Great Theoretical Ideas In Computer Science

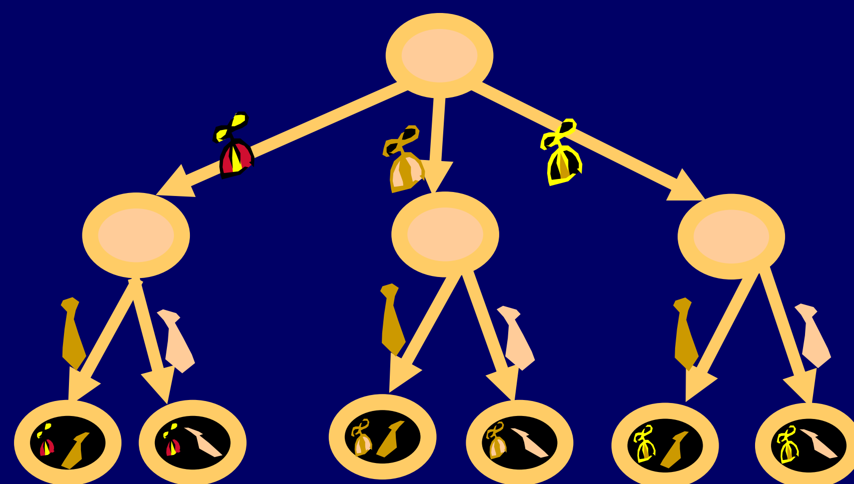
Anupam Gupta

CS 15-251 Fall 2006

Lecture 6 Sept 14, 2006

Carnegie Mellon University

Counting I: One-To-One Correspondence and Choice Trees



How many seats in this auditorium?

Hint:
Count without counting!

If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

Addition Rule

Let A and B be two disjoint finite sets.

The size of $A \cup B$ is the sum of
the size of A and the size of B.

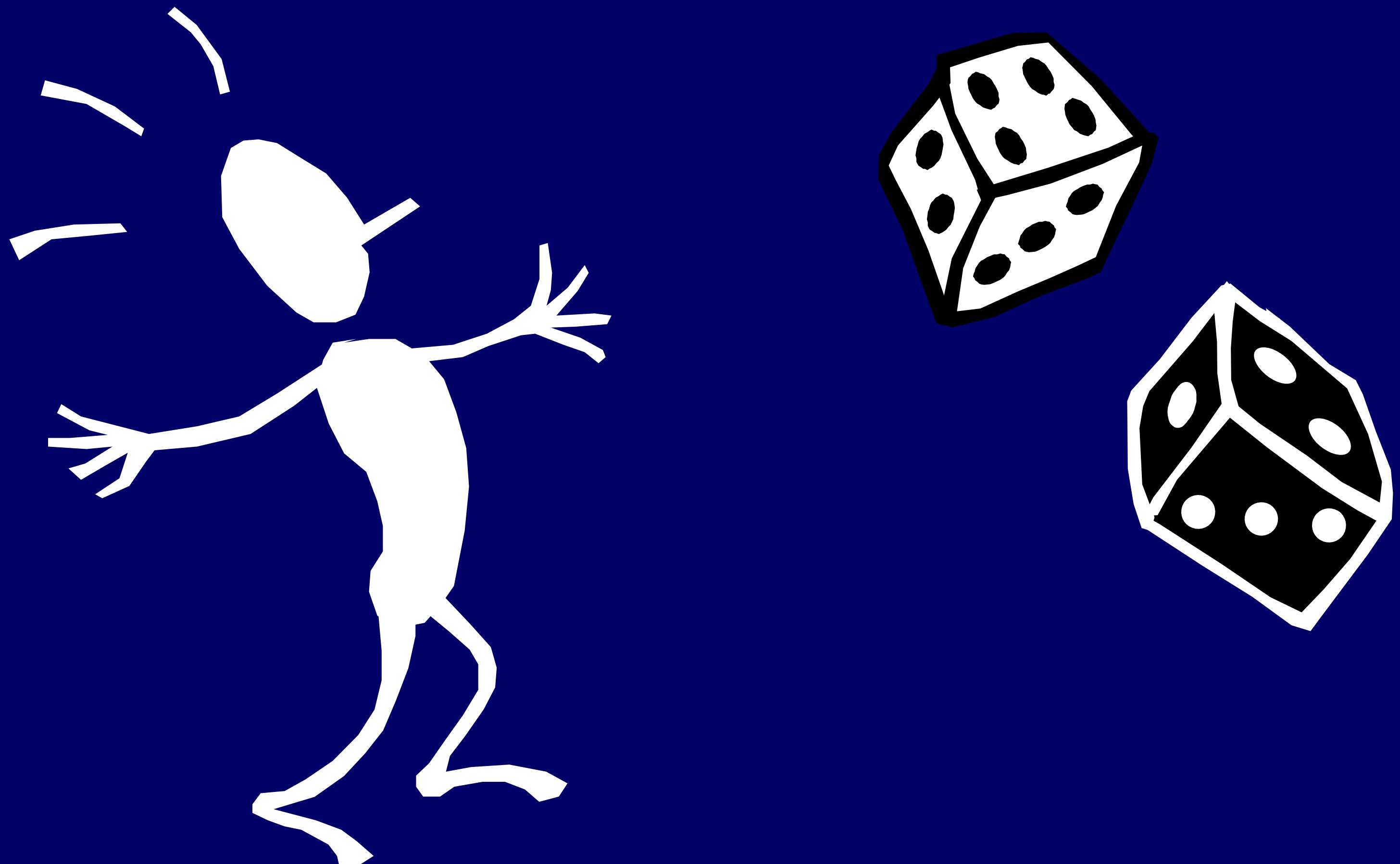
$$|A \cup B| = |A| + |B|$$

Corollary (by induction)

Let A_1, A_2, A_3, …, A_n be disjoint, finite sets.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Suppose I roll a white die and a black die.



$S \equiv$ Set of all outcomes where the dice show different values.

$|S| = ?$

$S \equiv$ Set of all outcomes where the dice show different values.

$|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something else.

$$|S| = \left|\bigcup_{i=1}^{6} A_i\right| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

$S \equiv$ Set of all outcomes where the dice show different values.

$|S| = ?$

$T \equiv$ set of outcomes where dice agree.

$| S \cup T | = \#$ of outcomes $= 36$

$|S| + |T| = 36$

$|T| = 6$

$|S| = 36 - 6 = 30.$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

S $\equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

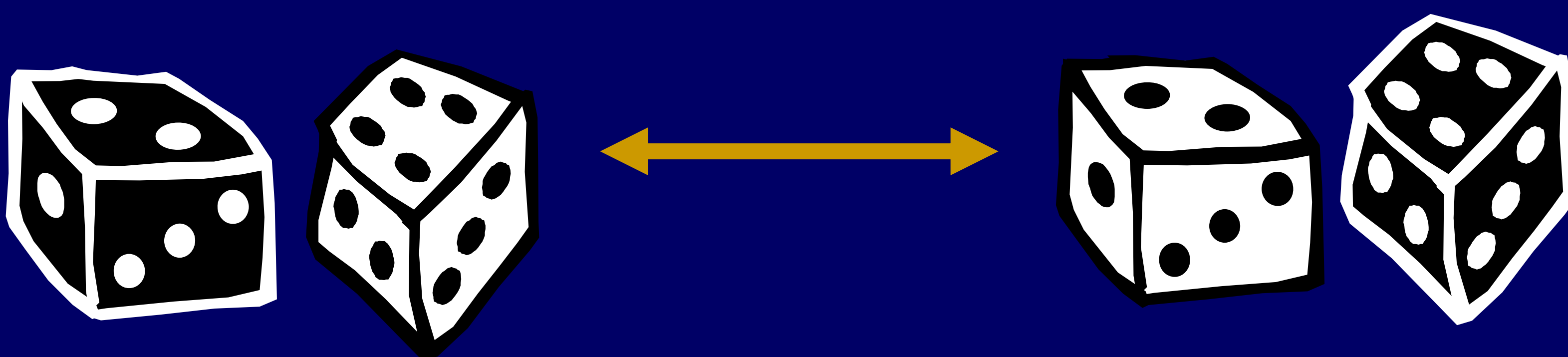
It is clear by symmetry that $|S| = |L|$.

Therefore $|S| = 15$

"It is clear by symmetry that $|S| = |L|$."

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.



S L

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.

Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: $|S| = |L|$.

