15-251

Some ~~Great~~ Theoretical Ideas for ~~in~~ Computer Science

Complexity Theory: The P vs NP question

Lecture 27 (April 27, 2010)

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The $1M Questions

The Clay Mathematics Institute
Millenium Prize Problems

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equations
4. P vs NP
5. Poincaré Conjecture ← solved!
6. Riemann Hypothesis
7. Yang-Mills Theory

The P versus NP problem

Is perhaps the biggest open problem
in computer science (and mathematics!) today.

(Even featured in the TV show NUMB3RS)

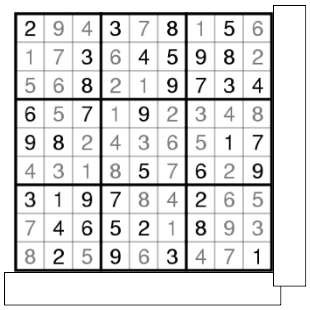
But what is the P-NP problem?

Sudoku

2			3		8		5	
		3		4	5	9	8	
		8			9	7	3	4
6		7		9				
9	8						1	7
				5		6		9
3	1	9	7			2		
	4	6	5	2		8		
	2		9		3			1

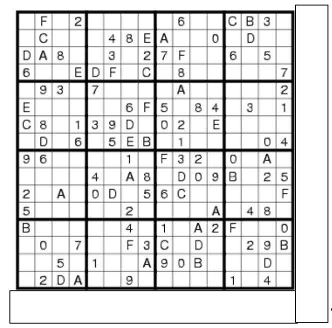
3 x 3 x 3

Sudoku



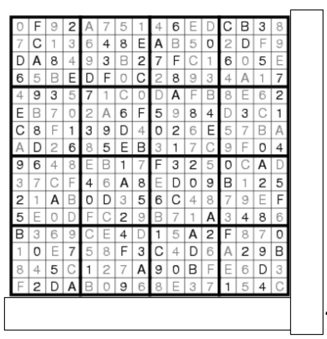
3 x 3 x 3

Sudoku



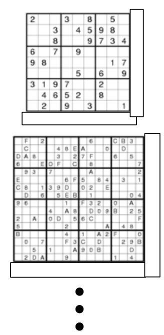
4 x 4 x 4

Sudoku



4 x 4 x 4

Sudoku



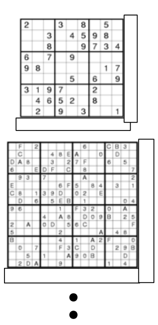
n x n x n

Suppose it takes you S(n) to solve n x n x n

V(n) time to verify the solution

Fact: $V(n) = O(n^2 \times n^2)$

Question: is there some constant c such that $S(n) \leq n^c$?



n x n x n

P vs NP problem

=

Does there exist an algorithm for n x n x n Sudoku that runs in time p(n) for some polynomial p( ) ?

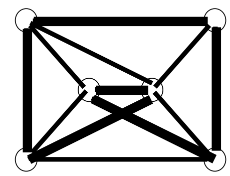
The P versus NP problem (informally)

Is proving a theorem much more difficult than checking the proof of a theorem?

Let’s start at the beginning…

Hamilton Cycle

Given a graph G = (V,E), a cycle that visits all the nodes exactly once



The Problem “HAM”

Input: Graph G = (V,E)

Output: YES if G has a Hamilton cycle

NO if G has no Hamilton cycle

The Set “HAM”

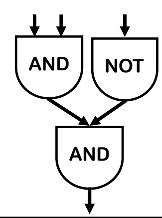
HAM = { graph G | G has a Hamilton cycle }

Circuit-Satisfiability

Input: A circuit C with one output

Output: YES if C is satisfiable

NO if C is not satisfiable



The Set “SAT”

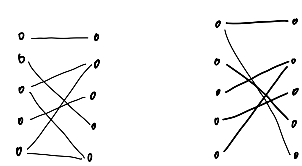
SAT = { all satisfiable circuits C }

Bipartite Matching

Input: A bipartite graph G = (U,V,E)

