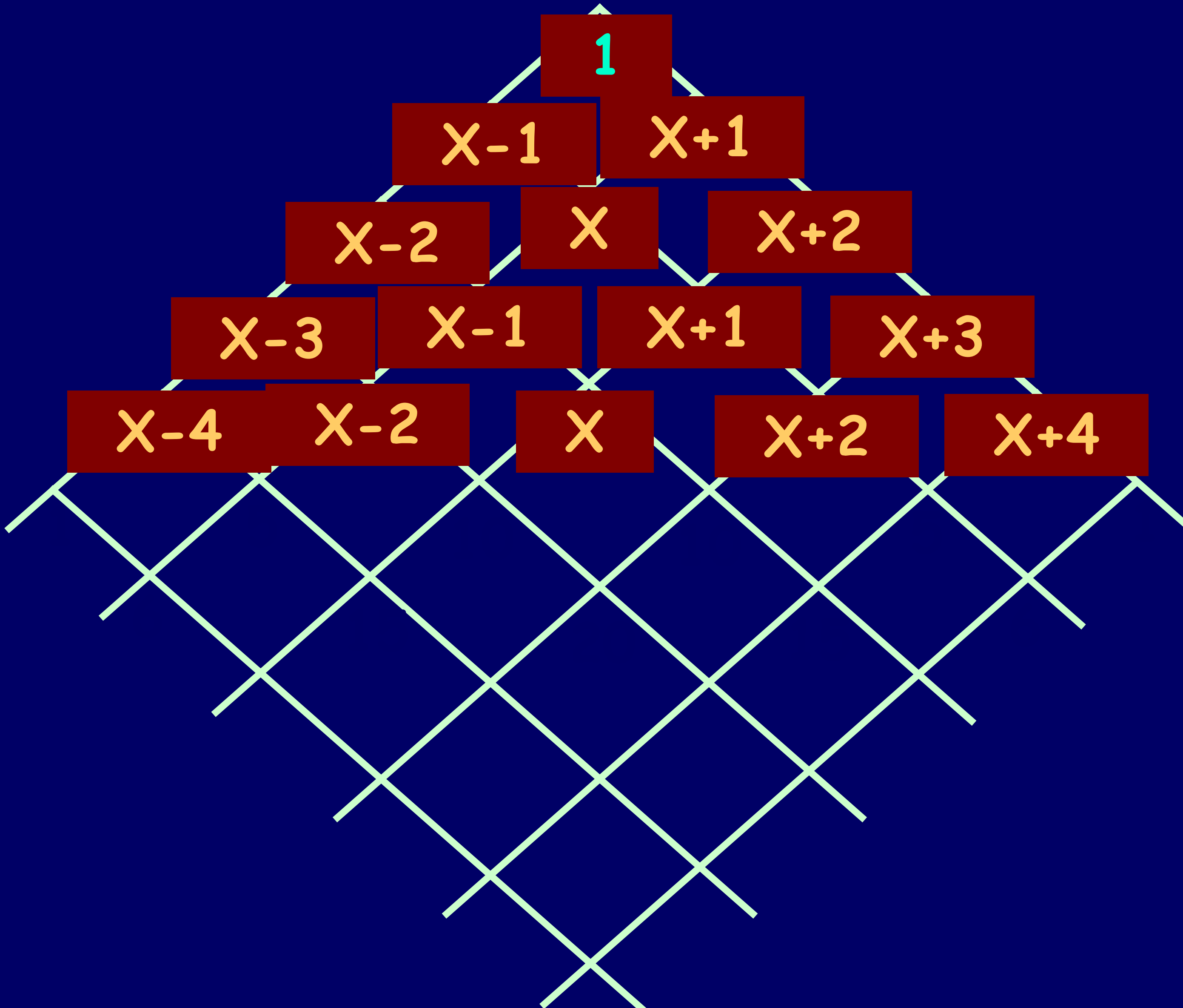
X+3
X-4
X-2
X
X+2
X+4


1
X-1
X+1
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X+2
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X+3
X-4
X-2
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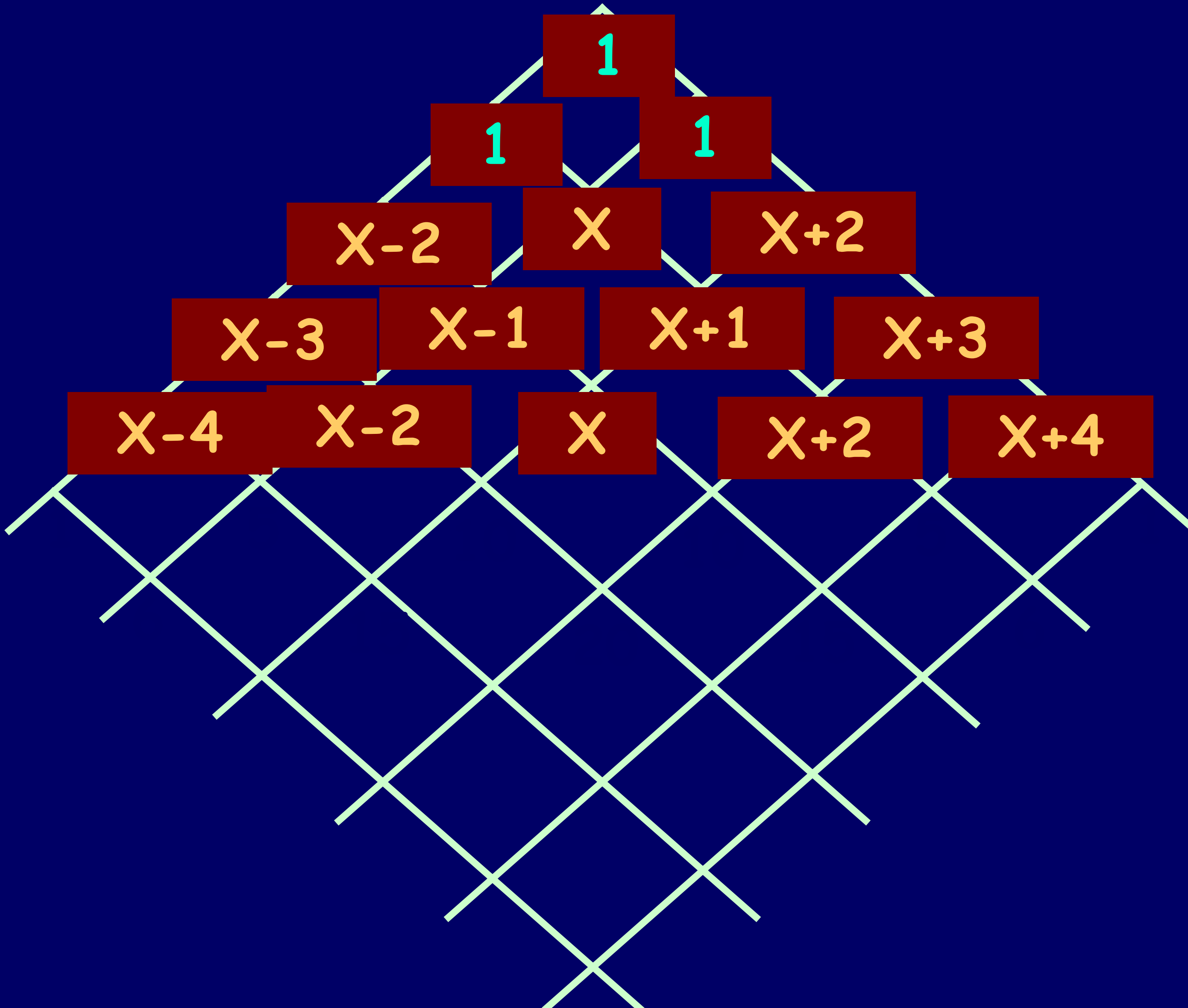
X+2
X+4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X+2
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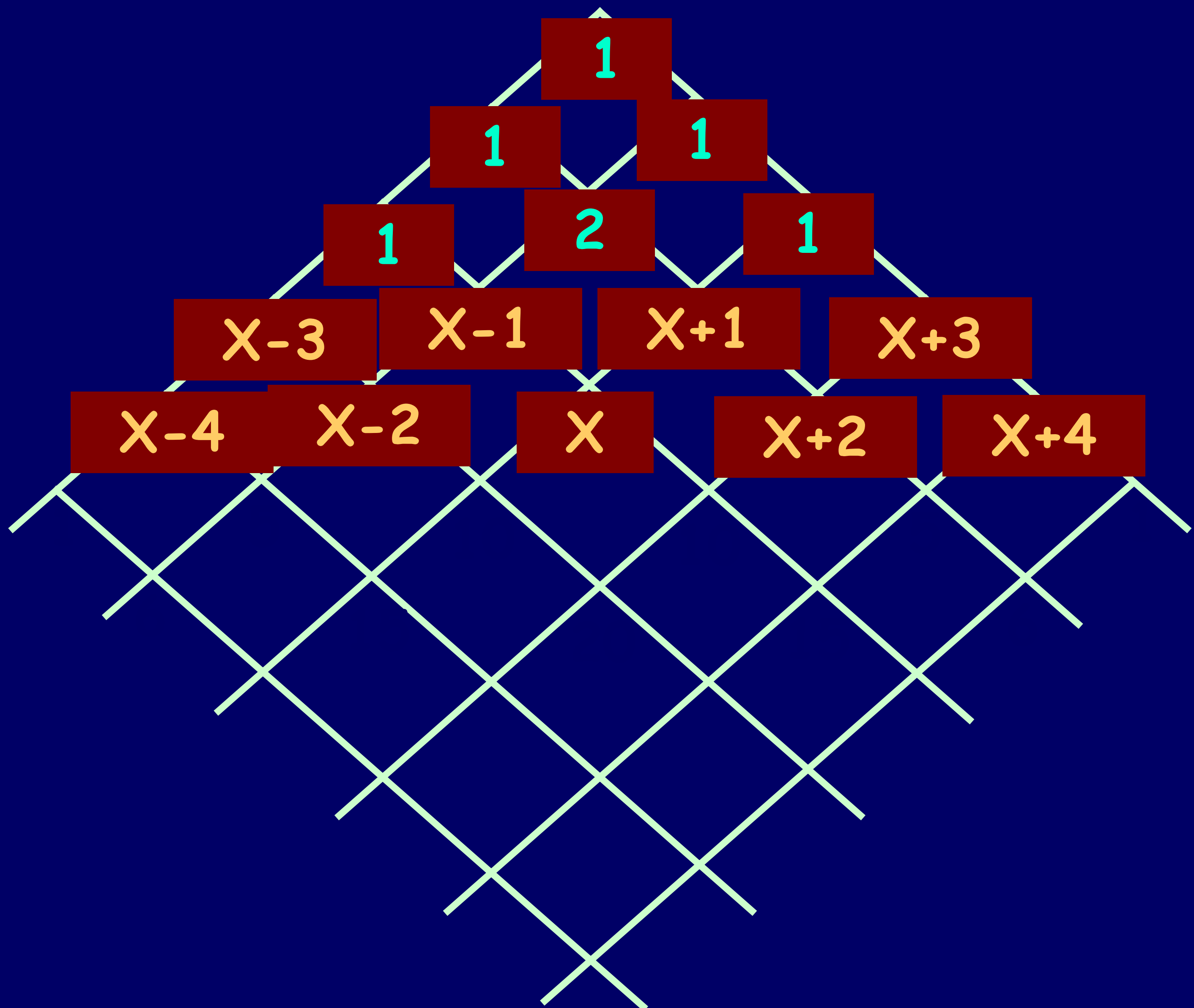
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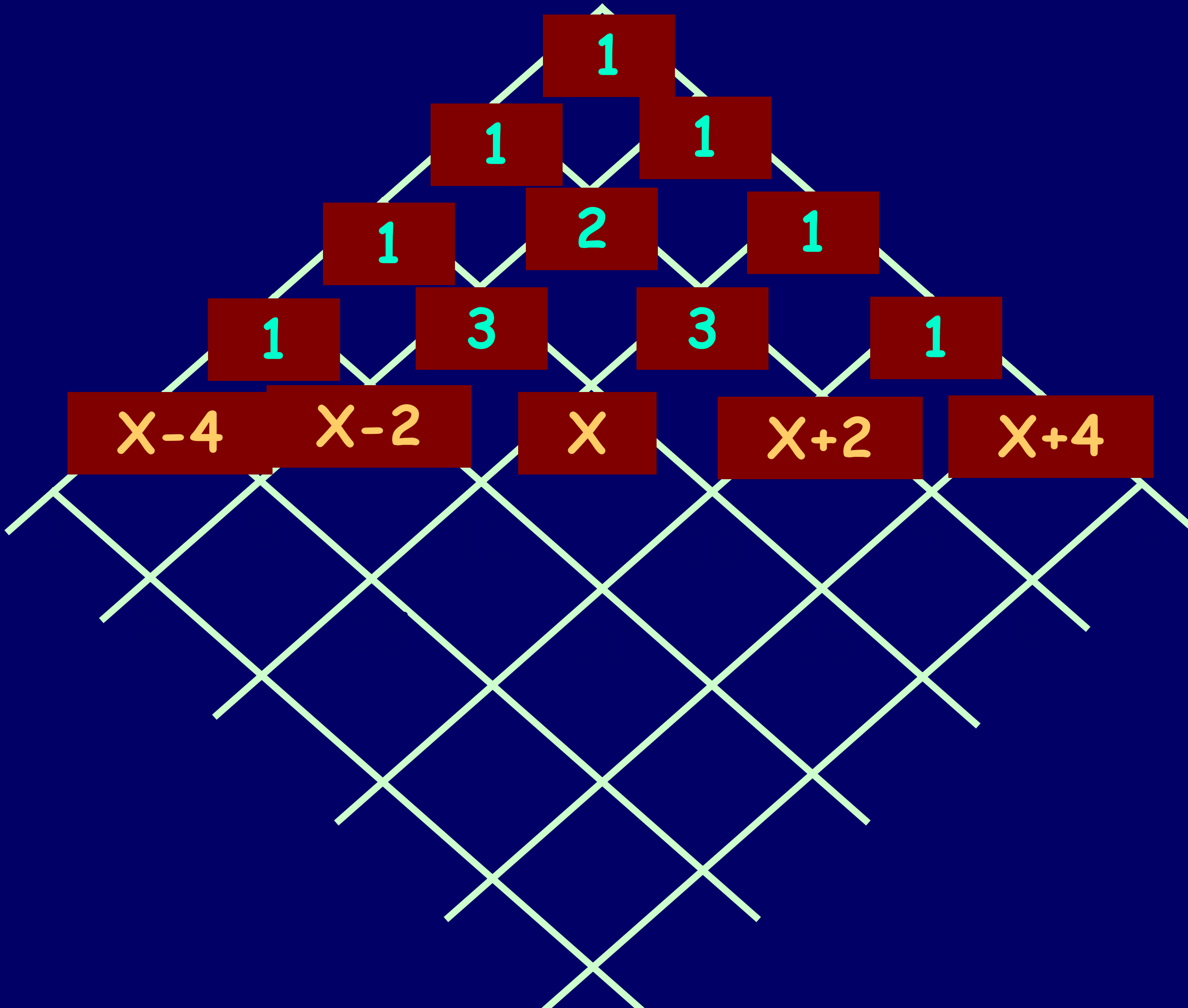
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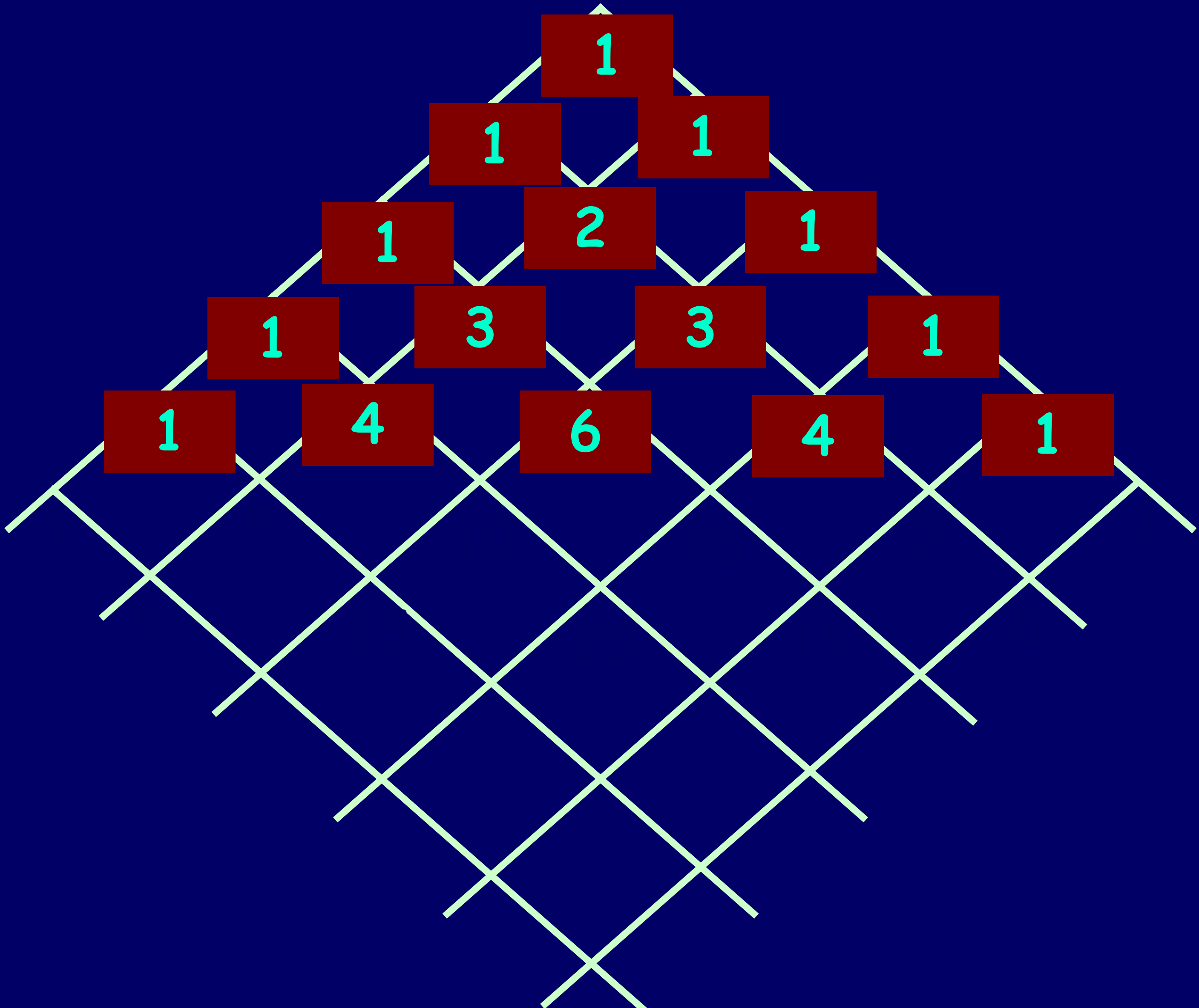
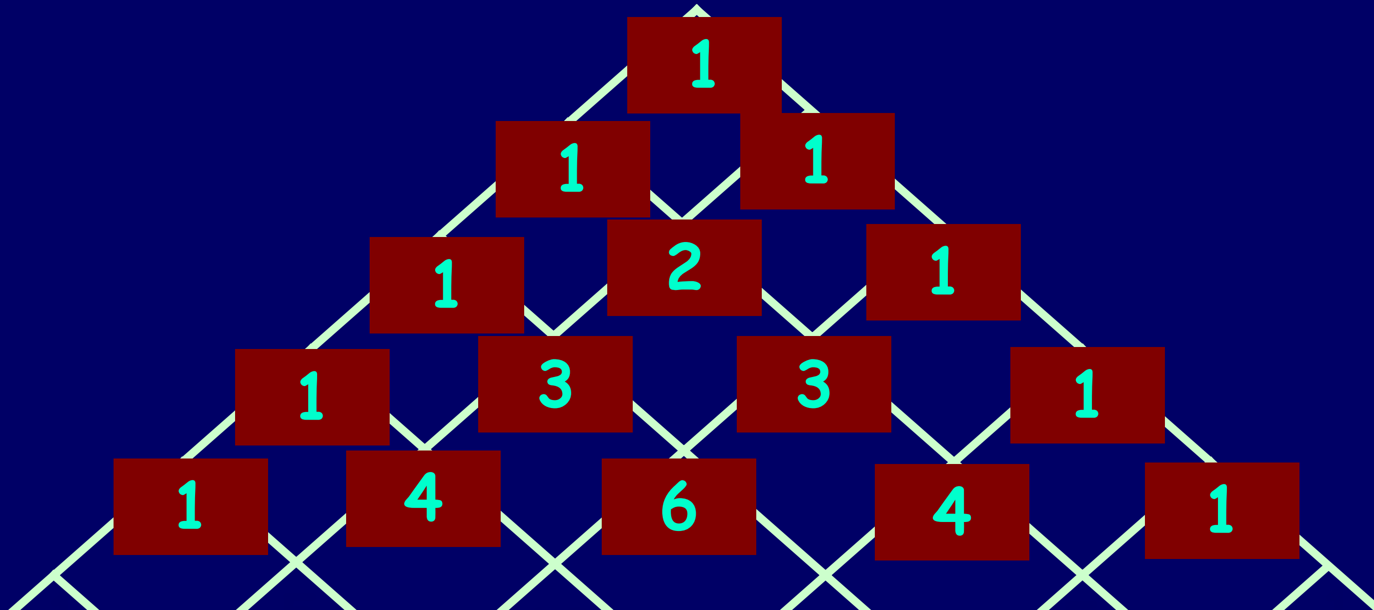
3
3
1
X-4
X-2
X
X+2
X+4