Let $f:A\to B$ be a function from a set A to a set B.

f is 1-1 if and only if

$$\forall x,y\in A,\ \ x\neq y \Rightarrow f(x)\neq f(y)$$

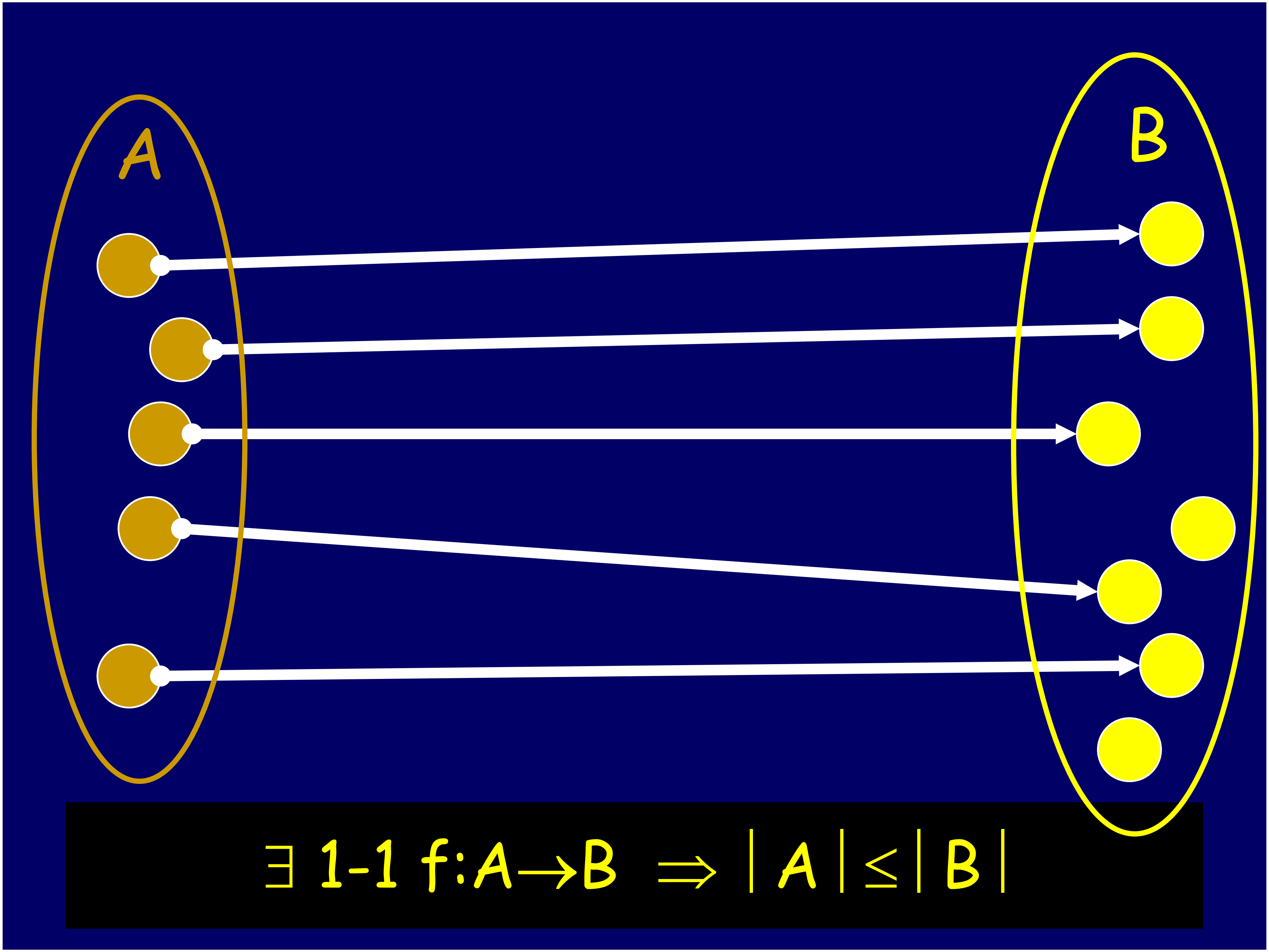
f is onto if and only if

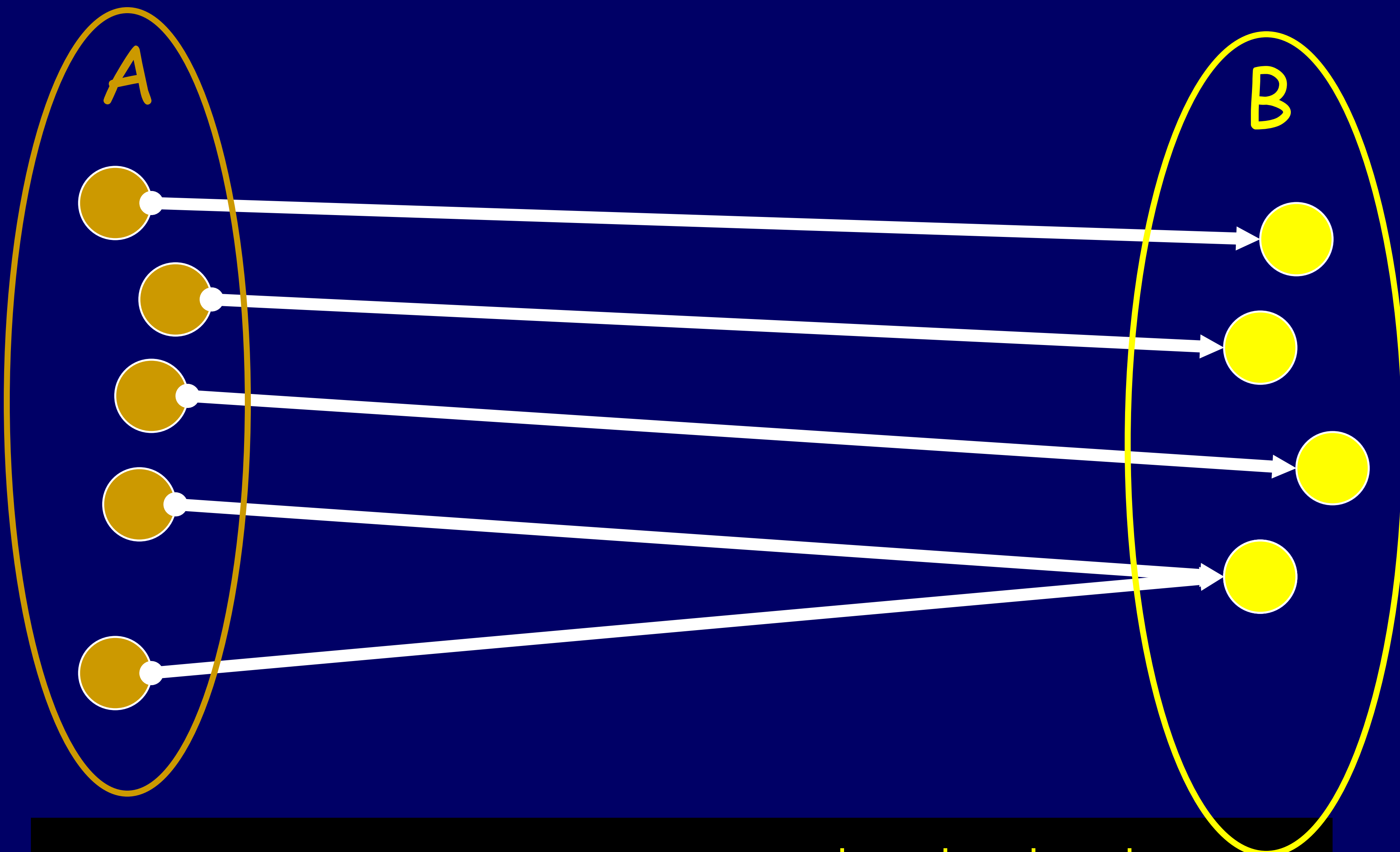
$$\forall z\in B\ \ \exists x\in A\ \ f(x) = z$$

There Exists

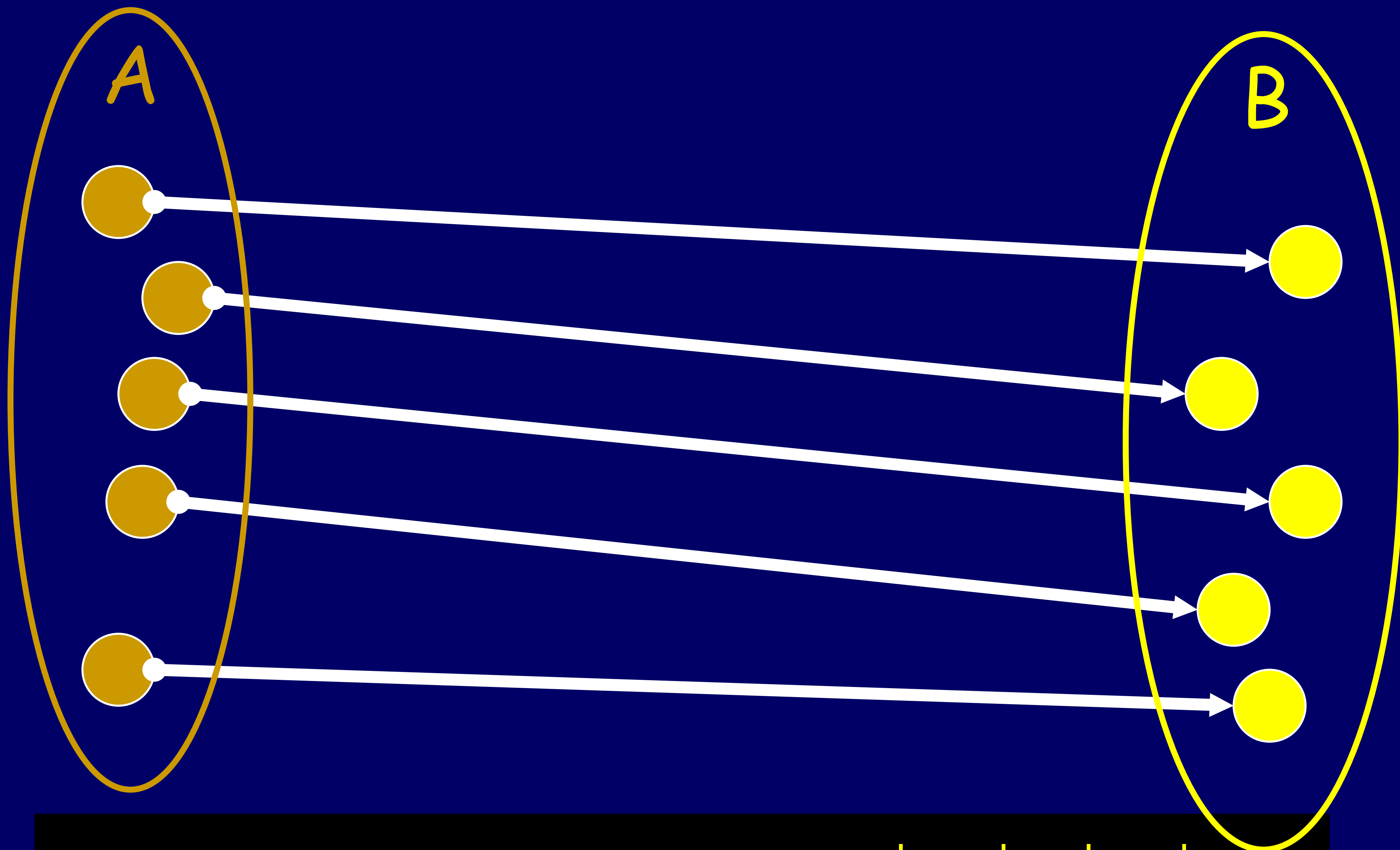
For Every

Let's restrict our attention to finite sets.


A
B
$\exists$ 1-1 $f:A\to B \Rightarrow |A| \leq |B|$

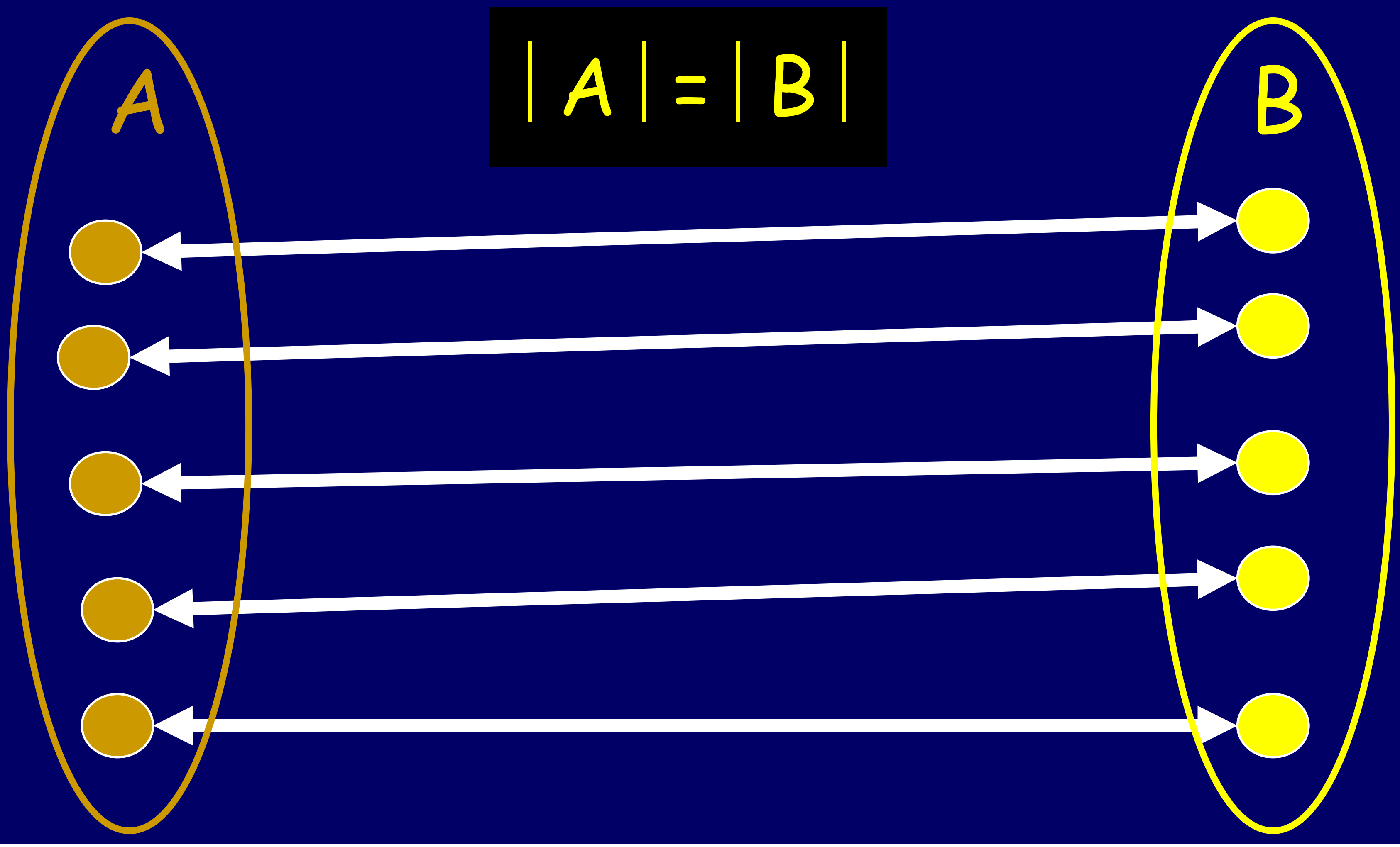


$\exists$ onto $f: A \rightarrow B \;\Rightarrow\; |A| \geq |B|$



$\exists$ 1-1 onto $f{:}A \rightarrow B \;\Rightarrow\; |A| = |B|$

1-1 onto Correspondence
(just "correspondence" for short)

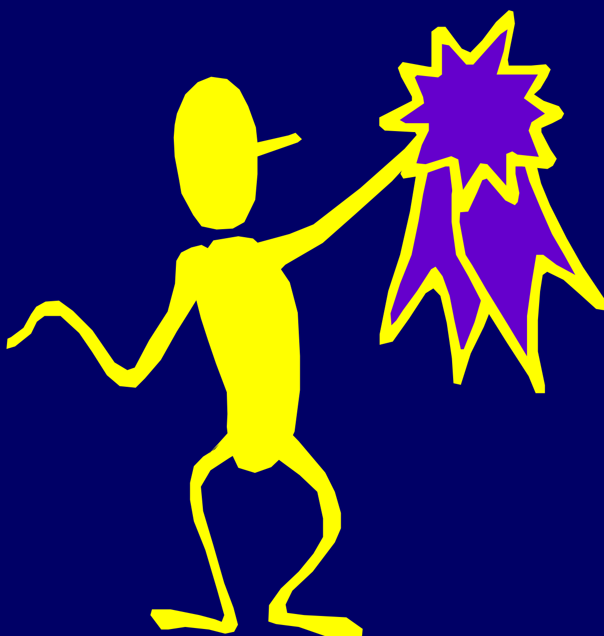


Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Correspondence Principle

If two finite sets can be placed into
1-1 onto correspondence,
then they have the same size.



It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

000000	$\leftarrow\rightarrow$	0
000001	$\leftarrow\rightarrow$	1
000010	$\leftarrow\rightarrow$	2
000011	$\leftarrow\rightarrow$	3
	...	
1...11111	$\leftarrow\rightarrow$	2^n-1

Each sequence corresponds to a unique number from 0 to 2^n-1. Hence 2^n sequences

A = {a,b,c,d,e} has many subsets.

{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, ...

The entire set and the empty set are subsets with all the rights and privileges pertaining thereto.

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

$\{b \quad c \quad e\}$

1 means "TAKE IT"
0 means "LEAVE IT"

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

$A = \{a_1, a_2, a_3, \ldots, a_n\}$

B = set of all n-bit strings

Let's construct a correspondence

$f: B \to A$

a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

For the bit string $b = b_1 b_2 b_3 \ldots b_n$

Let $f(b) = \{ a_i \mid b_i = 1 \}$

a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

$$f(b) = \{a_i \mid b_i=1\}$$

Claim: f is 1-1

Any two distinct binary sequences b and b' have a position i at which they differ.

Hence, f(b) is not equal to f(b') because they disagree on element a_i.

a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

$$f(b) = \{a_i \mid b_i=1\}$$

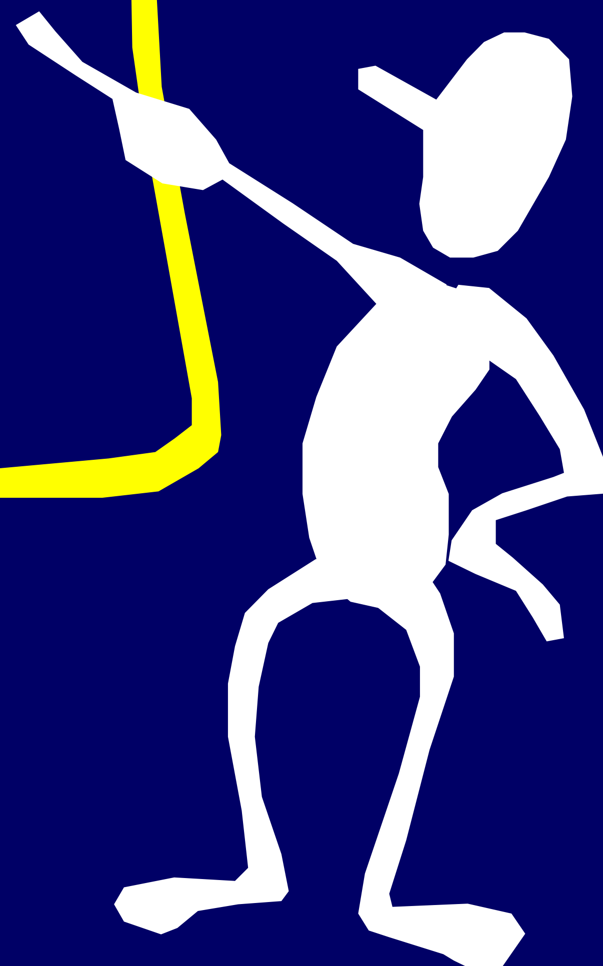
Claim: f is onto

Let S be a subset of $\{a_1,\ldots,a_n\}$.