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1



In n^{th} generation, there will be 2^n males, each with one of n+1 different heights:

$h_0 < h_1 < \ldots < h_n$.

$h_i = (X - n + 2i)$ occurs with proportion

$$\frac{\binom{n}{i}}{2^n}$$

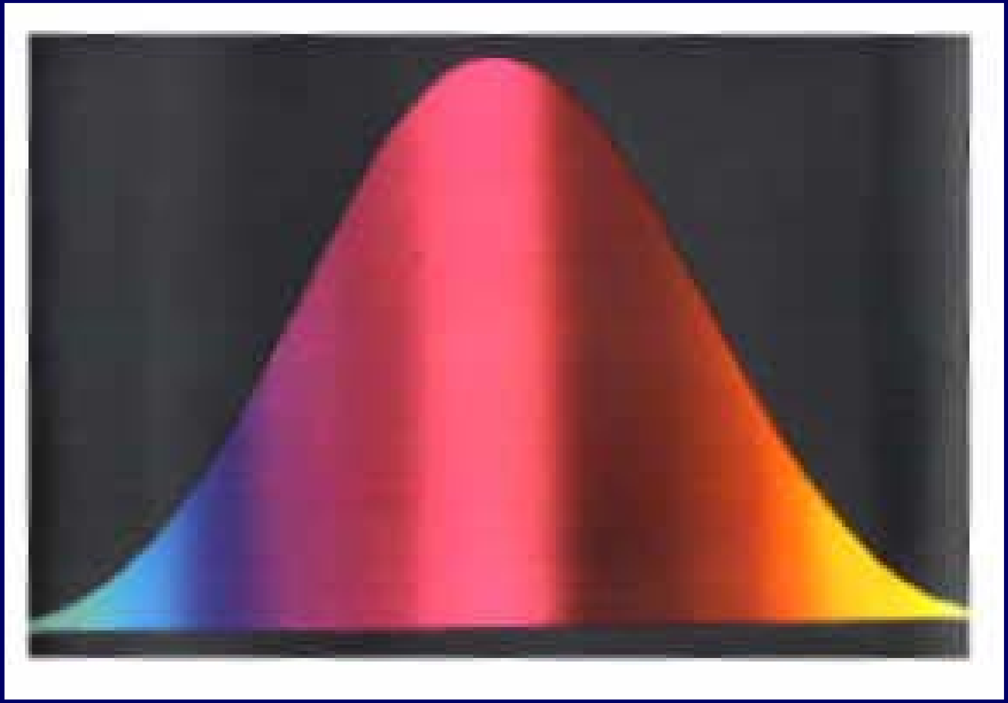
Unbiased Binomial Distribution On n+1 Elements.

Let S be any set $\{h_0, h_1, \ldots, h_n\}$ where each element h_i has an associated probability

$$\frac{\binom{n}{i}}{2^n}$$

Any such distribution is called a (unbiased) Binomial Distribution.

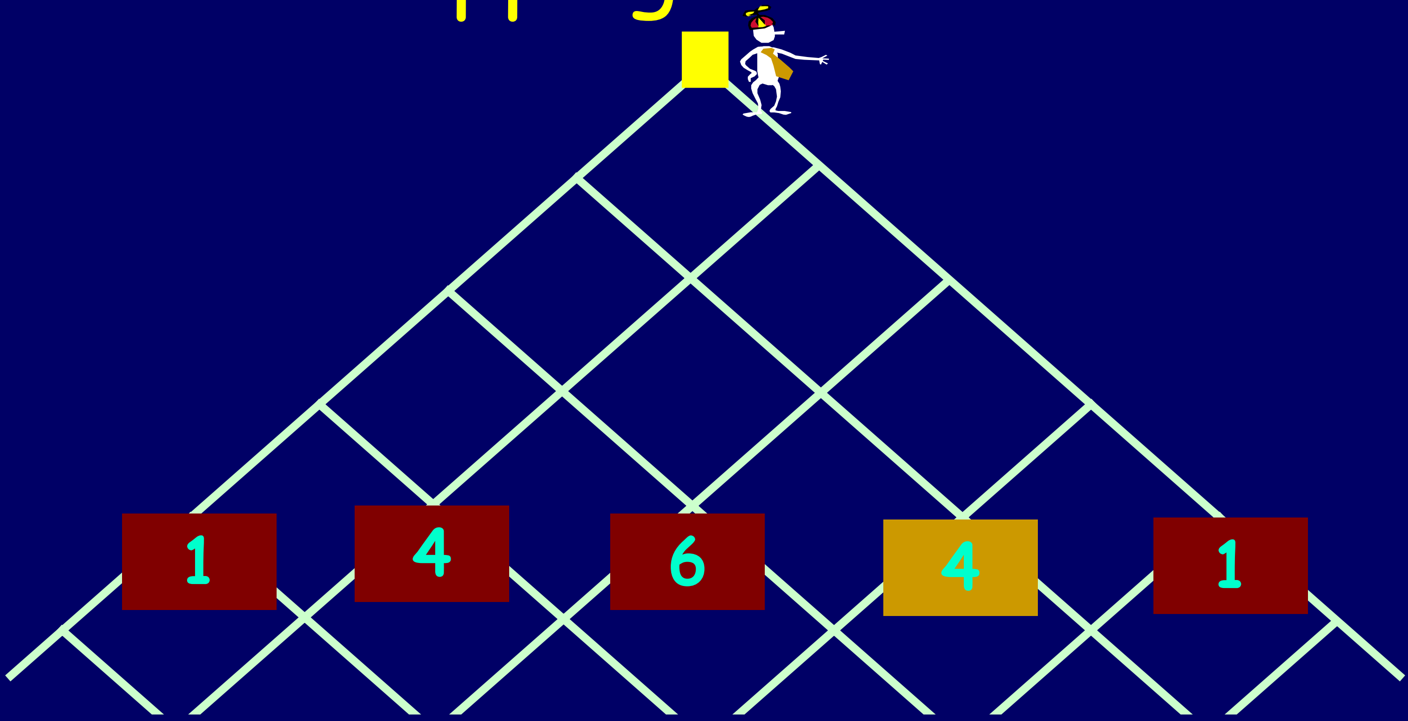
As the number of elements gets larger, the shape of the unbiased binomial distribution converges to a Normal (or Gaussian) distribution.



Mean

As the number of elements gets larger, the shape of the unbiased binomial distribution converges to a Normal (or Gaussian) distribution.

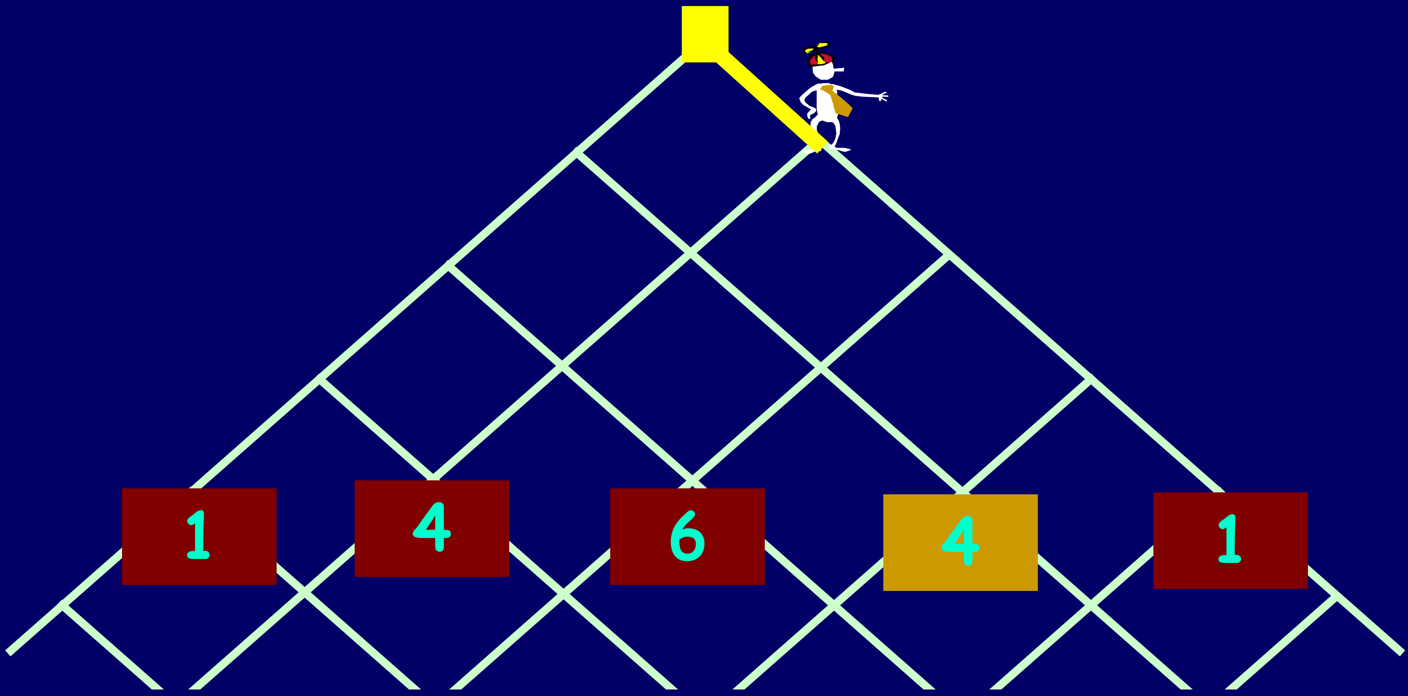
Coin Flipping in Manhattan



At each step, we flip a coin to decide which way to go.

What is the probability of ending at the intersection of Avenue i and Street (n-i) after n steps?

Coin Flipping in Manhattan



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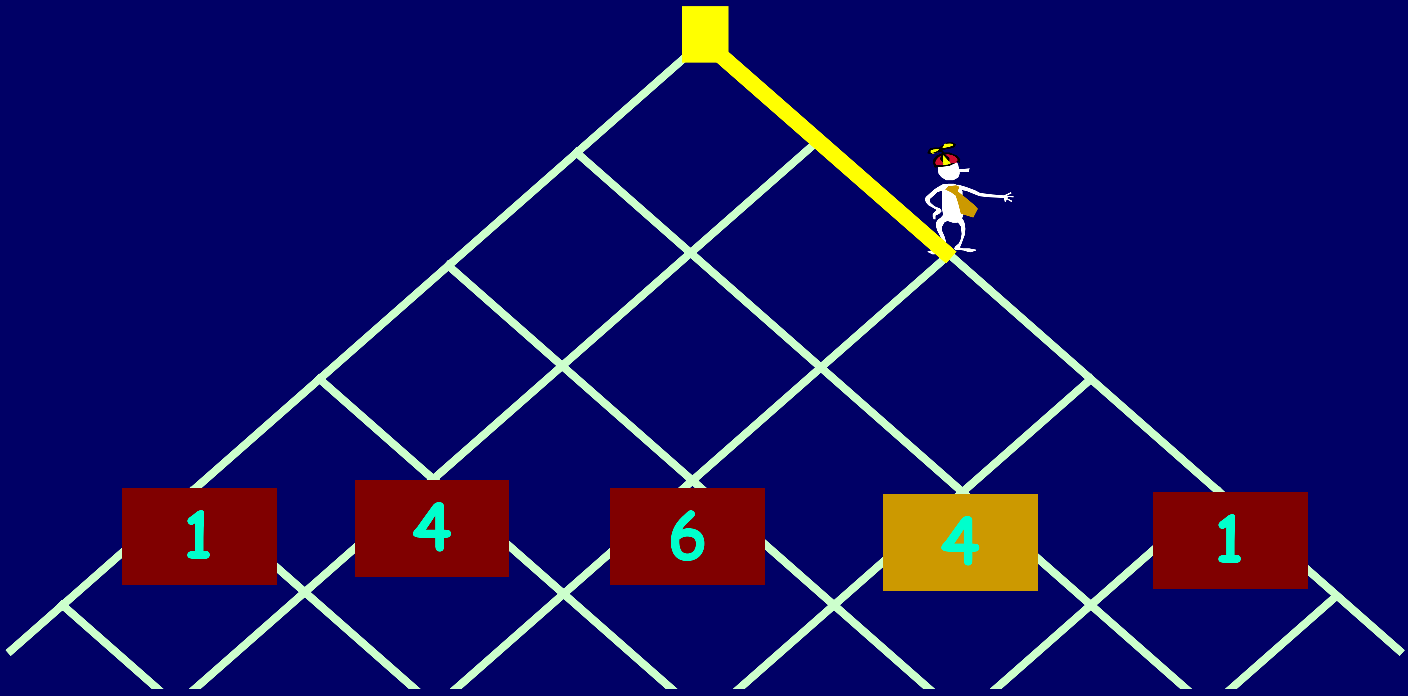