Define $b_k = 1$ if a_k in S and $b_k = 0$ otherwise.

Note that $f(b_1b_2\ldots b_n) = S$.

The number of subsets of an n-element set is 2^n.



Let $f:A\to B$

be a function from a set A to a set B.

f is 1-1 if and only if

$$\forall x,y\in A,\ \ x\neq y \Rightarrow f(x)\neq f(y)$$

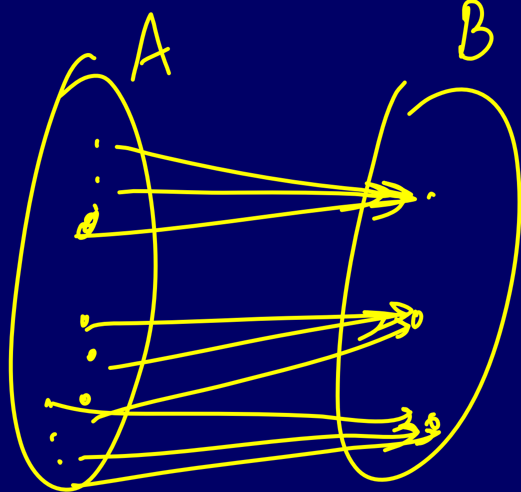
f is onto if and only if

$$\forall z\in B\ \ \exists x\in A\ \ f(x) = z$$

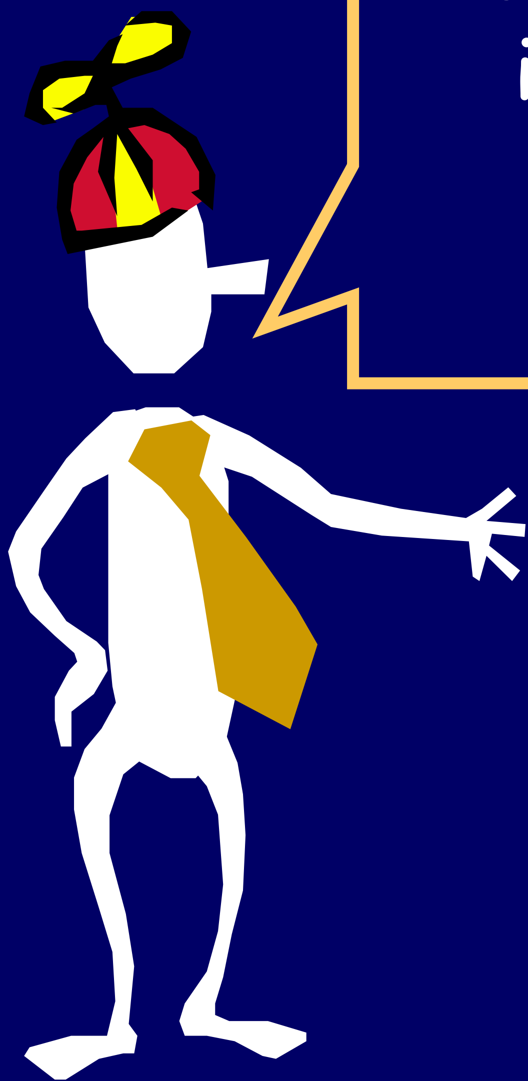
Let $f:A \to B$
be a function from a set A to a set B.

f is a 1 to 1 correspondence iff
$\forall z \in B \ \exists$ exactly one $x \in A$ s.t. $f(x) = z$

f is a k to 1 correspondence iff
$\forall z \in B \ \exists$ exactly k elements $x \in A$ s.t. $f(x)=z$



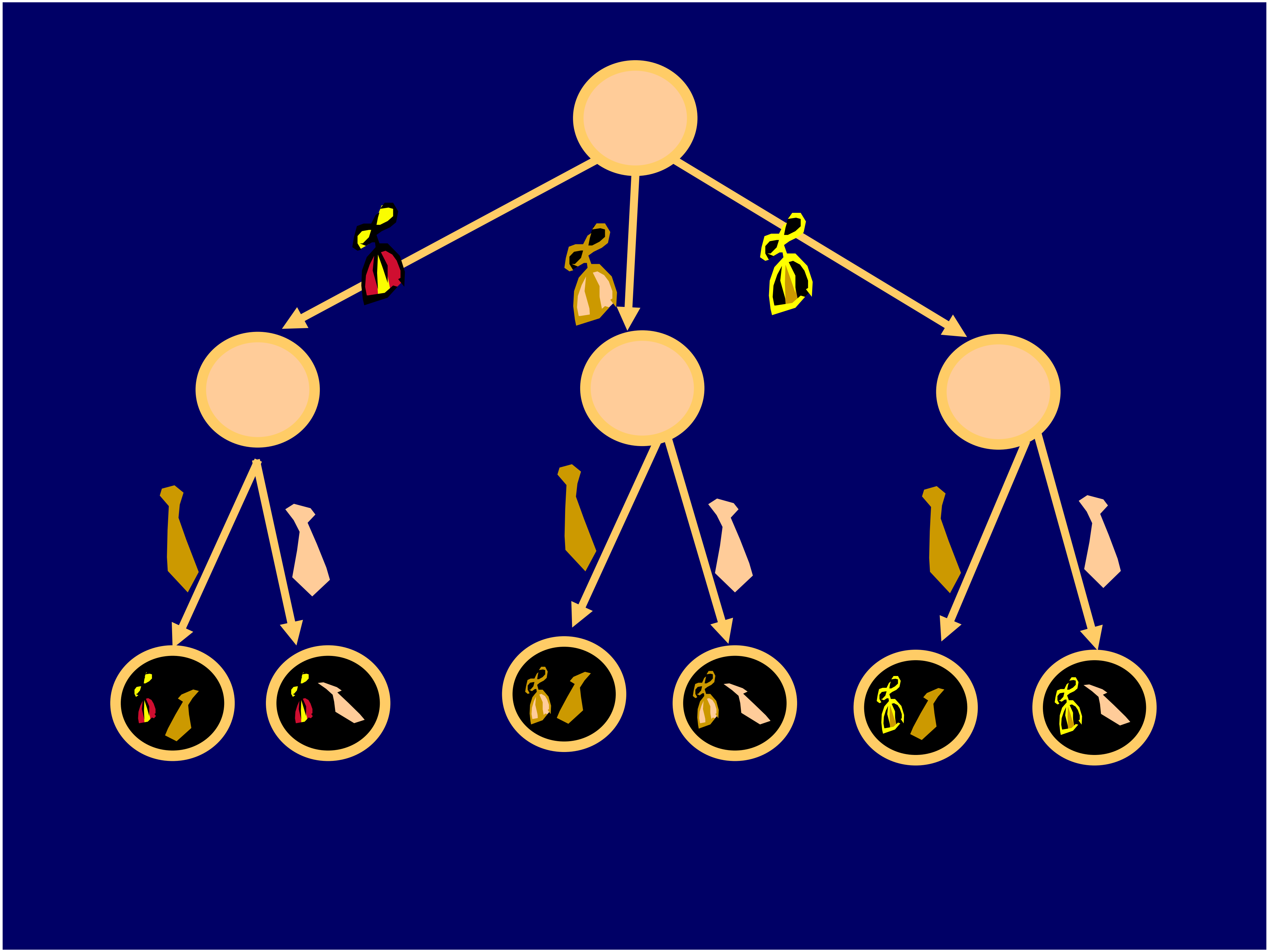
3-1 correspondence



To count the number of horses in a barn, we can count the number of hoofs and then divide by 4.

If a finite set A has a k-to-1 correspondence to finite set B, then $|B| = |A|/k$

I own 3 beanies and 2 ties.
How many different ways can I dress up in a beanie and a tie?



A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many items on the menu?

$5 + 6 + 3 + 7 = 21$

How many ways to choose a complete meal?

$5 \times 6 \times 3 \times 7 = 630$

A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many ways to order a meal if I am allowed to skip some (or all) of the courses?