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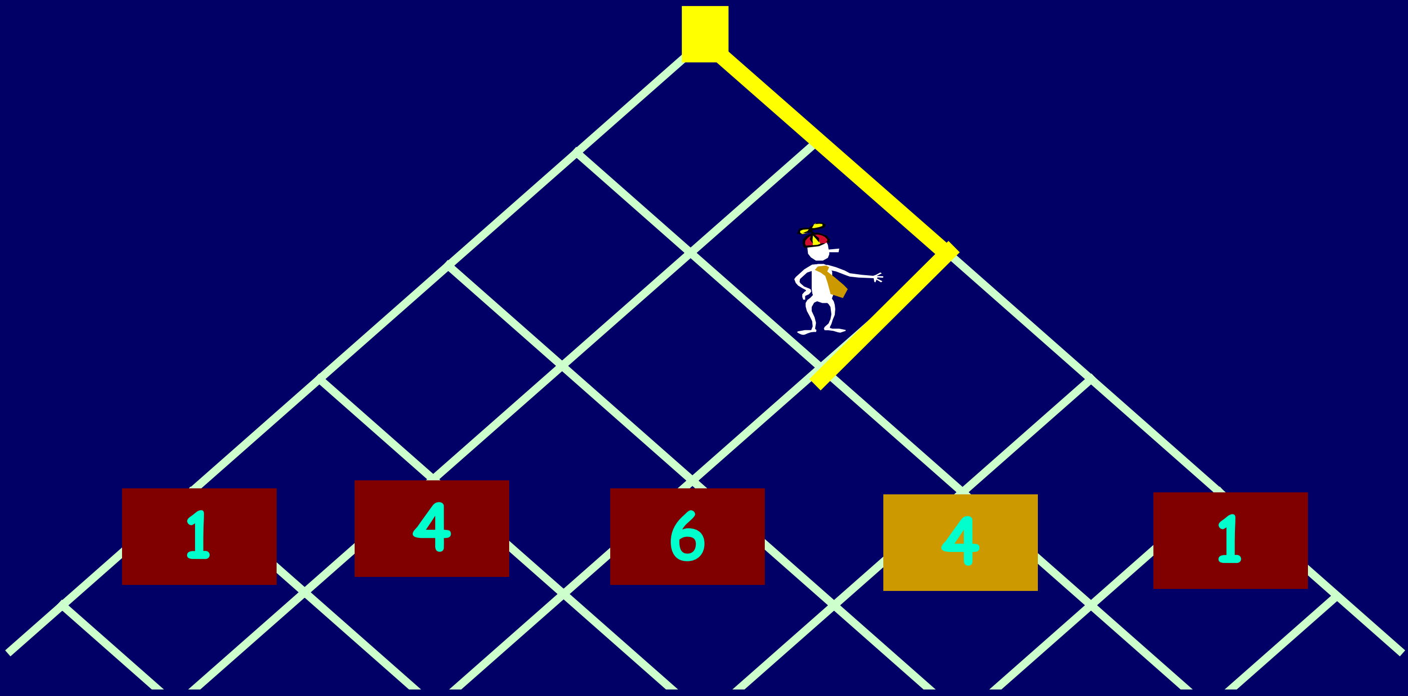

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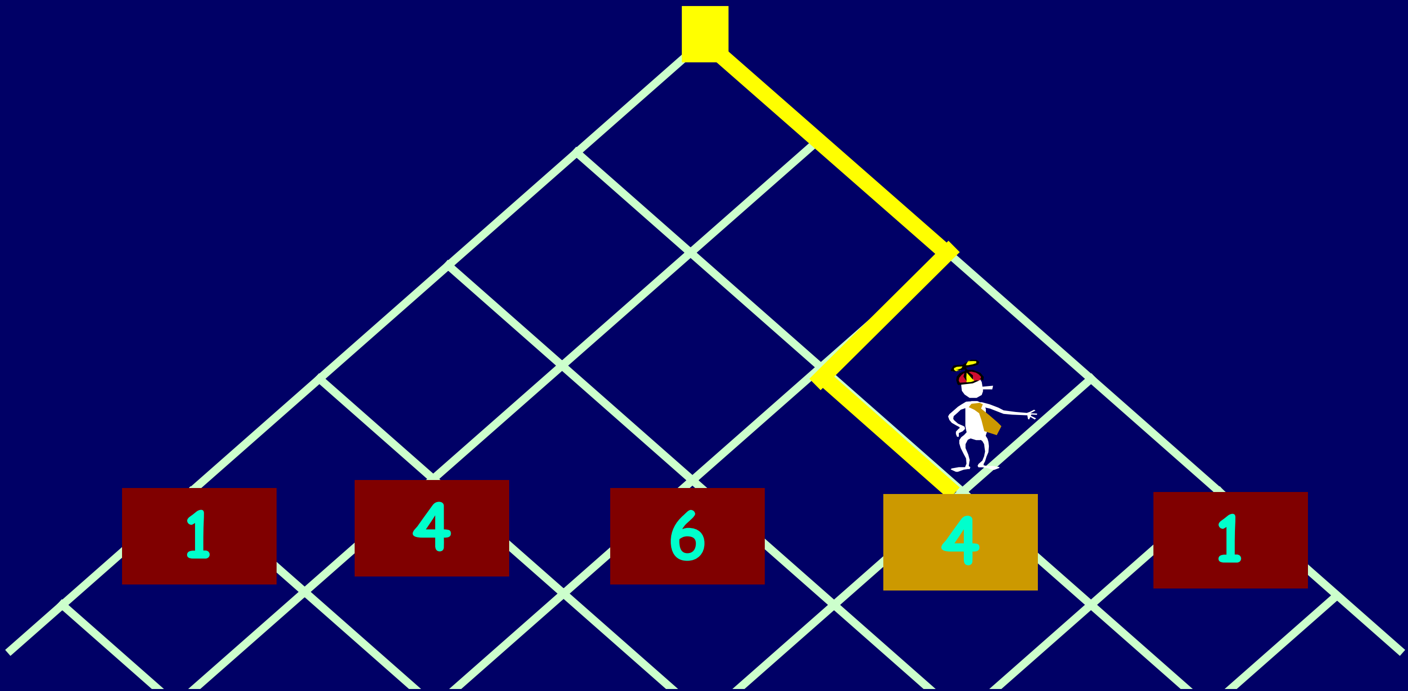

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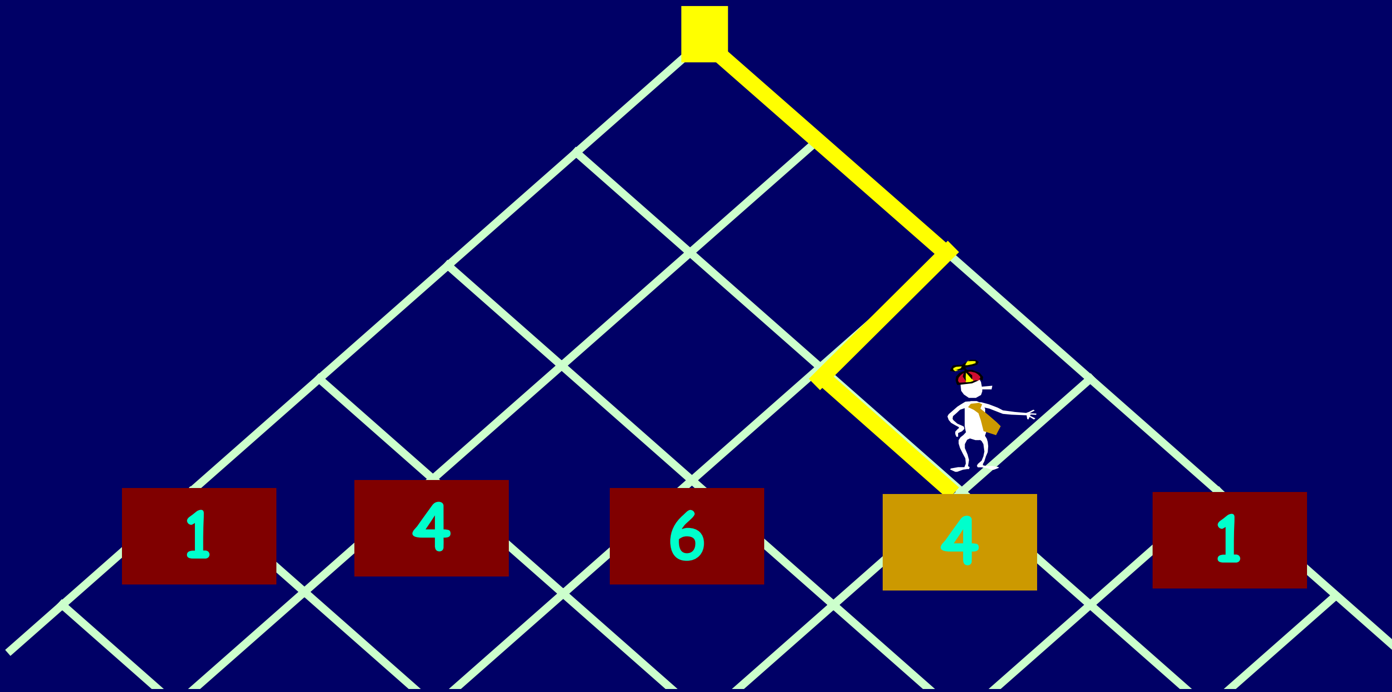
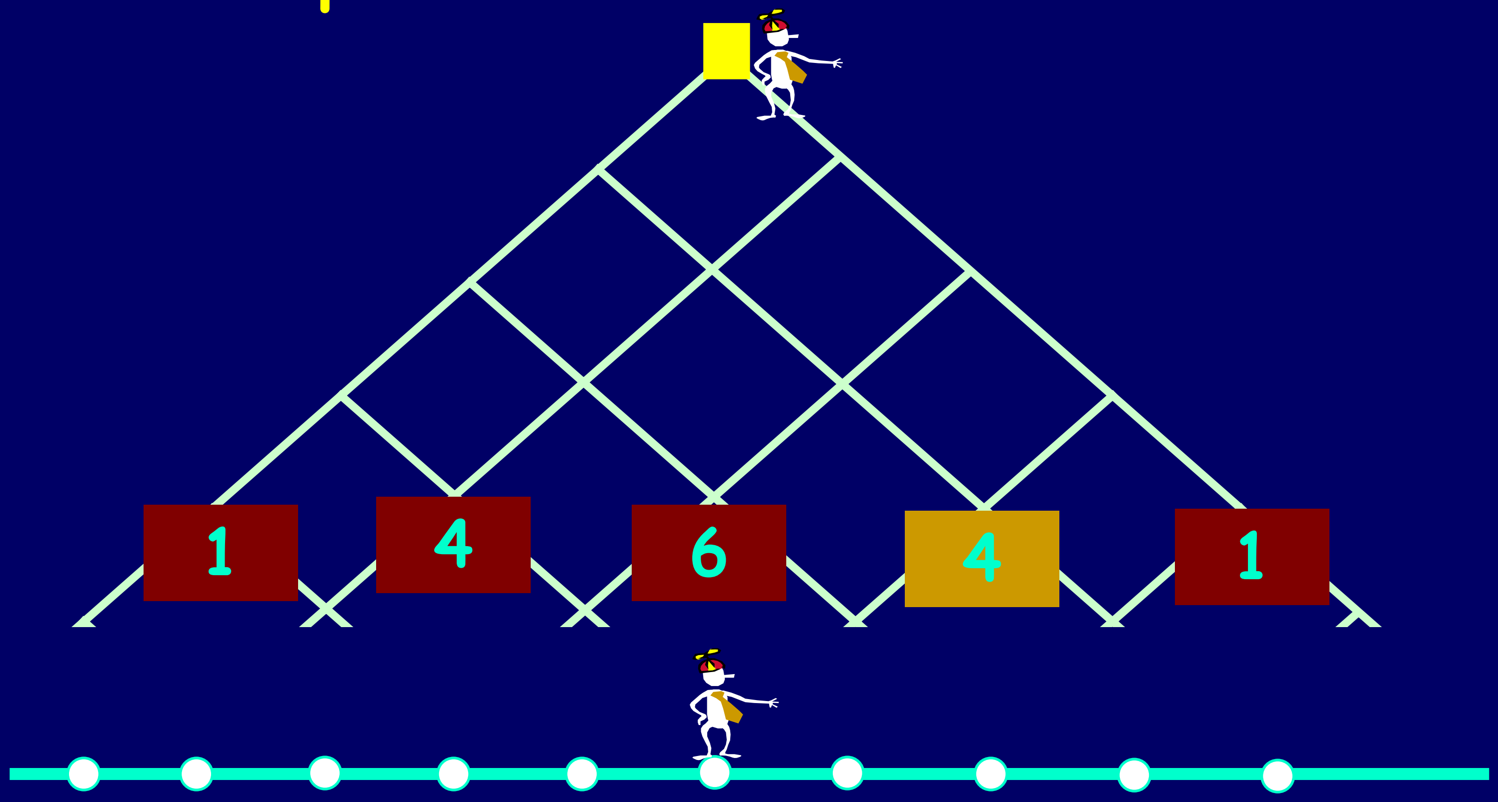


2^n different paths to level n, each equally likely.

The probability of i heads occurring on the path we generate is: $\frac{\binom{n}{i}}{2^n}$

n-step Random Walk on a line

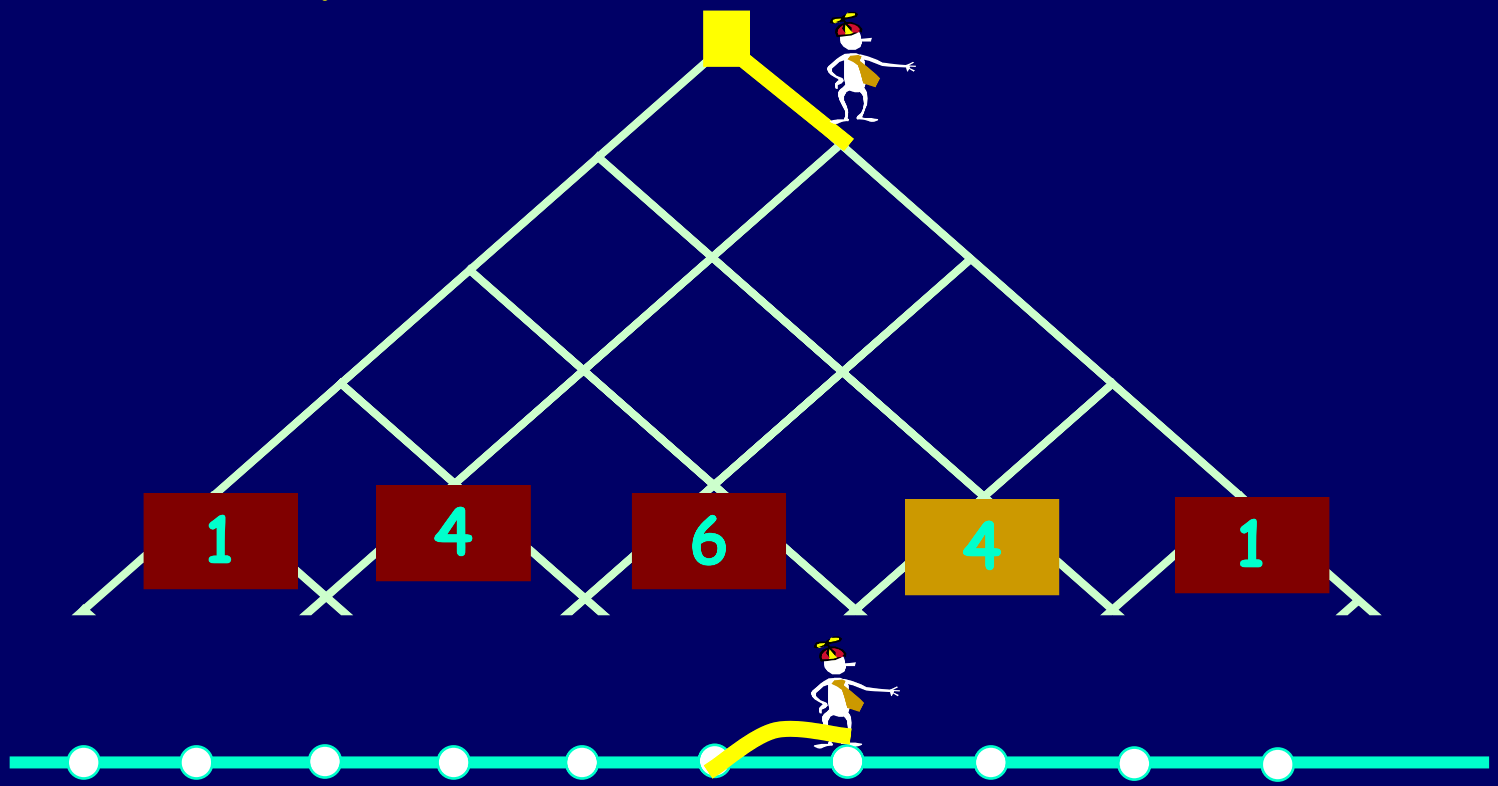


Start at the origin: at each point, flip an unbiased coin to decide whether to go right or left.