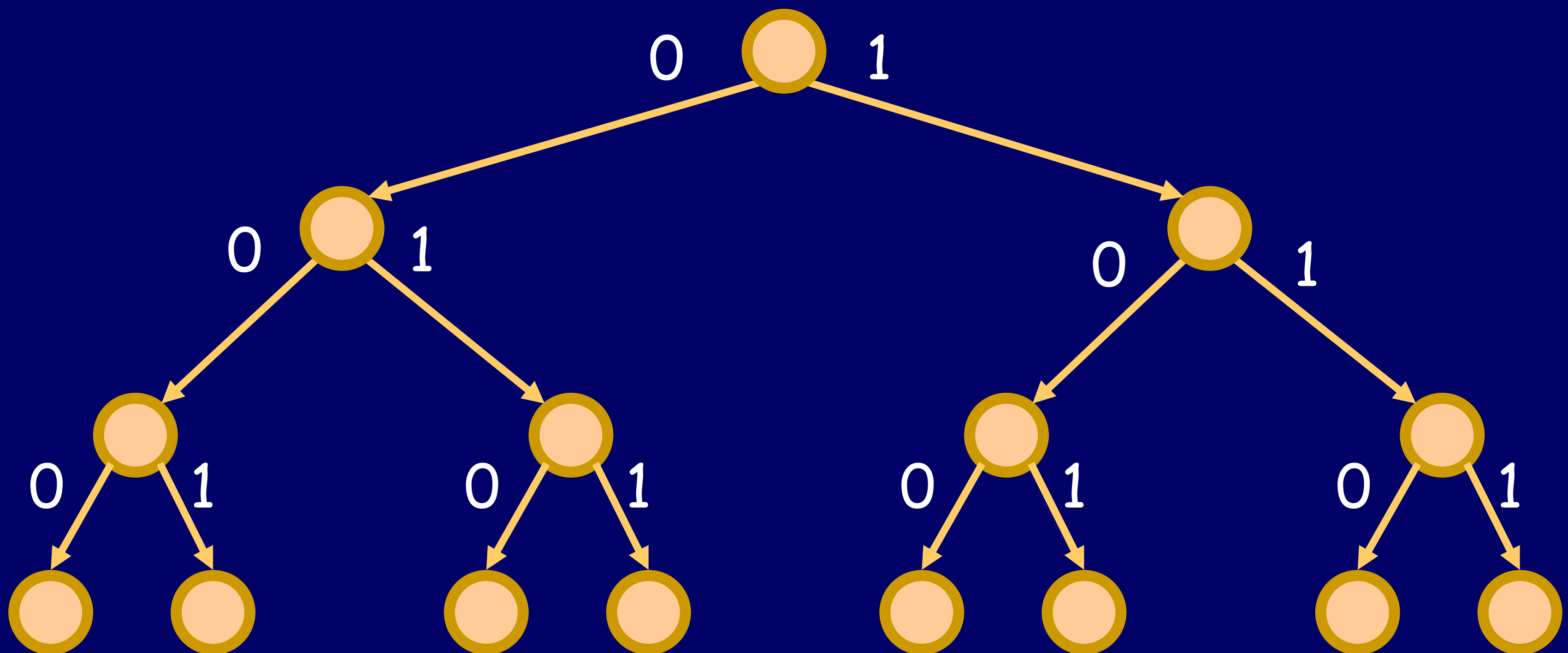
$$6 \times 7 \times 4 \times 8 = 1344$$

Hobson's restaurant has only 1 appetizer, 1 entree, 1 salad, and 1 dessert.

2^4 ways to order a meal if I might not have some of the courses.

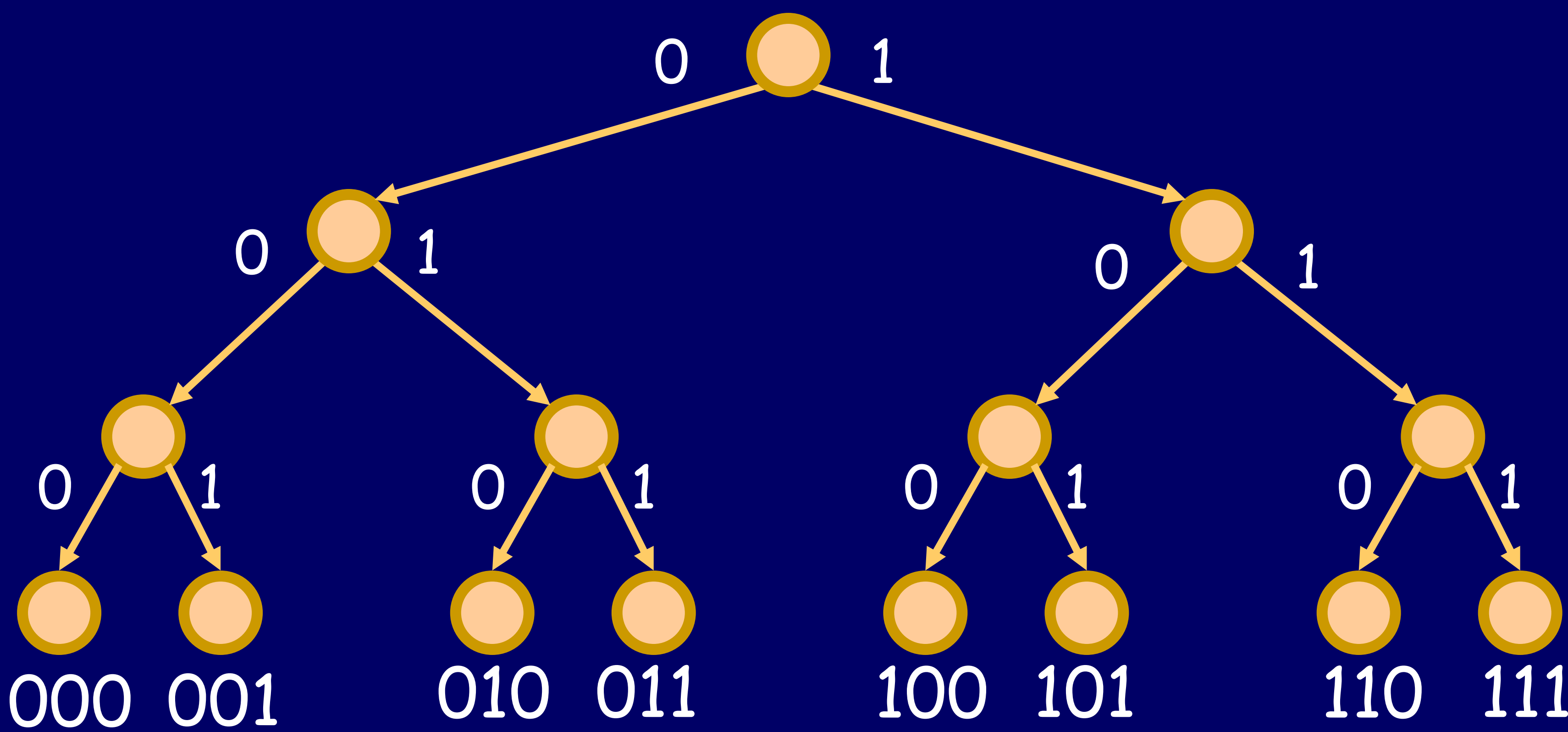
Same as number of subsets of the set {Appetizer, Entrée, Salad, Dessert}

Choice Tree for 2^n n-bit sequences

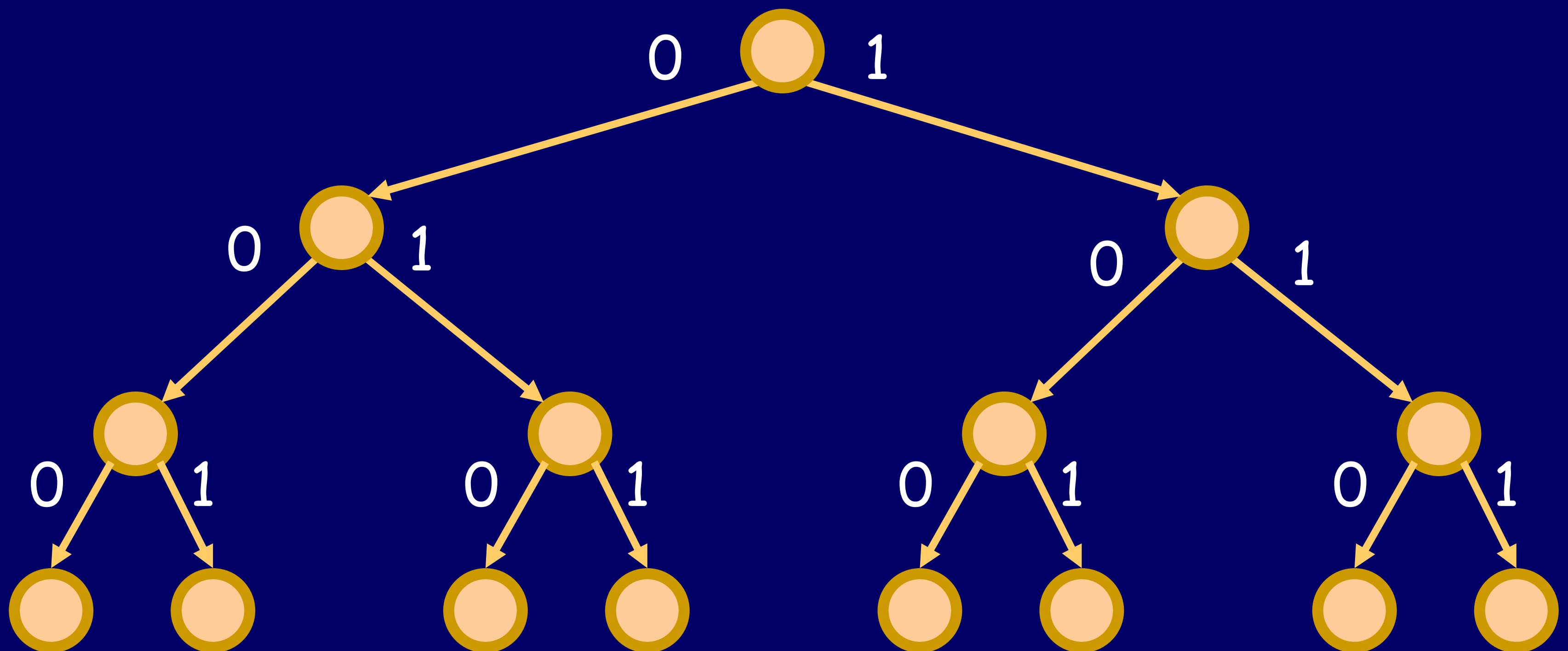


We can use a "choice tree" to represent the construction of objects of the desired type.

2^n n-bit sequences



Label each leaf with the object constructed by the choices along the path to the leaf.



2 choices for first bit
× 2 choices for second bit
× 2 choices for third bit
…
× 2 choices for the nth

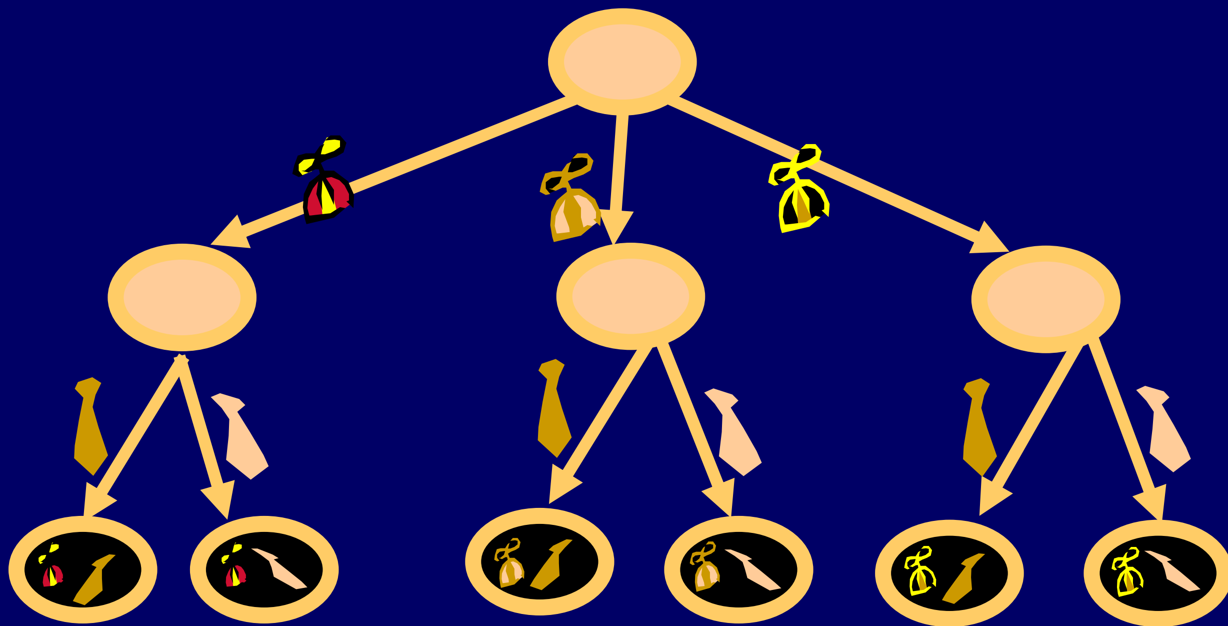
Leaf Counting Lemma

Let T be a depth-n tree when each node at depth $0 \leq i \leq n-1$ has P_{i+1} children.

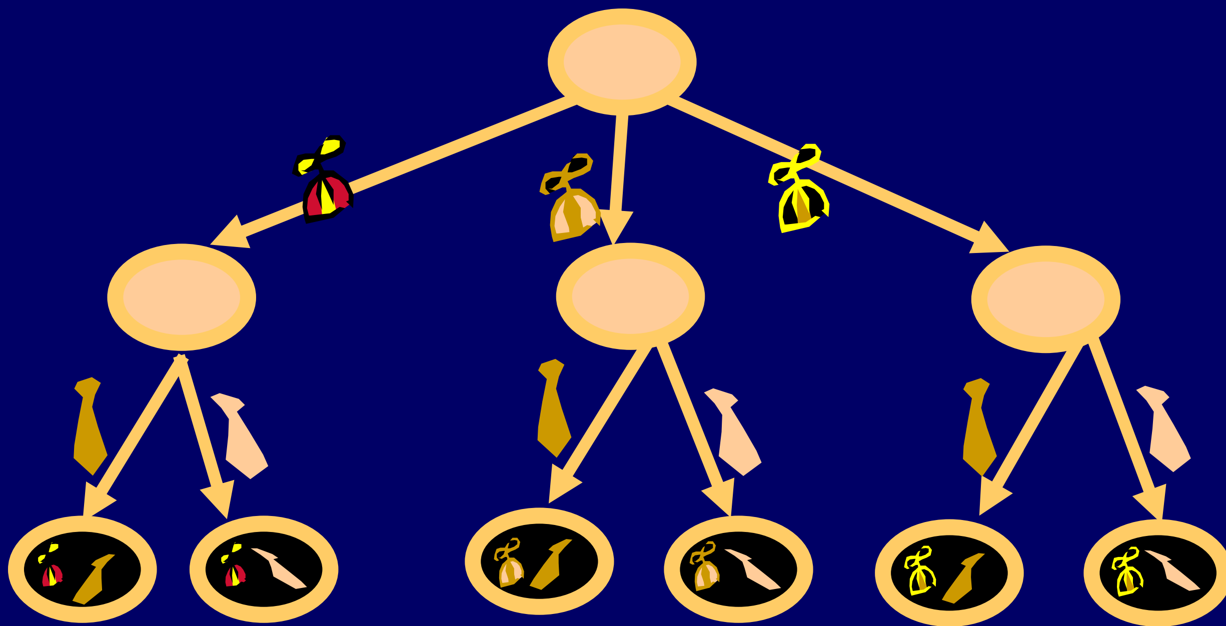
The number of leaves of T is given by:

$$P_1 P_2 \ldots P_n$$

Choice Tree

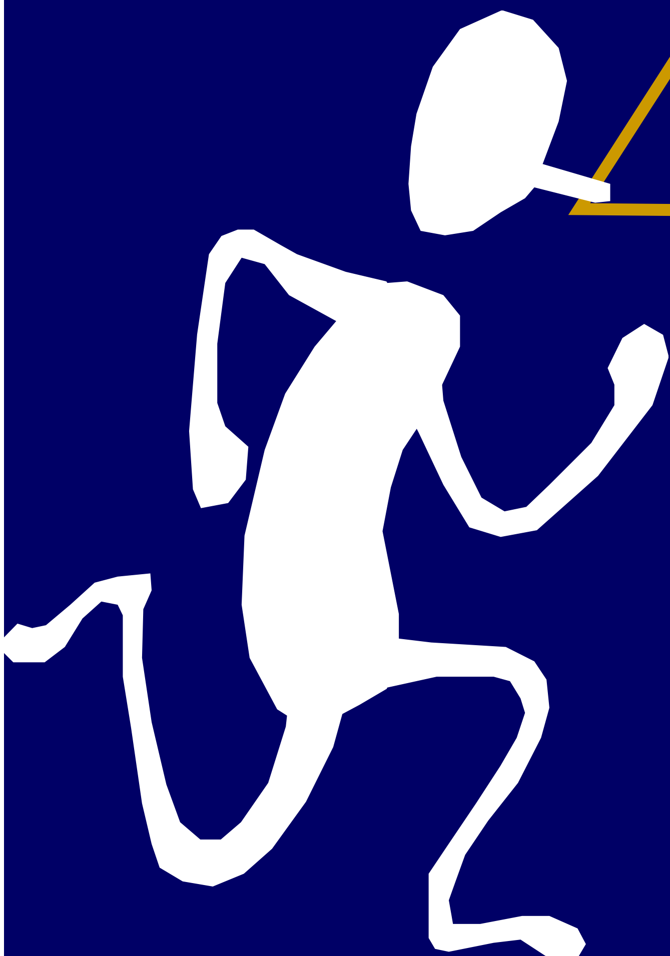


A <u>choice tree</u> is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a “choice tree representation” of a set S, if

1) Each leaf label is in S, and each element of S is some leaf label
2) No two leaf labels are the same



We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

IF set S has a choice tree representation with
P_1 possibilities for the first choice,
P_2 for the second, P_3 for the third,
and so on,

THEN

there are $P_1P_2P_3 \ldots P_n$ objects in S

Proof:

There are $P_1P_2P_3 \ldots P_n$ leaves of the choice tree which are in 1-1 onto correspondence with the elements of S.

Product Rule (rephrased)

Suppose <u>every</u> object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1) Each sequence of choices constructs an object of type S, AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

How many different orderings of deck with 52 cards?

What type of object are we making?

Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

50 possible choices for the third card;

…

1 possible choice for the 52^{nd} card.

By the product rule: $52 \times 51 \times 50 \times \ldots \times 3 \times 2 \times 1 = 52!$

A <u>permutation</u> or <u>arrangement</u> of n objects is an ordering of the objects.

The number of permutations of n distinct objects is n!