The probability that, in n steps, we take i steps to the right and n-i to the left (so we are at position 2i-n) is:

$$\frac{\binom{n}{i}}{2^n}$$

n-step Random Walk on a line



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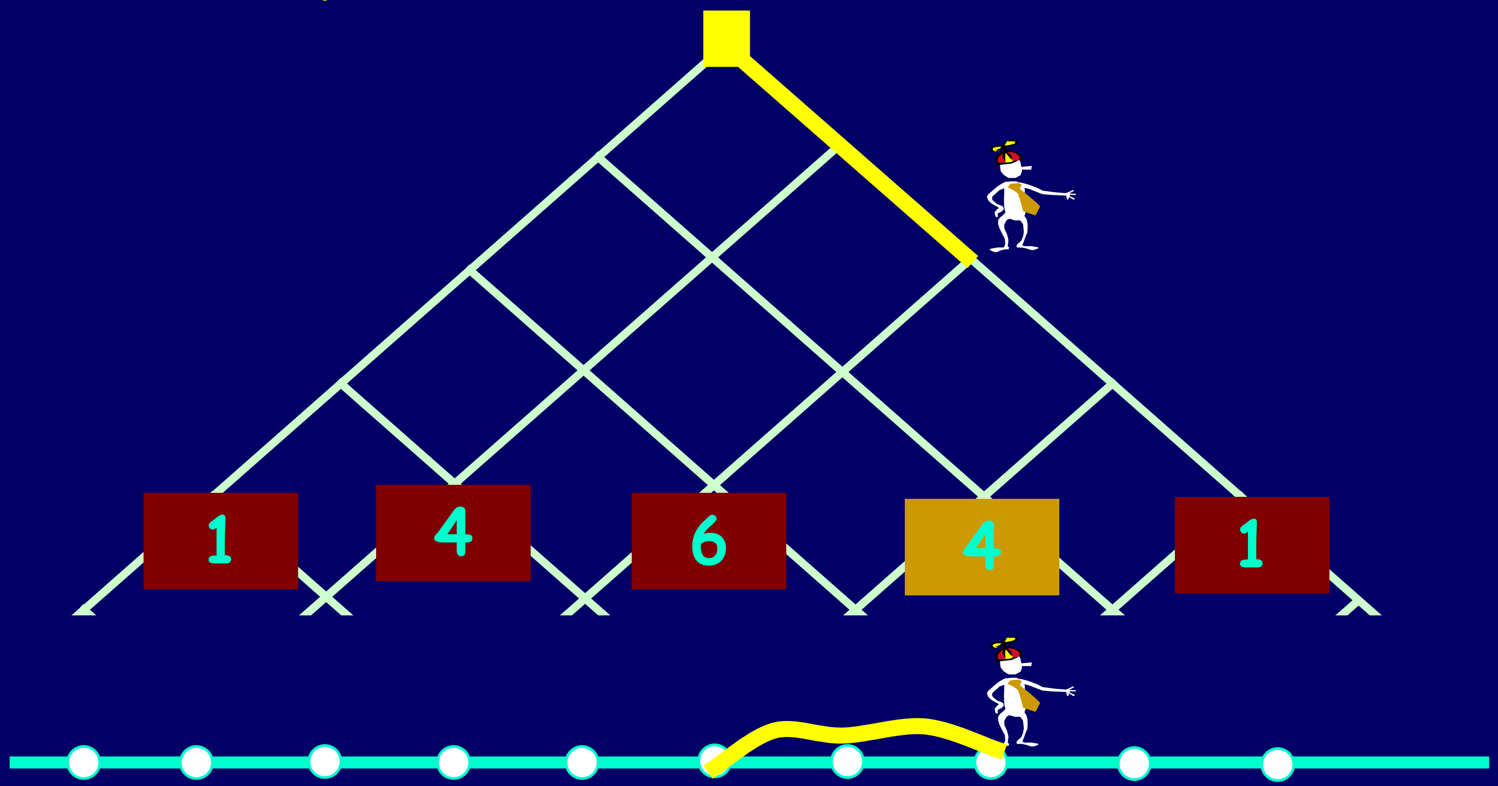

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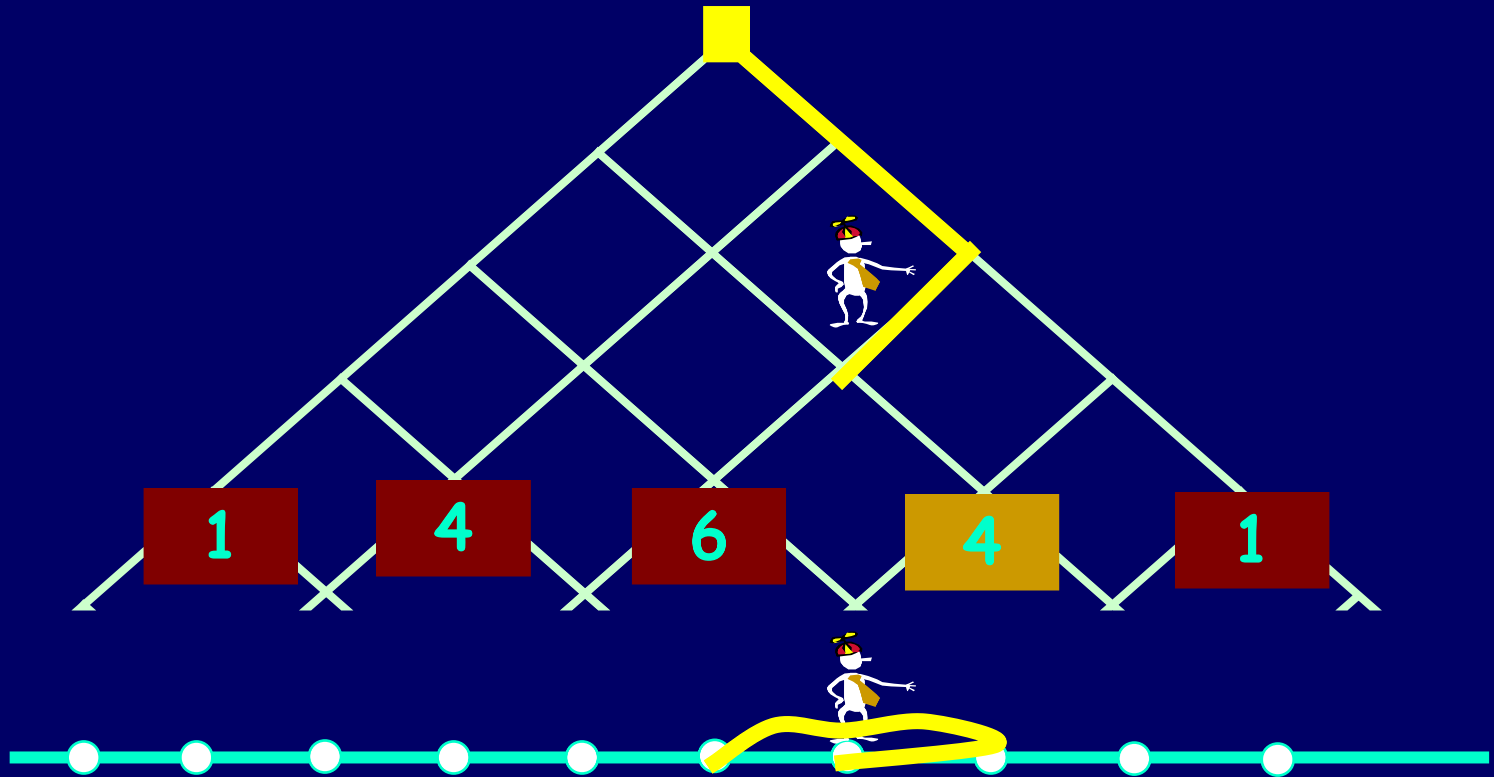

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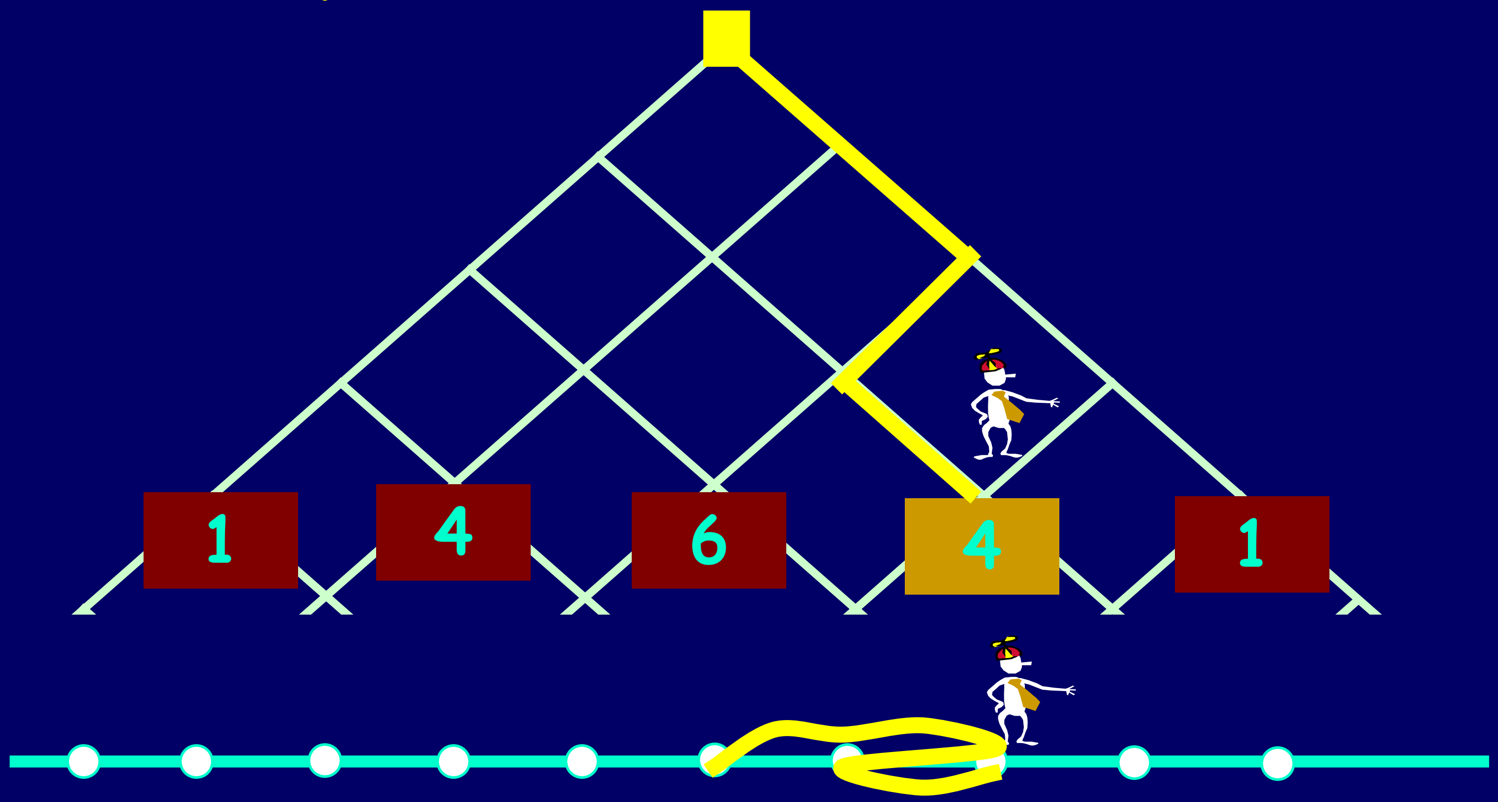

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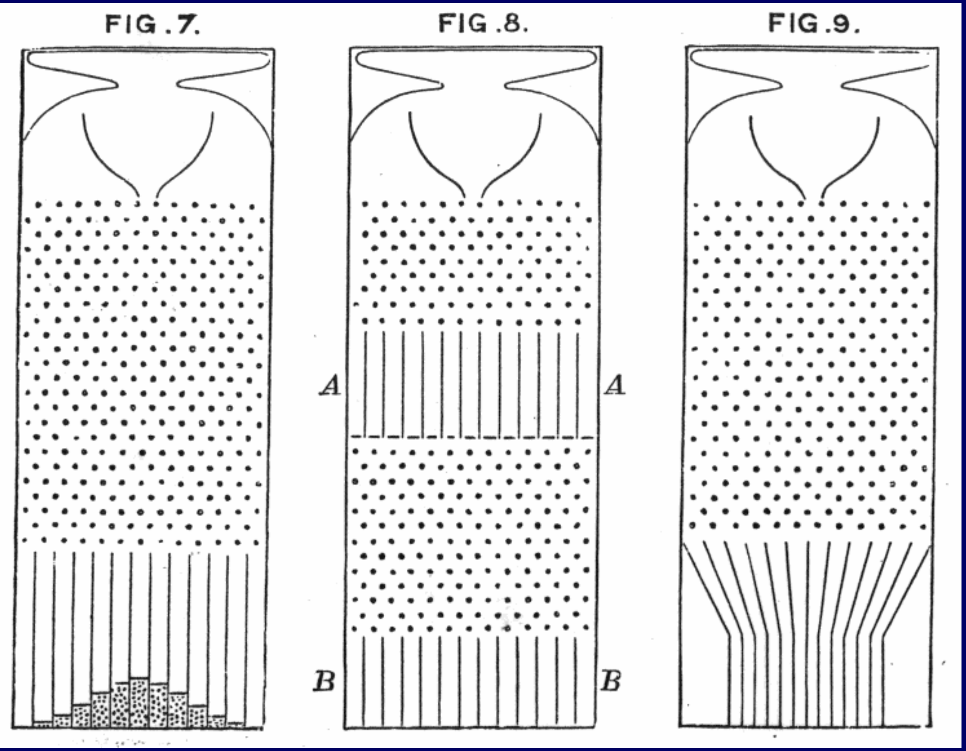
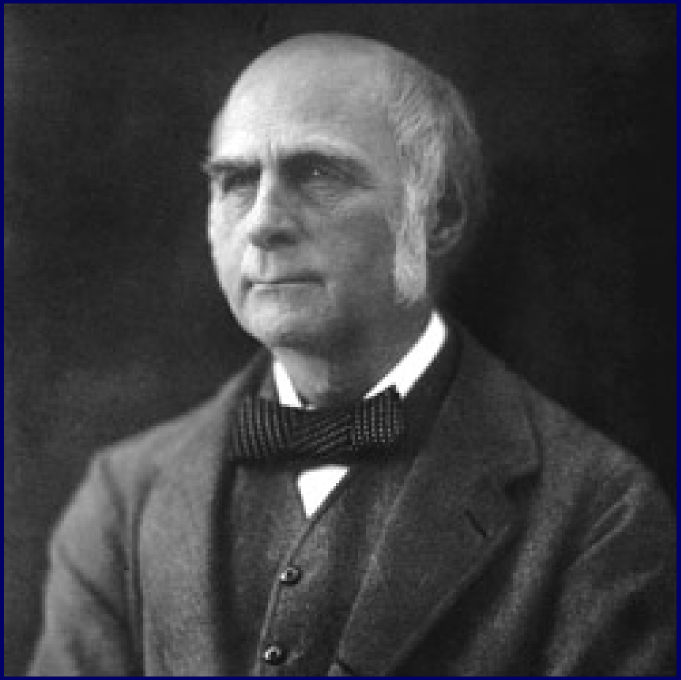
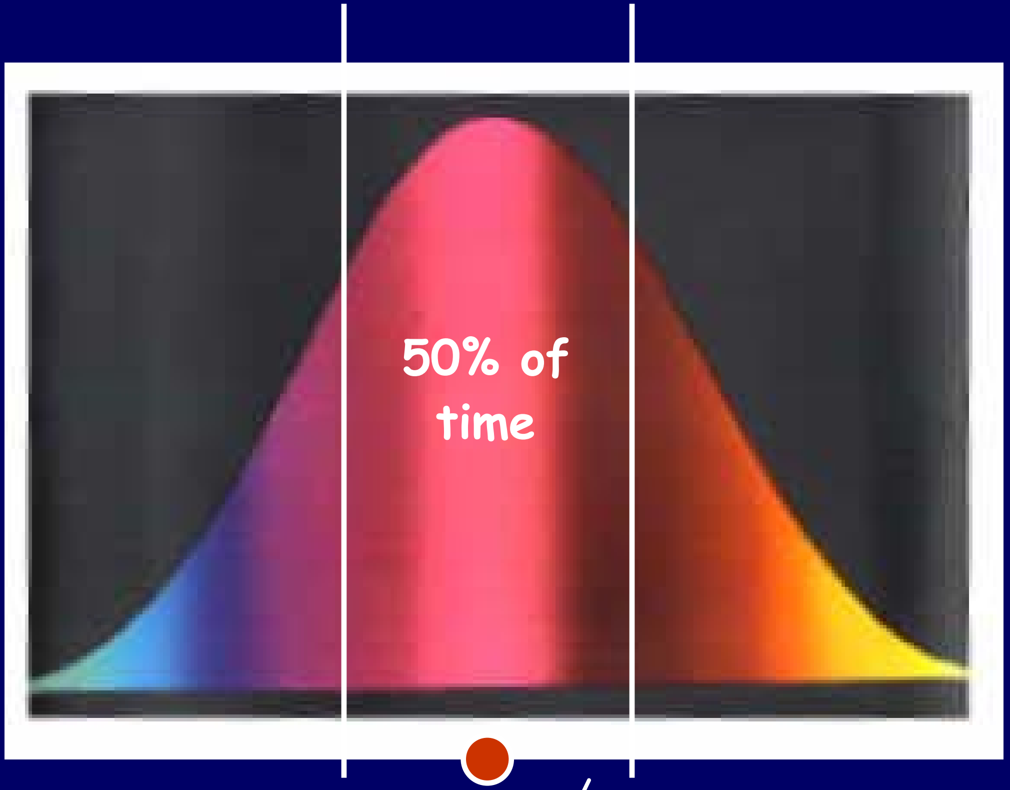
Galton's Quincunx Machine





Again, a Normal or Gaussian distribution!

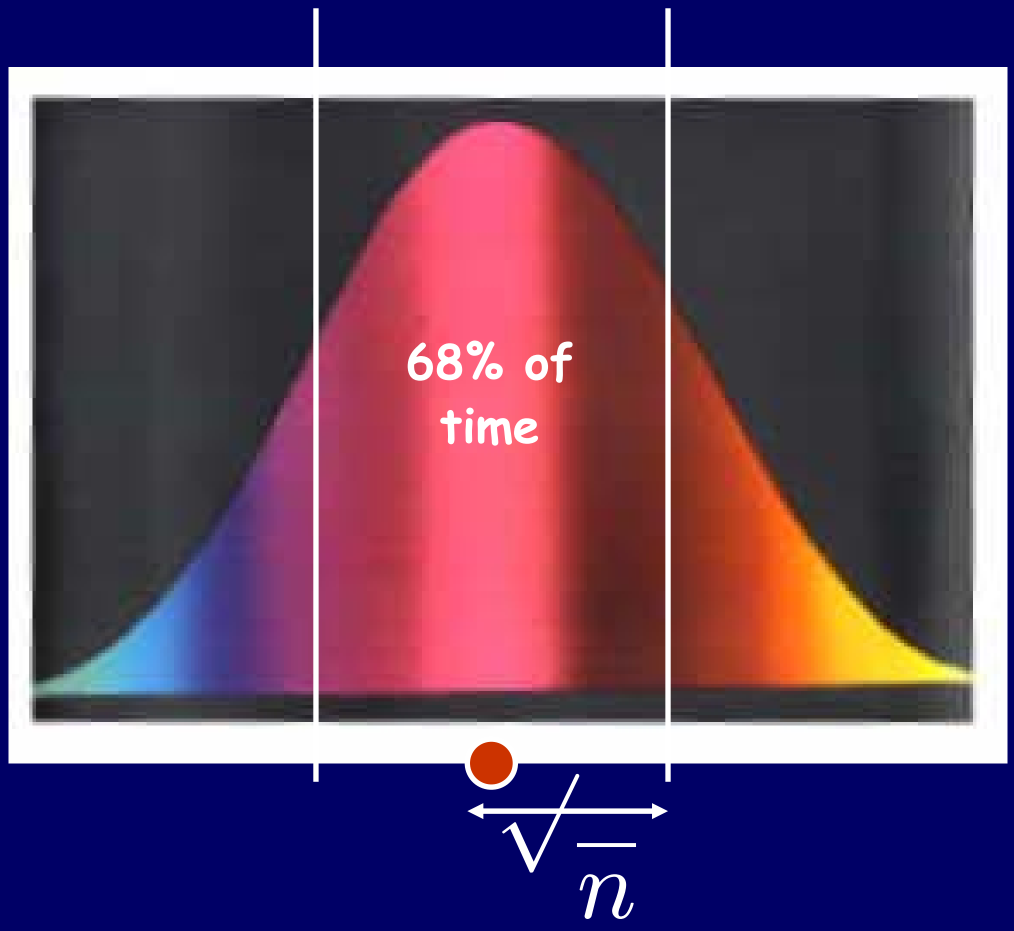
Let's look after n steps



$$\frac{2}{3}\sqrt{n}$$

Again, a Normal or Gaussian distribution!