How many sequences of 7 letters are there?

$$26^7$$

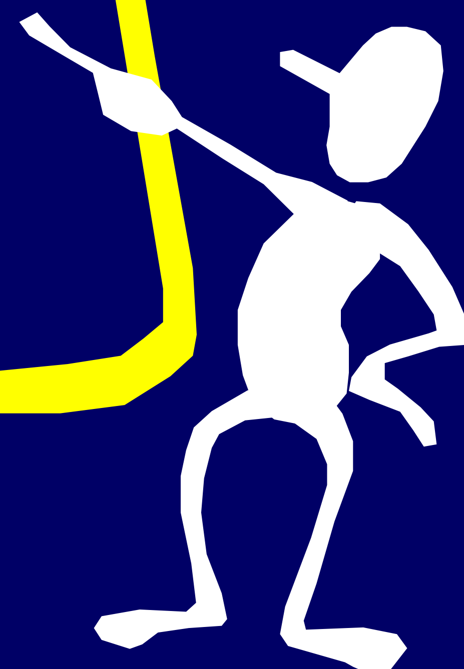
26 choices for each of the 7 positions

How many sequences of 7 letters contain at least two of the same letter?

How many sequences of 7 letters contain <u>at least</u> two of the same letter?

$$26^7 - \underbrace{26\times25\times24\times23\times22\times21\times20}_{\text{number of sequences containing all different letters}}$$

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.



Helpful Advice:

In logic, it can be useful to represent a statement in the contrapositive.

In counting, it can be useful to represent a set in terms of its complement.

If 10 horses race, how many orderings of the top three finishers are there?

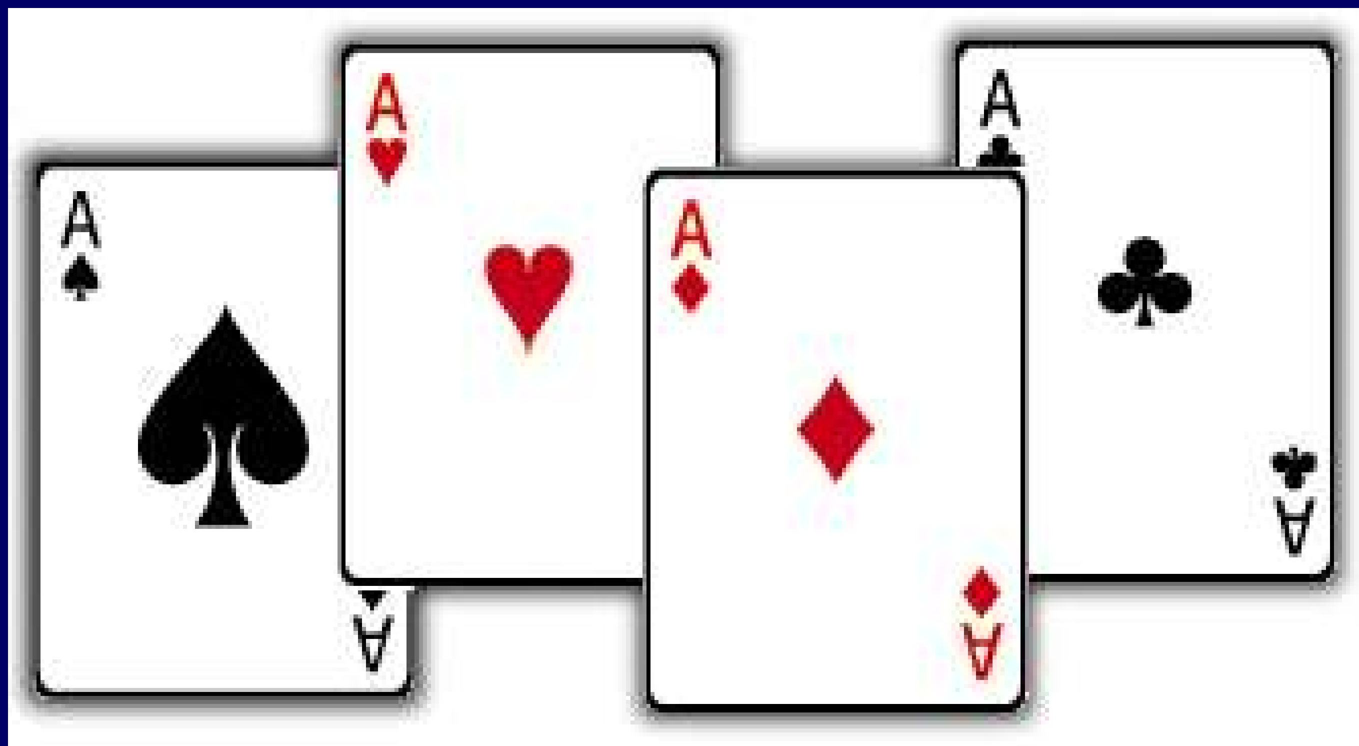
$$10 \times 9 \times 8 = 720$$

The number of ways of ordering, permuting, or arranging r out of n objects.

n choices for first place, n-1 choices for second place, . . .

$$n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$



Ordered Versus Unordered

From a deck of 52 cards how many **ordered** pairs can be formed?

52×51

How many **unordered** pairs?

$52 \times 51 / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs.

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$52 \times 51 / 2$ ← divide by overcount

We have a 2-1 map from ordered pairs to unordered pairs.
Hence #unordered-pairs = (#ordered pairs)/2

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

$52 \times 51 \times 50 \times 49 \times 48$

How many orderings of 5 cards?

$5!$

How many unordered 5 card hands?

$(52\times51\times50\times49\times48)/5! = 2{,}598{,}960$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects.

The number of r combinations of n objects:

$$\frac{n!}{(n-r)!r!} = \binom{n}{r}$$

${}^{n}C_{r}$

n "choose" r

The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Product Rule (rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1) Each sequence of choices constructs an object of type S, AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

How many 8 bit sequences have 2 0's and 6 1's?

Tempting, but incorrect:

8 ways to place first 0, times

7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and then position j for the second 0
gives same sequence as
choosing position j for the first 0 and position i for the second 0.

two ways of generating the same object!

How many 8 bit sequences have 2 0's and 6 1's?

1) Choose the set of 2 positions to put the 0's.
   The 1's are forced.

$$\binom{8}{2} \times 1 = \binom{8}{2}$$

2) Choose the set of 6 positions to put the 1's.
The 0's are forced.

$$\binom{8}{6} \times 1 = \binom{8}{6}$$

Symmetry in the formula:

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$$

"number of ways to pick r out of n elements"
is the same as
"number of ways to choose the (n-r) elements to omit"

How many hands have at least 3 aces?

4560

7104

2352

4704

$\binom{4}{3} = 4$ ways of picking 3 out of the 4 aces

$\binom{49}{2} = 1176$ ways of picking 2 cards out of the remaining 49 cards

14112

$4 \times 1176 = 4704$

4752

How many hands have at least 3 aces?

How many hands have <u>exactly</u> 3 aces?

$\binom{4}{3} = 4$ ways of picking 3 out of the 4 aces

$\binom{48}{2} = 1128$ ways of picking 2 cards out of the 48 non-ace cards

4 × 1128 = 4512

How many hands have <u>exactly</u> 4 aces?

$\binom{4}{4} = 1$ ways of picking 4 out of the 4 aces

$\binom{48}{1} = 48$ ways of picking 1 cards out of the 48 non-ace cards

1 × 48 = 48

$\boxed{\text{Total: } 4512 + 48 = 4560}$

$4704 \neq 4560$

At least one of the two counting arguments is not correct.