Let's look after n steps



Again, a Normal or Gaussian distribution!

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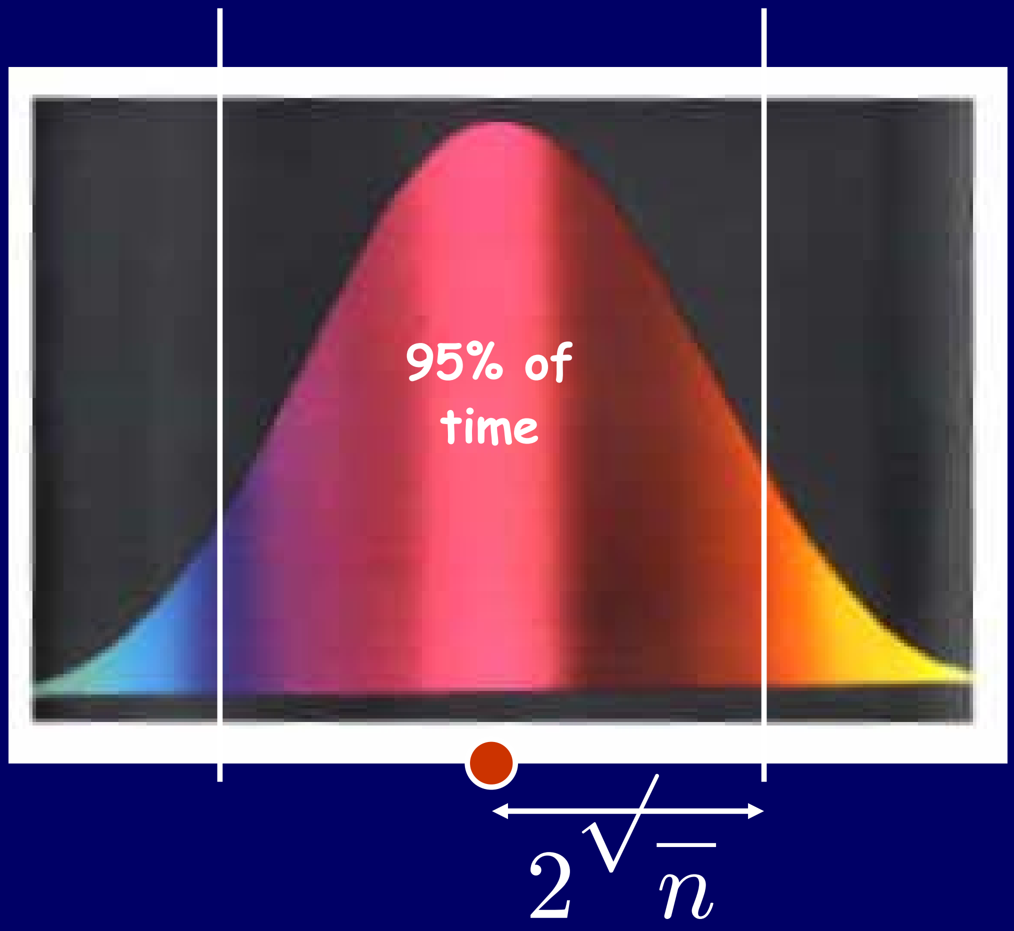

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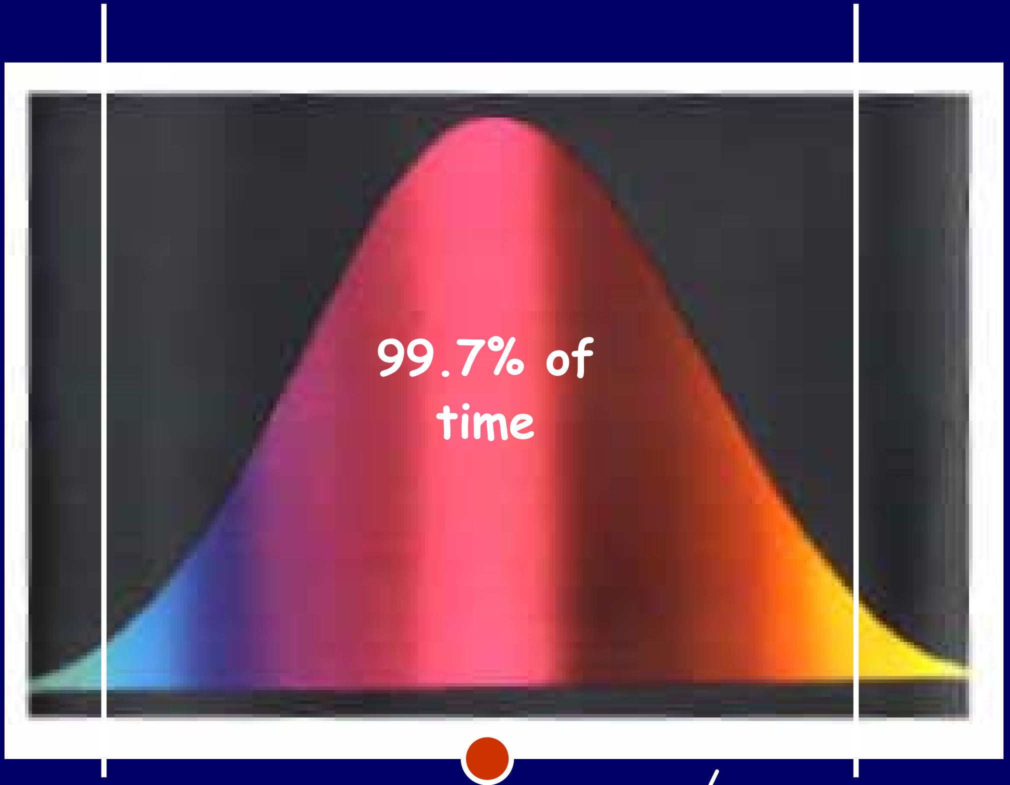

$$3\sqrt{n}$$

Probabilities and Counting
are intimately related ideas…

Probabilities and counting

Say we want to count

the number of X's with property P

One way to do it is to ask

"if we pick an X at random,
what is the probability it has property P?"

and then multiply by the number of X's.

$$\begin{bmatrix} \text{Probability of X} \\ \text{with prop. P} \end{bmatrix} = \frac{\text{(\# of X with prop. P)}}{\text{(total \# of X)}}$$

Probabilities and counting

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"if we pick an X at random,
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and then multiply by the number of X's.

$$\times \left[\begin{array}{c}\text{Probability of X}\\ \text{with prop. P}\end{array}\right] = \frac{(\text{\# of X with prop. P})}{(\text{total \# of X})}$$

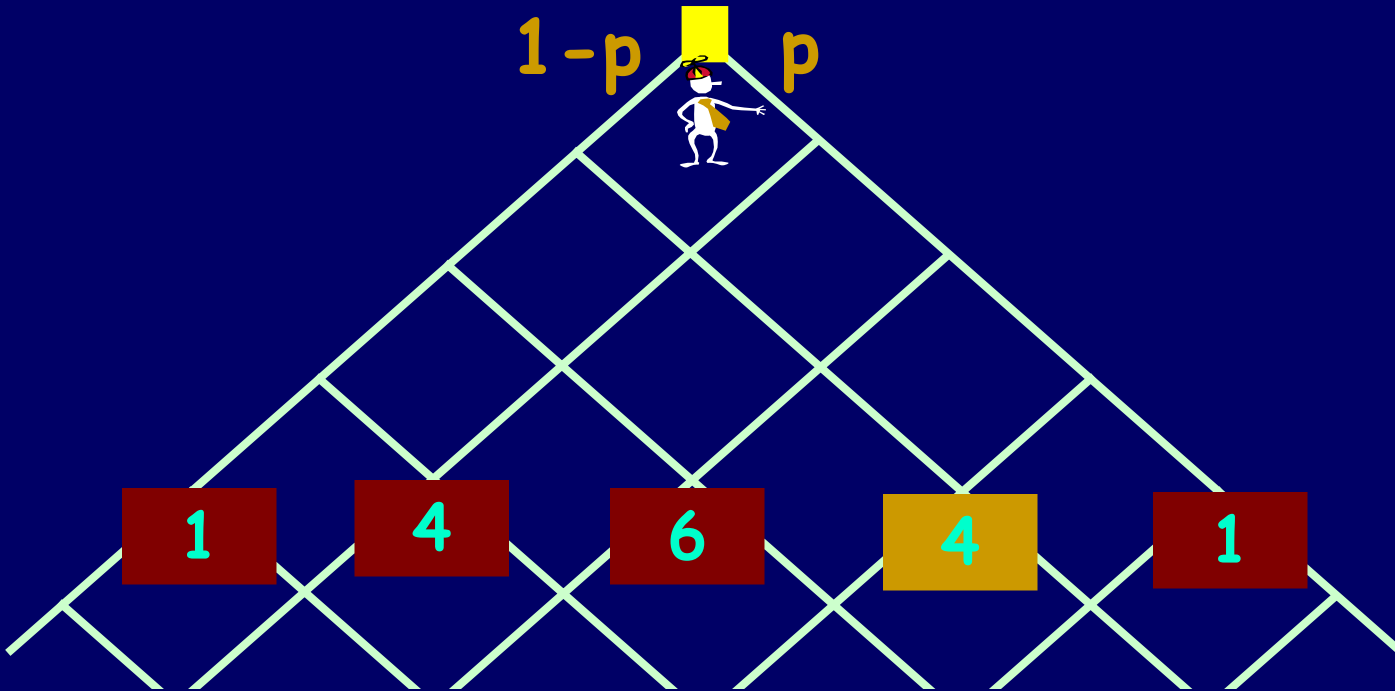
How many n-bit strings have an even number of 1's?

If you flip a coin n times, what is the probability you get an even number of heads? Then multiply by 2^n.

Say prob was q after $n-1$ flips.

$$(\text{total \# of X}) \times \left[\begin{array}{c}\text{Probability of X}\\ \text{with prop. P}\end{array}\right] = (\text{\# of X with prop. P})$$

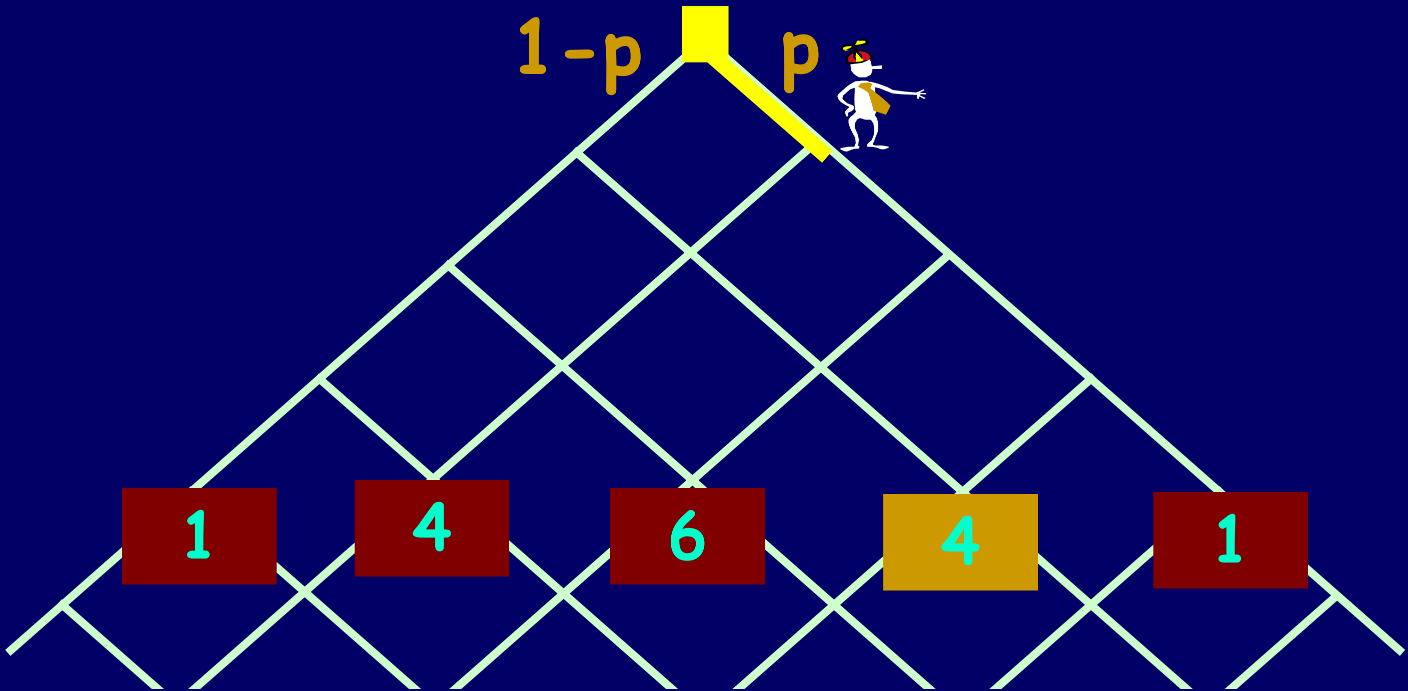
Binomial distribution with bias p



Start at the top. At each step, flip a coin with a bias p of heads to decide which way to go.

What is the probability of ending at the intersection of Avenue i and Street (n-i) after n steps?

Binomial distribution with bias p



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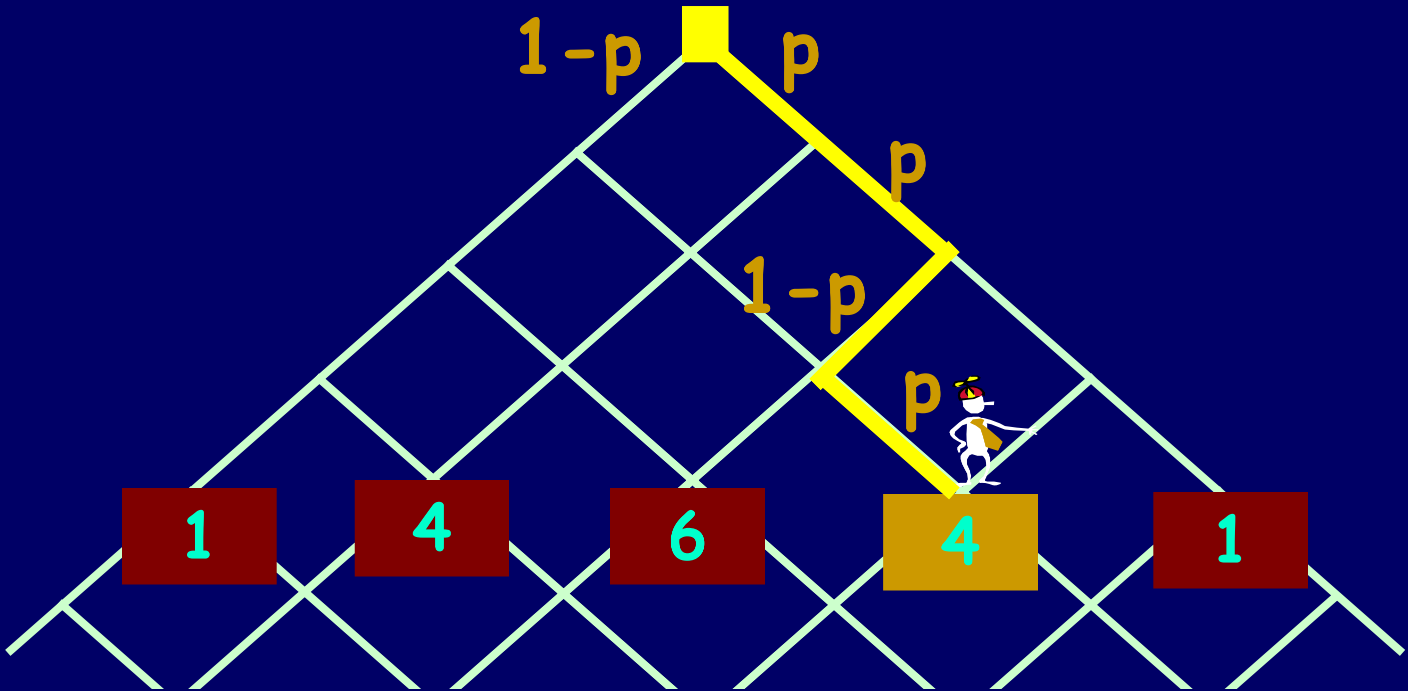