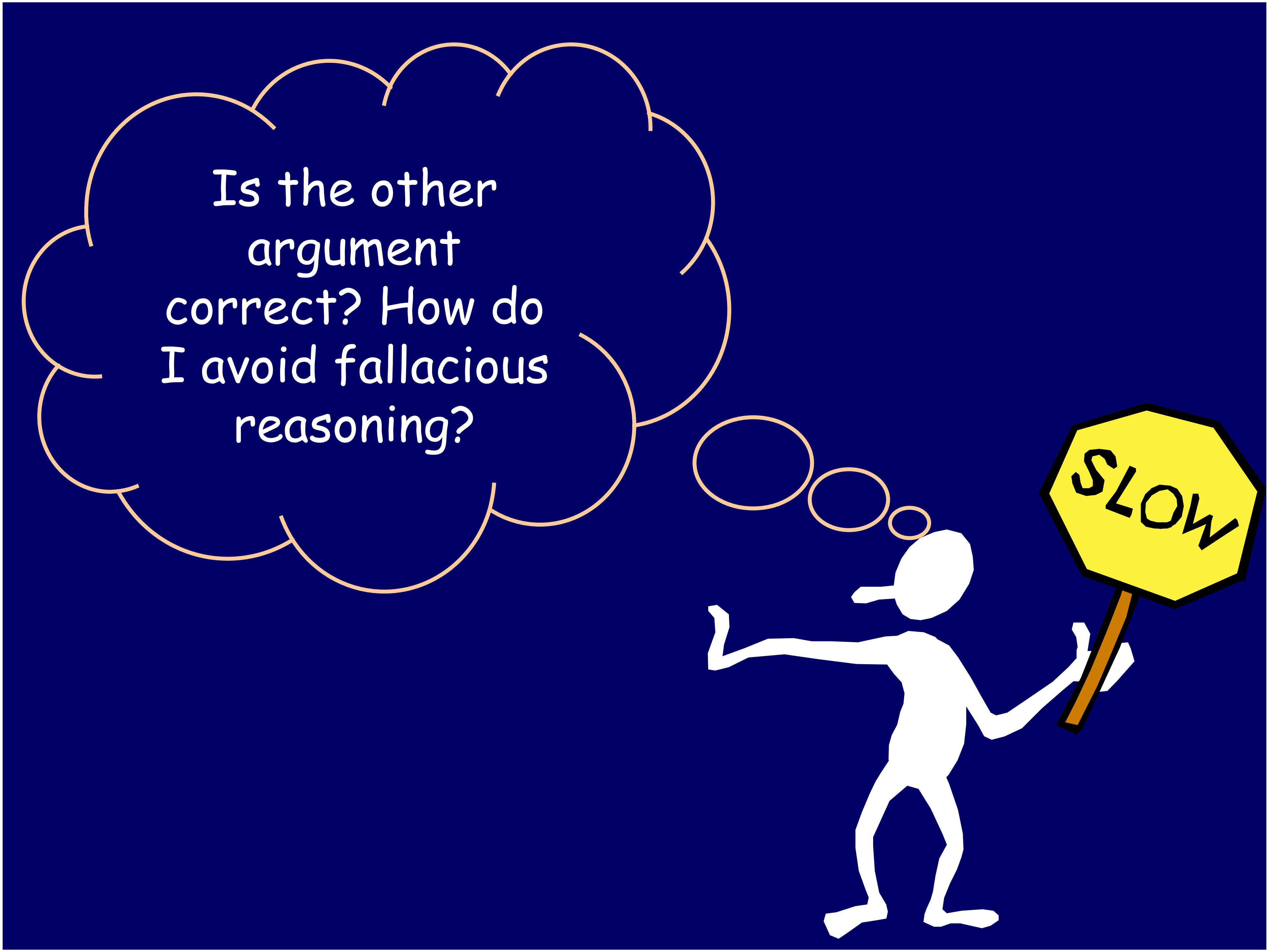
Four different sequences of choices produce the same hand

$\binom{4}{3} = 4$ ways of picking 3 out of the 4 aces

$\binom{49}{2} = 1176$ ways of picking 2 cards out of the remaining 49 cards

4 × 1176 = 4704

A♣	A♦	A♥	A♠	K♦
A♣	A♦	A♠	A♥	K♦
A♣	A♠	A♥	A♦	K♦
A♠	A♦	A♥	A♣	K♦


Is the other argument correct? How do I avoid fallacious reasoning?
SLOW

The Sleuth's Criterion

There should be a unique way to create an object in S.

in other words:

For any object in S, it should be possible to reconstruct <u>the</u> (unique) sequence of choices which lead to it.

Scheme I

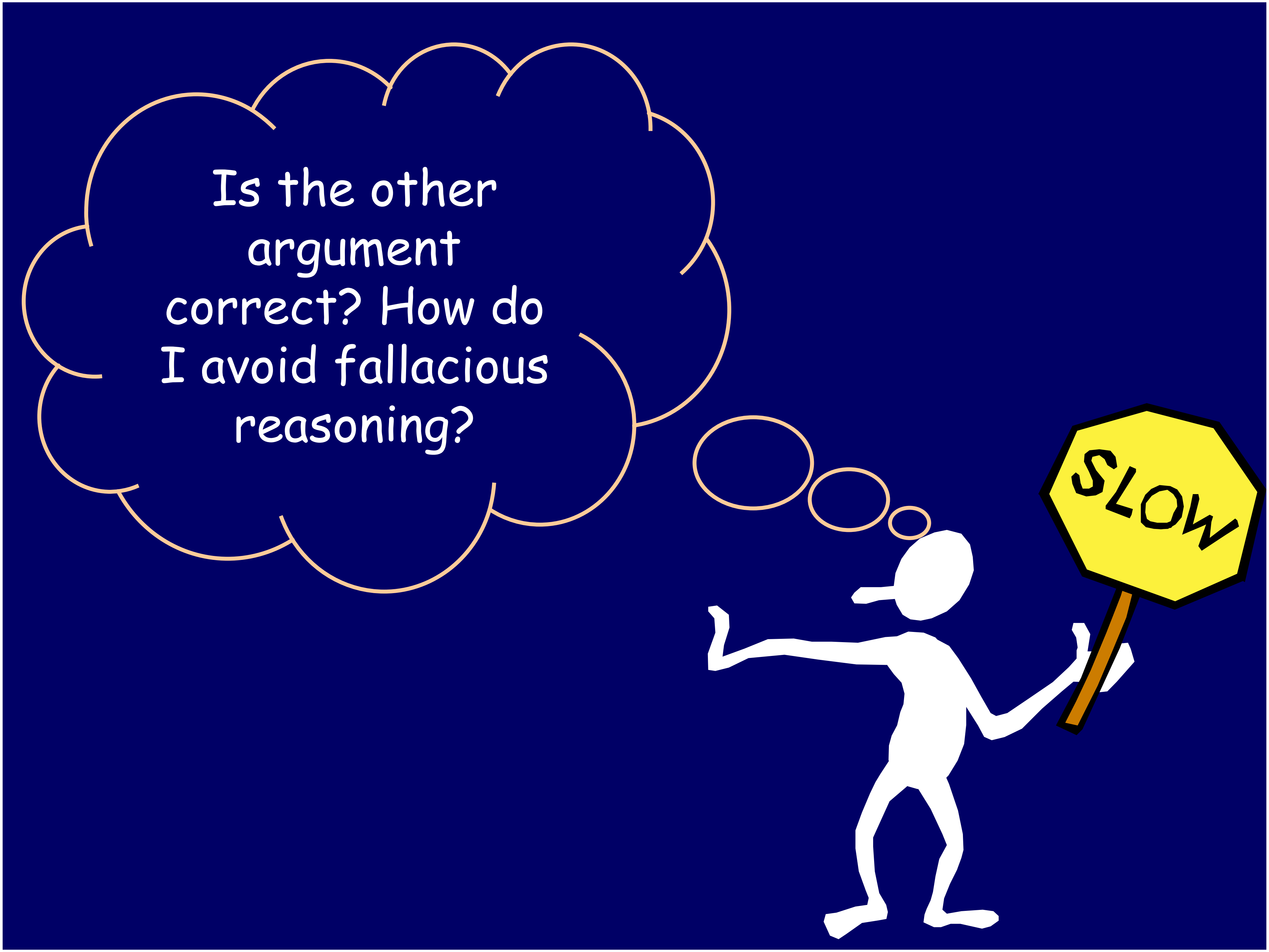
1) Choose 3 of 4 aces

2) Choose 2 of the remaining cards

A♣ A♦ A♥ A♠ K♦

Sleuth can't determine which cards came from which choice.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦


Is the other argument correct? How do I avoid fallacious reasoning?
SLOW

Scheme II

1) Choose 3 out of 4 aces

2) Choose 2 out of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

Sleuth infers: Aces came from choices in (1) and others came from choices in (2)

Scheme II
1) Choose 4 out of 4 aces
2) Choose 1 out of 48 non-ace cards

A♣ A♦ A♥ A♠ K♦

Sleuth infers: Aces came from choices in (1)
and others came from choices in (2)

Product Rule (rephrased)

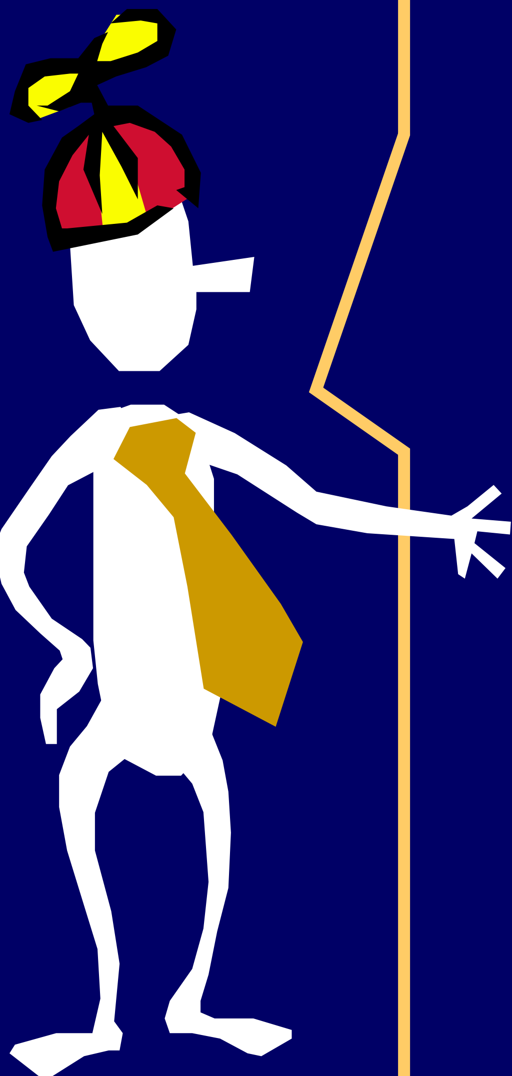
Suppose <u>every</u> object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1) Each sequence of choices constructs an object of type S, AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.



DEFENSIVE THINKING

ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

Study Bee

Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

Choice Tree

Product Rule

two conditions

Counting by complementing

it's sometimes easier to count the "opposite" of something

Binomial coefficient

Number of r subsets of an n set