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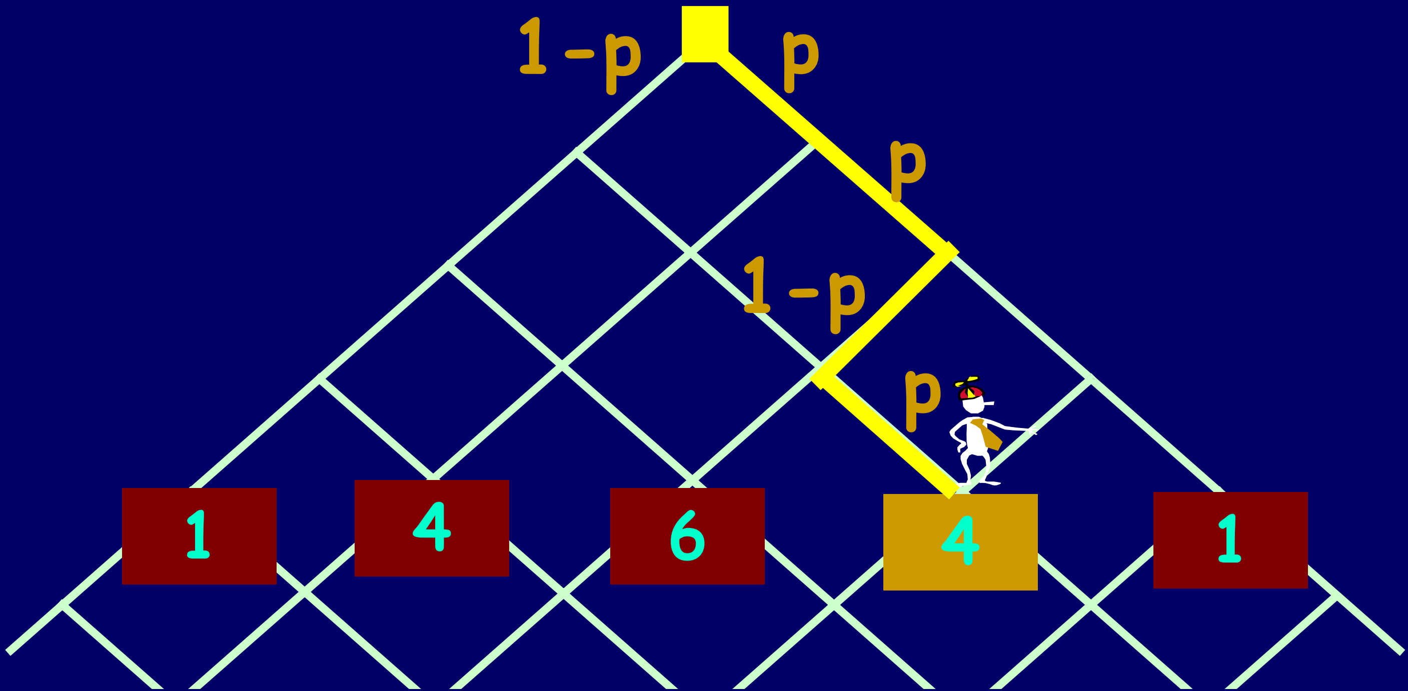
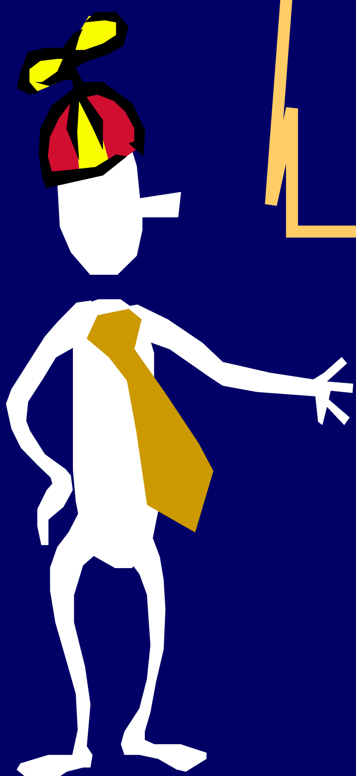


Start at the top. At each step, flip a coin with a bias p of heads to decide which way to go.

The probability of any fixed path with i heads (n-i tails) being chosen is: $p^i (1-p)^{n-i}$

Overall probability we get i heads is: $\binom{n}{i} p^i (1-p)^{n-i}$



Bias p coin flipped n times. Probability of exactly i heads is:

$$\binom{n}{i} p^i (1-p)^{n-i}$$

How many n-trit strings have even number of 0's?

If you flip a bias 1/3 coin n times, what is the probability q_n you get an even number of heads? Then multiply by 3^n. [Why is this right?]

Then $q_0=1$.

Say probability was q_{n-1} after $n-1$ flips.

Then, $q_n = (2/3)q_{n-1} + (1/3)(1-q_{n-1})$.

Rewrite as: $q_n - \frac{1}{2} = 1/3(q_{n-1} - \frac{1}{2})$

$p_n = q_n - \frac{1}{2}$

$\Rightarrow p_n = 1/3\ p_{n-1}$ and $p_0 = \frac{1}{2}$.

So, $q_n - \frac{1}{2} = (1/3)^n \frac{1}{2}$. Final count $= \frac{1}{2} + \frac{1}{2}3^n$

Some puzzles

Teams A and B are equally good.

In any one game, each is equally likely to win.

What is most likely length of a “best of 7” series?

Flip coins until either 4 heads or 4 tails. Is this more likely to take 6 or 7 flips?

Actually, 6 and 7 are equally likely

To reach either one, after 5 games, it must be 3 to 2.

½ chance it ends 4 to 2. ½ chance it doesn't.

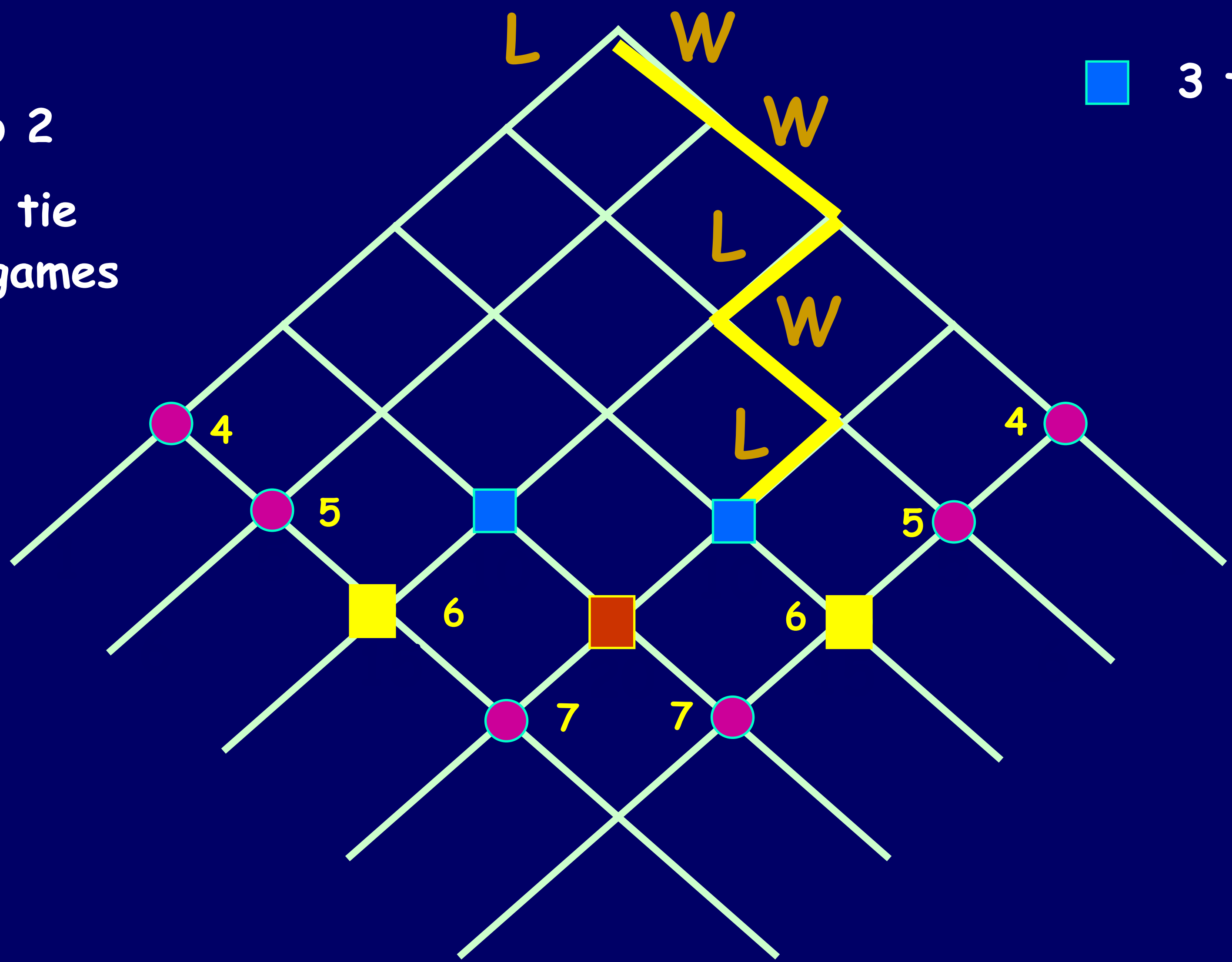
Another view

4 to 2

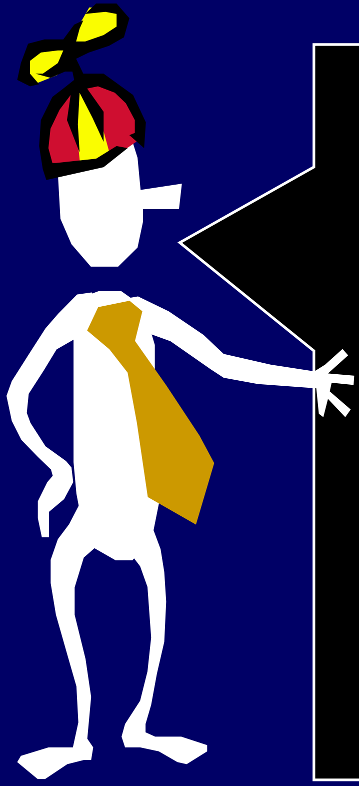
3-3 tie

⇒ 7 games

3 to 2



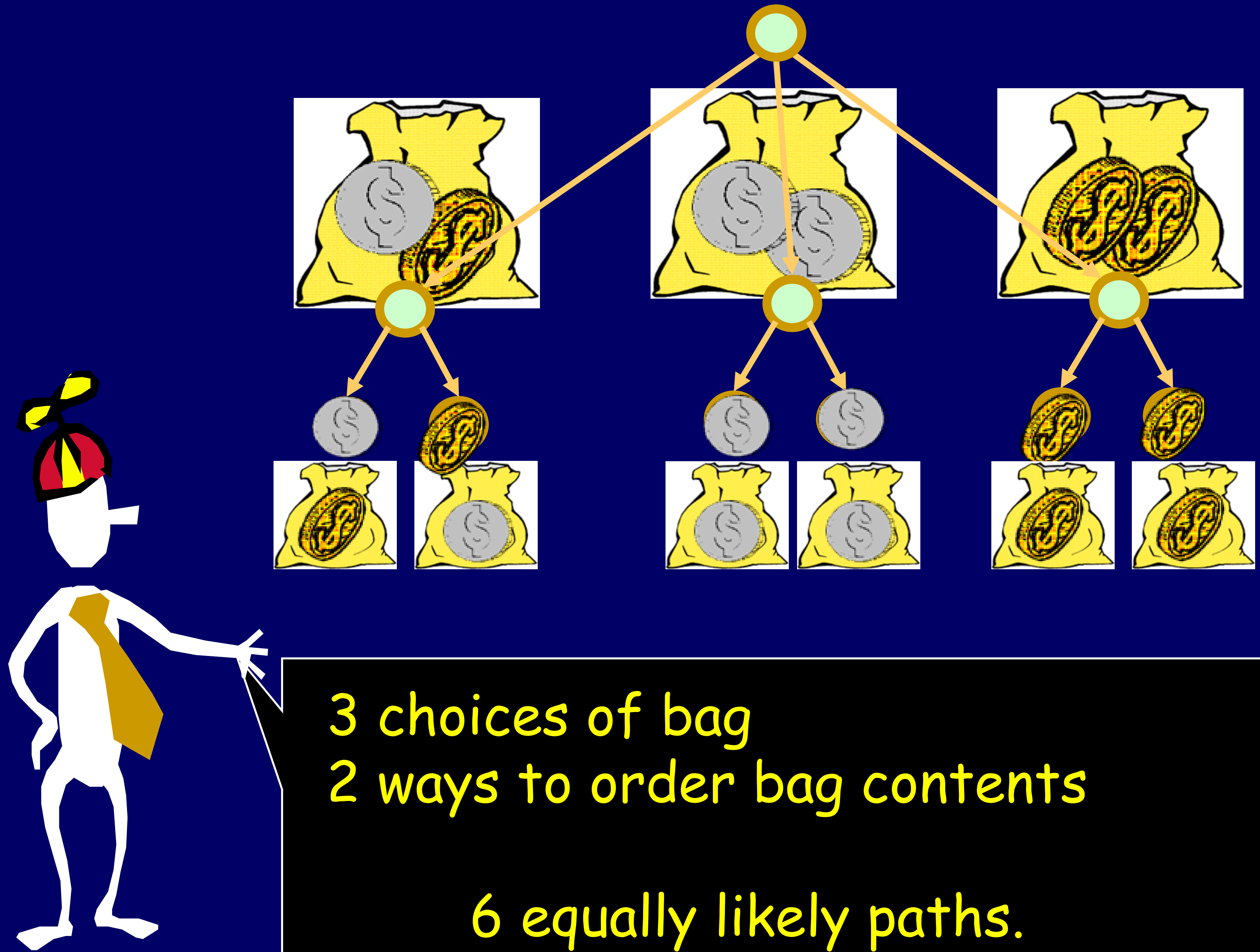
Silver and Gold



One bag has two silver coins, another has two gold coins, and the third has one of each.

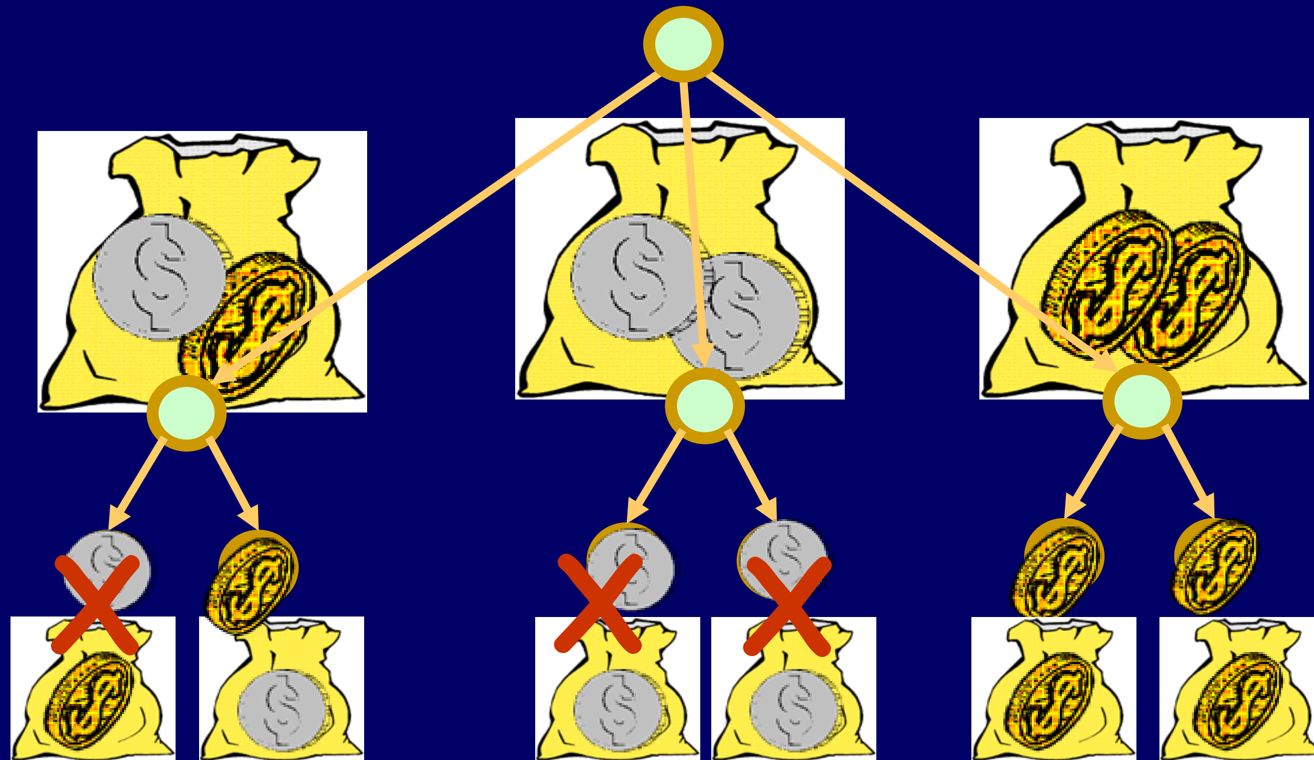
One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

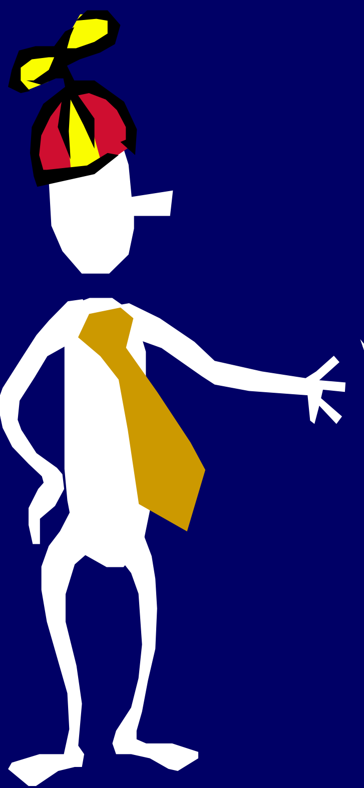
What is the probability that the other coin is gold?



3 choices of bag
2 ways to order bag contents

6 equally likely paths.

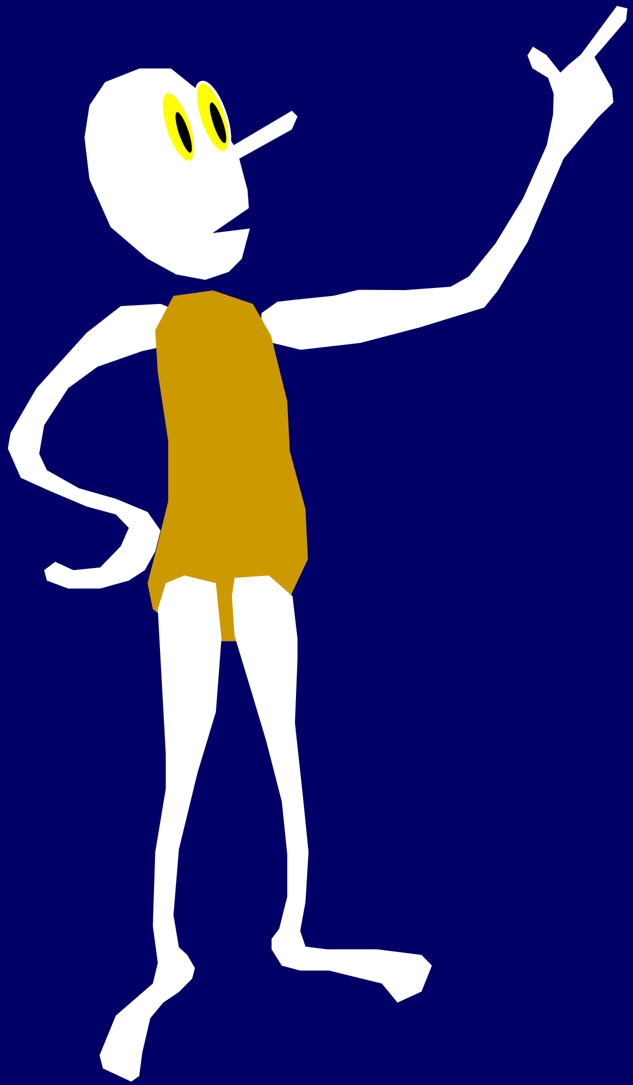




Given you see a [gold coin], 2/3 of remaining paths have a second gold.

So, sometimes, probabilities can be counter-intuitive

Language Of Probability



The formal language of probability is a very important tool in describing and analyzing probability distributions.

Finite Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

The weights must satisfy:

$$\sum_{x \in S} p(x) = 1$$

Finite Probability Distribution

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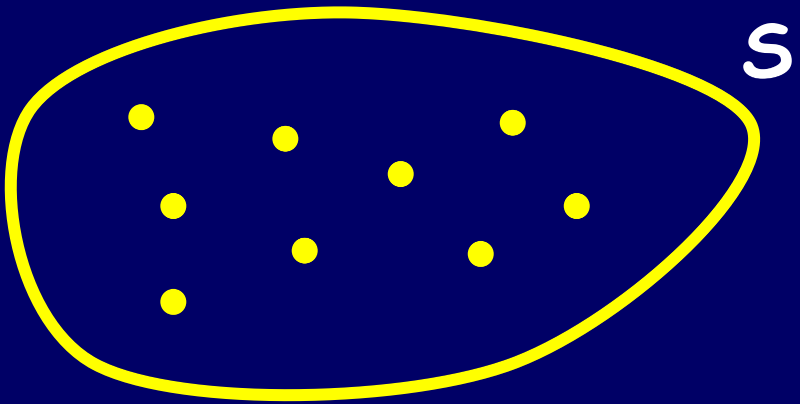
For notational convenience we will define D(x) = p(x).

S is often called the sample space and elements x in S are called samples.

Sample space

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

Sample space



Probability

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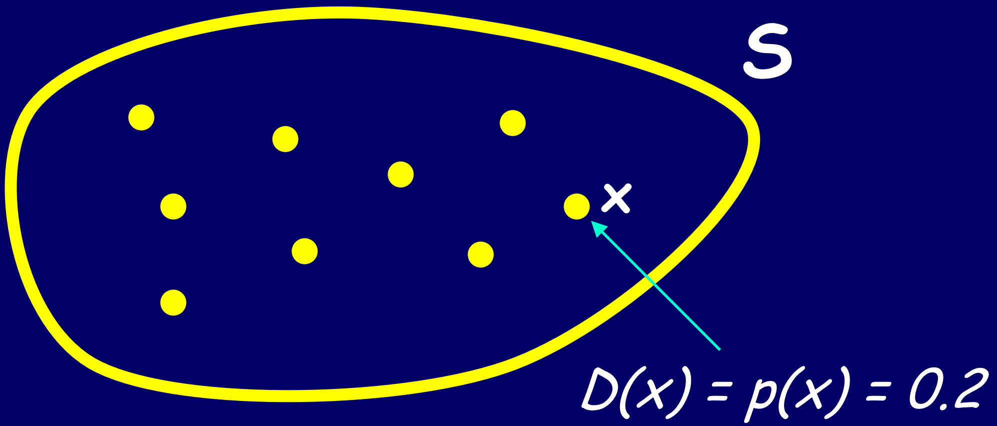
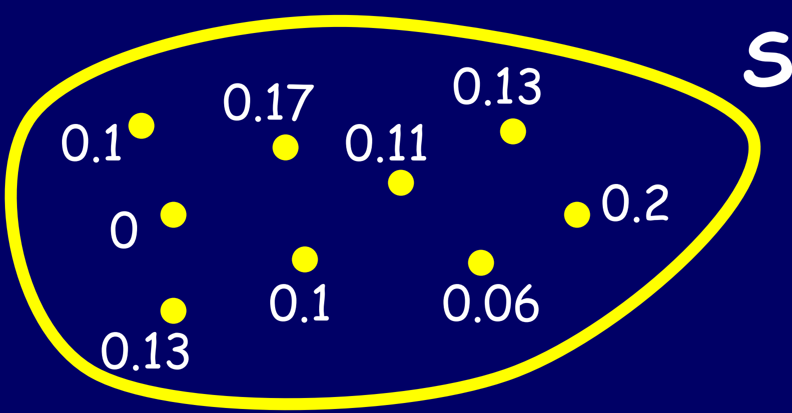


weight or probability of x

Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

weights must sum to 1



Events

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

Any subset E ≀{ S is called an event.
The probability of event E is

$$Pr_D[E] = \sum_{x \in E} p(x)$$

Events

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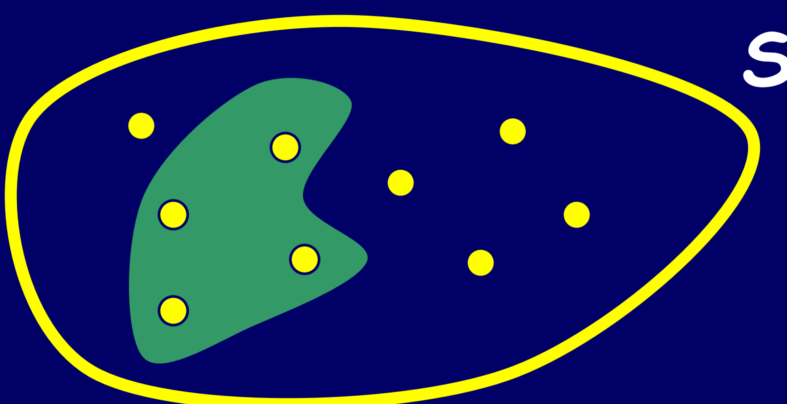
Event E



Events

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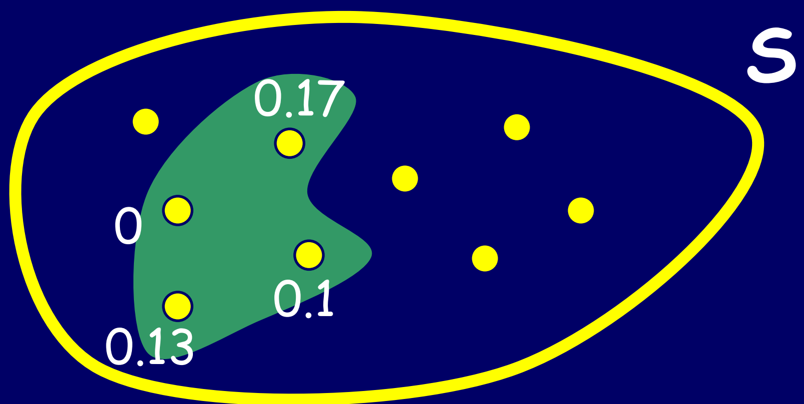
$Pr_D[E] = 0.4$



Uniform Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

If each element has equal probability, the distribution is said to be uniform.

$$Pr_D[E] = \sum_{x \in E} p(x) = \frac{|E|}{|S|}$$

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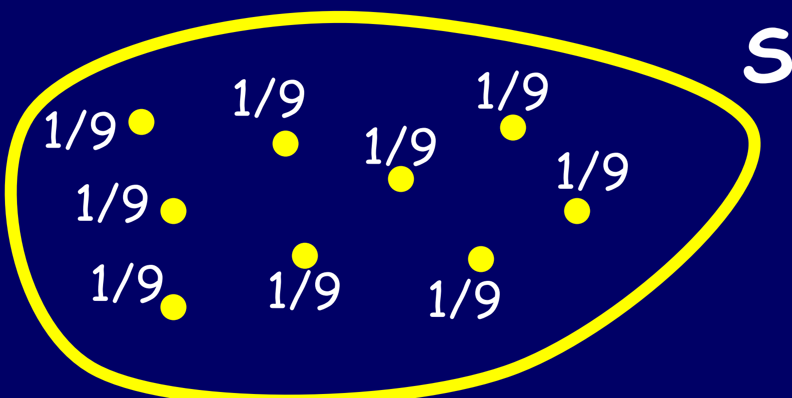

Each p(x) = 1/9.

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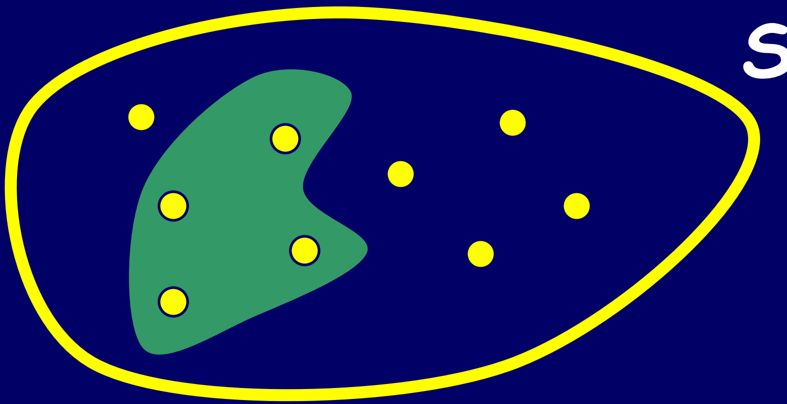
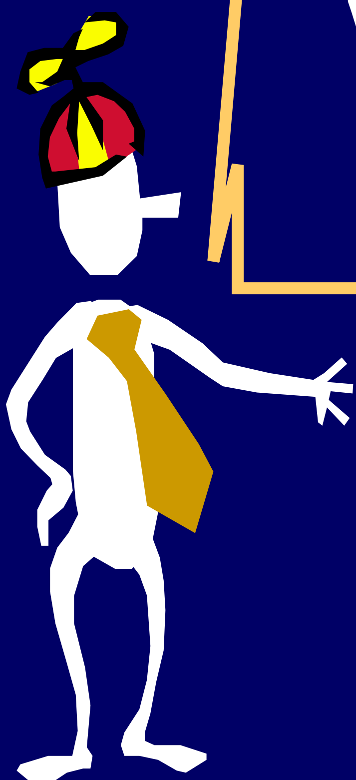


$Pr_D[E] = |E|/|S|$
$= 4/9$



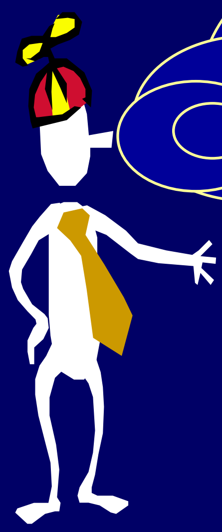
A fair coin is tossed 100 times in a row.

What is the probability that we get exactly half heads?

Using the Language

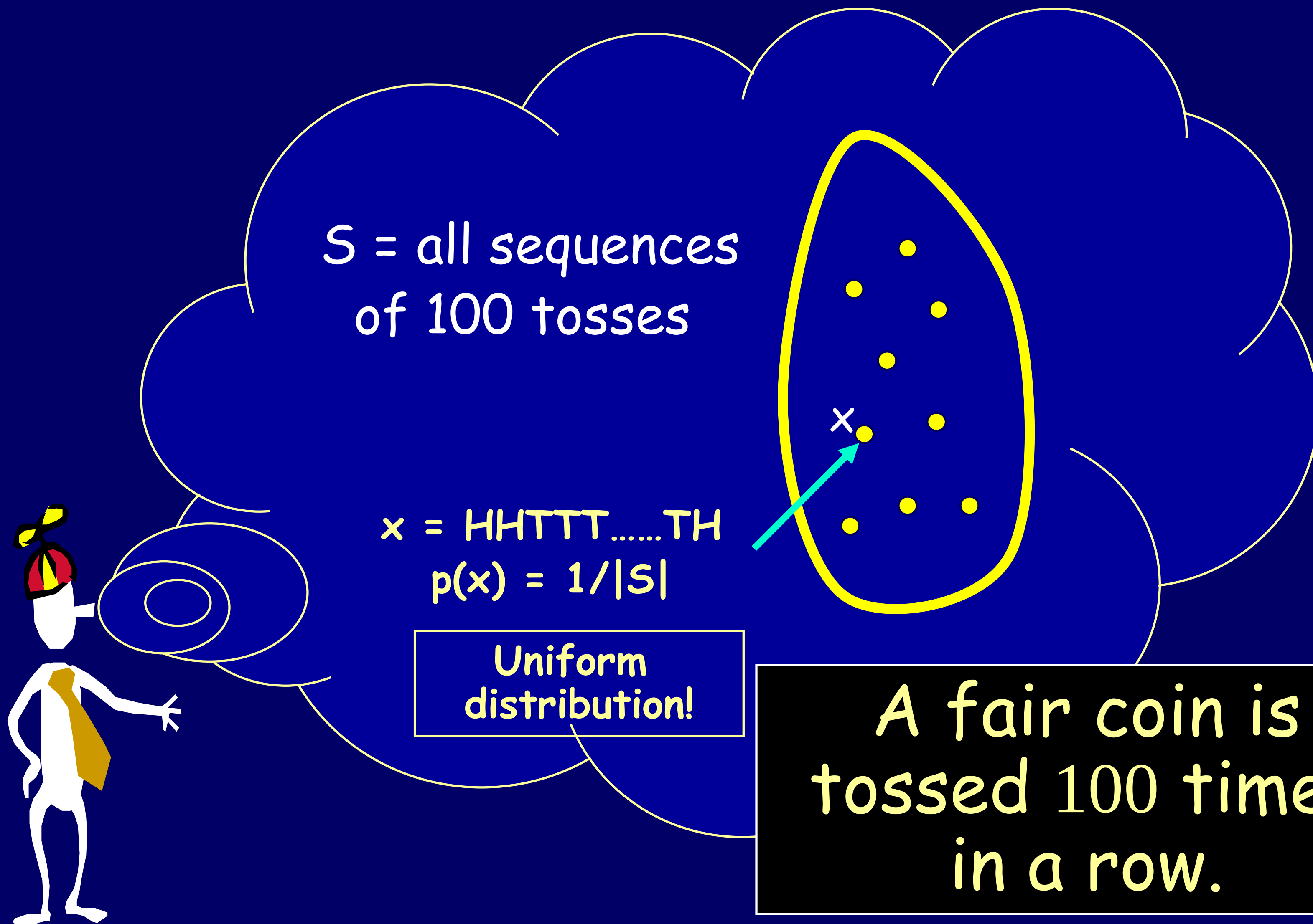
The sample space S is the set of all outcomes $\{H,T\}^{100}$.

Each sequence in S is equally likely, and hence has probability $1/|S|=1/2^{100}$.



A fair coin is tossed 100 times in a row.

Using the Language: visually

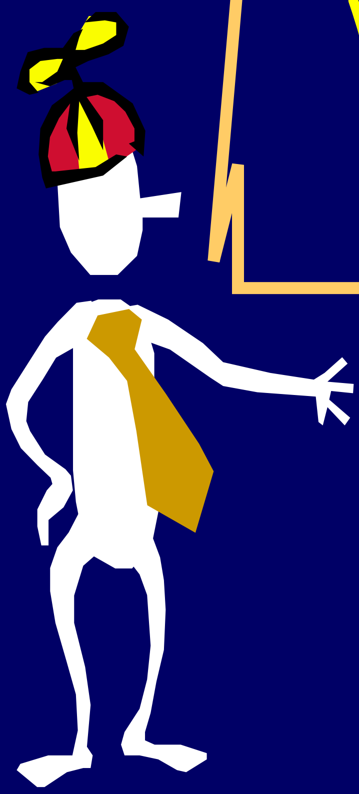


A fair coin is tossed 100 times in a row.

A fair coin is tossed 100 times in a row.

OK

What is the probability that we get exactly half heads?

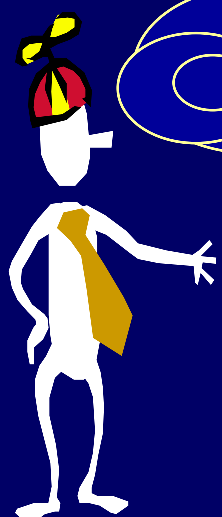


Using the Language

The event that we see half heads is

$E = \{x \in S \mid x \text{ has 50 heads}\}$

And $|E| = \binom{100}{50}$

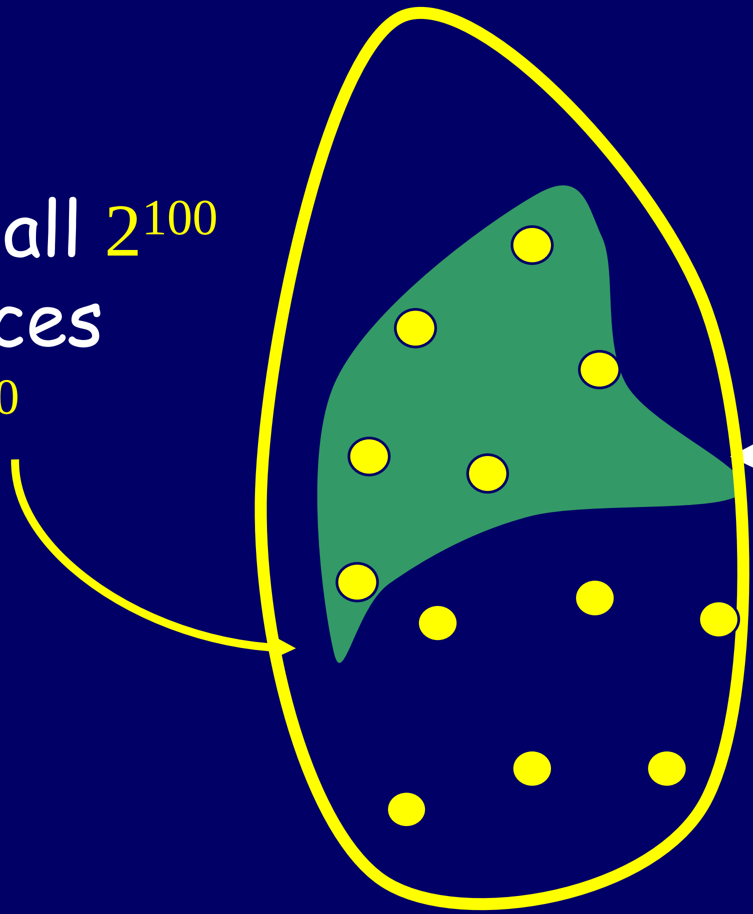


Probability of exactly half tails?

Picture

Event E = Set of sequences with 50 H's and 50 T's

Set of all 2^{100} sequences $\{H,T\}^{100}$



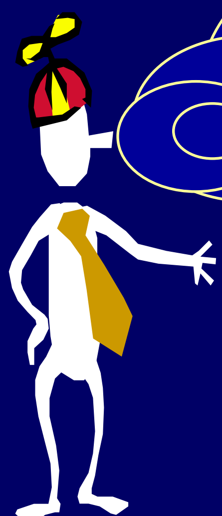
Probability of event E = proportion of E in S $= \dfrac{|E|}{|S|} = \dfrac{\binom{100}{50}}{2^{100}}$

Using the Language

Answer:

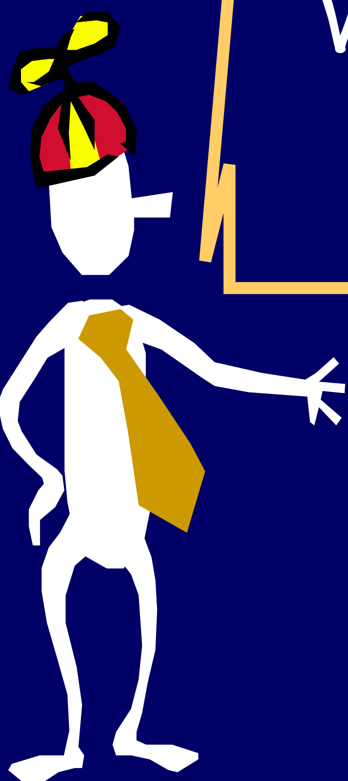
$Pr[E] = |E|/|S| = |E|/2^{100}$

$$\frac{|E|}{|S|} = \frac{\binom{100}{50}}{2^{100}} \approx 0.0795$$



Suppose we roll a ***white*** die and a ***black*** die.

What is the probability that sum is 7 or 11?



Same methodology!

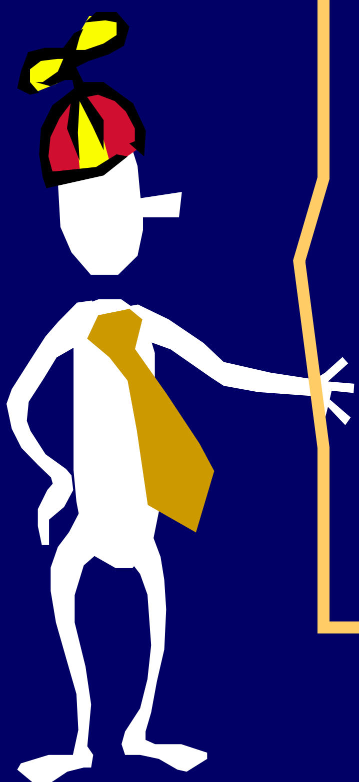
Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
(3,1),	(3,2),	(3,3),	(3,4),	(3,5),	(3,6),
(4,1),	(4,2),	(4,3),	(4,4),	(4,5),	(4,6),
(5,1),	(5,2),	(5,3),	(5,4),	(5,5),	(5,6),
(6,1),	(6,2),	(6,3),	(6,4),	(6,5),	(6,6) }

Pr(x) = 1/36
$\forall$ x $\in$ S

Event E = all (x,y) pairs with x+y = 7 or 11

Pr[E] = |E|/|S| = proportion of E in S = 8/36



23 people are in a room.

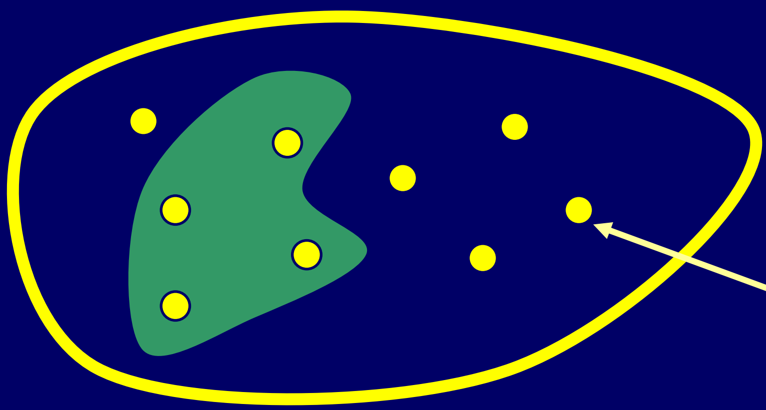
Suppose that all possible assignments of birthdays to the 23 people are equally likely.

What is the probability that two people will have the same birthday?

And the same methods again!

Sample space $\Omega = \{ 1, 2, 3, \ldots, 366\}^{23}$

Pretend it's always a leap year



$x = (17, 42, 363, 1, \ldots, 224, 177)$

23 numbers

Event $E = \{x \in \Omega \mid \text{two numbers in } x \text{ are same}\}$

What is $|E|$?

count $|\overline{E}|$ instead!

$\overline{E}$ = all sequences in Ω that have <u>no</u> repeated numbers

$$|\bar{E}| = 366 \cdot 365 \cdots 344$$

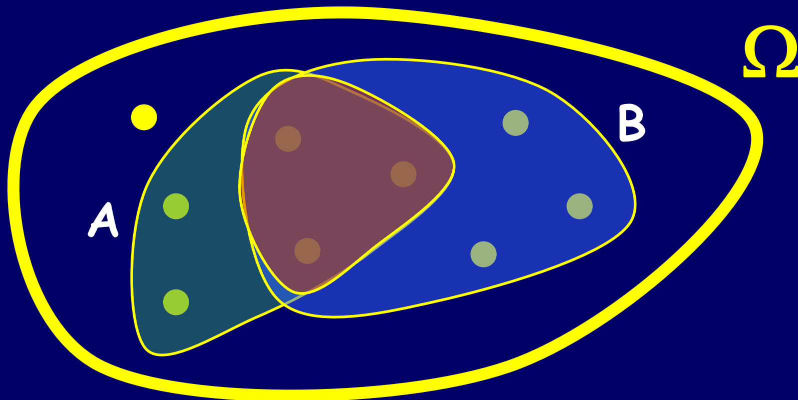
$$\frac{|\bar{E}|}{|\Omega|} = \frac{366 \cdots 344}{366^{23}} \approx .494$$

$$\frac{|E|}{|\Omega|} \approx .51$$

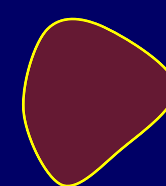
More Language Of Probability

The probability of event A <u>given event B</u> is written Pr[A | B]

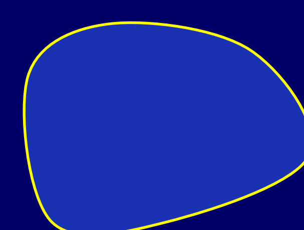
and is defined to be = $\frac{\Pr[A \cap B]}{\Pr[B]}$

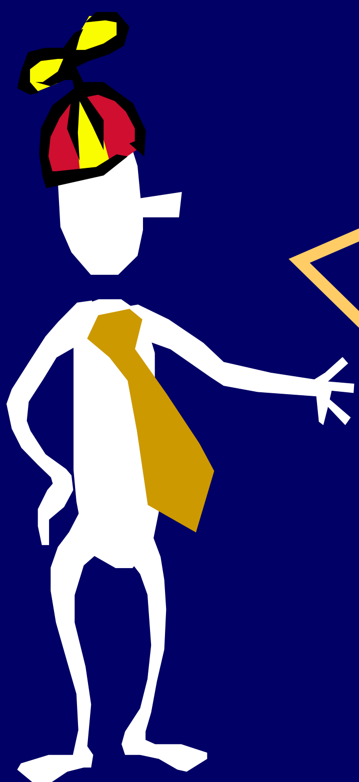


proportion of $A \cap B$



to B





Suppose we roll a ***white*** die and ***black*** die.

What is the probability that the white is 1 given that the total is 7?

event A = {white die = 1}

event B = {total = 7}

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

event A = {white die = 1}

event B = {total = 7}

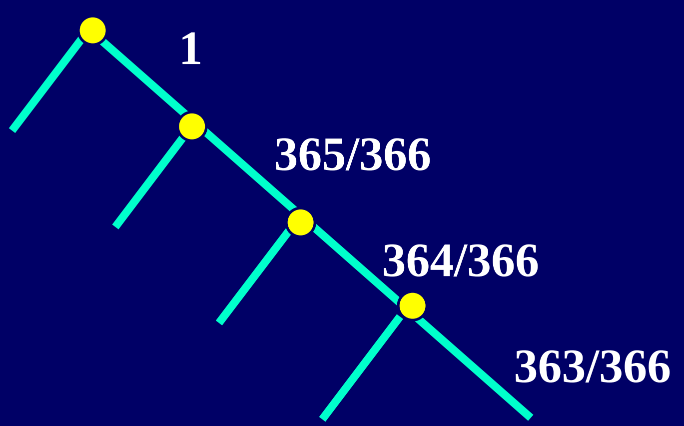
$$\frac{|A \cap B|}{|B|} = \Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{1/36}{1/6}$$

Can do this because Ω is uniformly distributed.

This way does not care about the distribution.

Another way to calculate Birthday probability Pr(no collision)

Pr(1st person doesn't collide) = 1.
Pr(2nd doesn't | no collisions yet) = 365/366.
Pr(3rd doesn't | no collisions yet) = 364/366.
Pr(4th doesn't | no collisions yet) = 363/366.
…
Pr(23rd doesn't| no collisions yet) = 344/366.



Independence!

A and B are <u>independent events</u> if

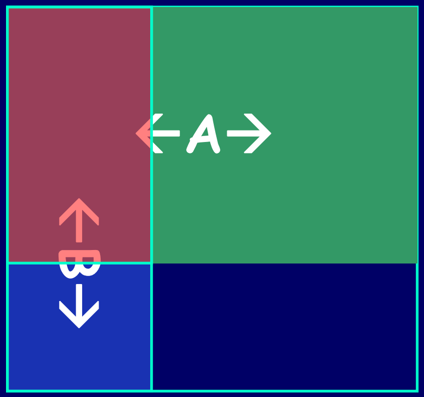
$$\Pr[\ A \mid B\ ] = \Pr[\ A\ ]$$

$$\Leftrightarrow$$

$$\Pr[\ A \cap B\ ] = \Pr[\ A\ ]\Pr[\ B\ ]$$

$$\Leftrightarrow$$

$$\Pr[\ B \mid A\ ] = \Pr[\ B\ ]$$



What about Pr[A | not(B)] ?

Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

E.g., {A_1, A_2, A_3} are independent events if:

$$Pr[A_1 \mid A_2 \cap A_3] = Pr[A_1]$$
$$Pr[A_2 \mid A_1 \cap A_3] = Pr[A_2]$$
$$Pr[A_3 \mid A_1 \cap A_2] = Pr[A_3]$$

$$Pr[A_1 \mid A_2] = Pr[A_1]$$
$$Pr[A_2 \mid A_1] = Pr[A_2]$$
$$Pr[A_3 \mid A_1] = Pr[A_3]$$

$$Pr[A_1 \mid A_3] = Pr[A_1]$$
$$Pr[A_2 \mid A_3] = Pr[A_2]$$
$$Pr[A_3 \mid A_2] = Pr[A_3]$$

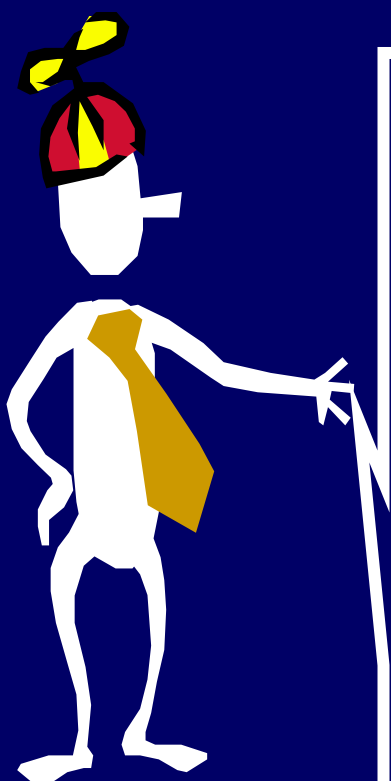
Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

$$Pr[A|X] = Pr[A]$$

{≀∇ ⊣⇕⇕ X a conjunction of any of the others (e.g., A_2 and A_6 and A_7)

Silver and Gold



One bag has two silver coins, another has two gold coins, and the third has one of each.

One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

What is the probability that the other coin is gold?

Let G_1 be the event that the first coin is gold.

$Pr[G_1] = 1/2$

Let G_2 be the event that the second coin is gold.

$Pr[G_2 \mid G_1] = Pr[G_1 \text{ and } G_2] / Pr[G_1]$

$= (1/3) / (1/2)$

$= 2/3$

Note: G_1 and G_2 are not independent.

Monty Hall problem

- Announcer hides prize behind
  one of 3 doors at random.
- You select some door.
- Announcer opens one of others with no prize.
- You can decide to keep or switch.

What to do?

Monty Hall problem

- Sample space Ω =

  { prize behind door 1,
  prize behind door 2,
  prize behind door 3 }.

Each has probability 1/3.

Staying

we win if we choose the correct door

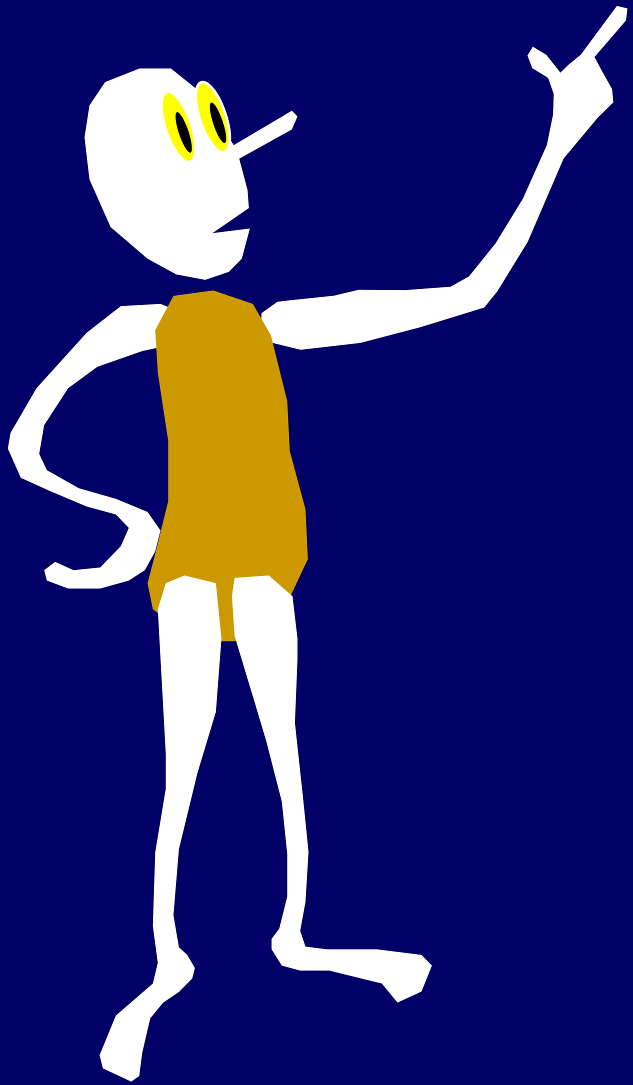
Pr[choosing correct door] = 1/3.

Switching

we win if we choose the <u>incorrect</u> door

Pr[choosing incorrect door] = 2/3.

why was this tricky?



We are inclined to think:

"After one door is opened, others are equally likely…"

But his action is not independent of yours!