Output: YES if G has a perfect matching

NO if G does not



The Set “BI-MATCH”

BI-MATCH = { all bipartite graphs that have a perfect matching }

Sudoku

Input: n x n x n sudoku instance

Output: YES if this sudoku has a solution

NO if it does not

The Set “SUDOKU”

SUDOKU = { All solvable sudoku instances }

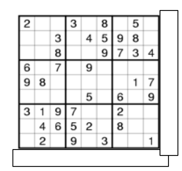
Decision Versus Search Problems

Decision Problem	Search Problem
YES/NO answers	
Does G have a Hamilton cycle?	Find a Hamilton cycle in G if one exists, else return NO
Can G be 3-colored ?	Find a 3-coloring of G if one exists, else return NO

Reducing Search to Decision

Given an algorithm for decision Sudoku, devise an algorithm to find a solution

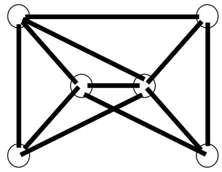
Idea:
Fill in one-by-one and use decision algorithm



Reducing Search to Decision

Given an algorithm for decision HAM, devise an algorithm to find a solution

Idea:
Find the edges of the cycle one by one



Decision/Search Problems

We’ll study decision problems because they are almost the same (asymptotically) as their search counterparts

Polynomial Time and The Class "P" of Decision Problems

What is an efficient algorithm?

Is an O(n) algorithm efficient?

How about O(n log n)?

$O(n^2)$?

$O(n^{10})$?

$\Big\}$ polynomial time

$O(n^c)$ for some constant c

$O(n^{\log n})$?

$O(2^n)$?

O(n!) ?

$\Big\}$ non-polynomial time

Does an algorithm running in $O(n^{100})$ time count as efficient?

We consider non-polynomial time algorithms to be inefficient.

And hence a necessary condition for an algorithm to be efficient is that it should run in poly-time.

Asking for a poly-time algorithm for a problem sets a (very) low bar when asking for efficient algorithms.

The question is: can we achieve even this for 3-coloring?
SAT?
Sudoku?
HAM?

The Class P

We say a set $L \subseteq \Sigma^*$ is in P if there is
- a program A and
- a polynomial p( )

such that for any x in Σ^*,

A(x) runs for at most p(|x|) time
and answers question "is x in L?" correctly.

The Class P

The class of all sets L that can be recognized in polynomial time.

The class of all decision problems that can be decided in polynomial time.

Why are we looking only at sets $\subseteq \Sigma^*$?

What if we want to work with graphs or boolean formulas?

Languages/Functions in P?

Example 1:
CONN = {graph G: G is a connected graph}

Algorithm A_1:
If G has n nodes, then run depth first search from any node, and count number of distinct nodes you see. If you see n nodes, $G \in$ CONN, else not.

Time: $p_1(|x|) = \Theta(|x|)$.

Languages/Functions in P?

HAM, SUDOKU, SAT are not known to be in P

CO-HAM = { G | G does NOT have a Hamilton cycle}

CO-HAM $\in$ P if and only if HAM $\in$ P

Onto the new class, NP

Verifying Membership

Is there a short "proof" I can give you for:

$G \in$ HAM?

$G \in$ BI-MATCH?

$C \in$ SAT?

$G \in$ CO-HAM?

NP

A set $L \in$ NP

if there exists an algorithm A and a polynomial p( )

For all $x \in L$	For all $x' \notin L$
there exists y with $\lvert y\rvert \le p(\lvert x\rvert)$	For all y' with $\lvert y'\rvert \le p(\lvert x'\rvert)$
such that A(x,y) = YES	we have A(x′,y′) = NO
in $p(\lvert x\rvert)$ time	in $p(\lvert x\rvert)$ time

Recall the Class P

We say a set $L \subseteq \Sigma^*$ is in **P** if there is
a program A and
a polynomial p()

such that for any x in Σ^*,

A(x) runs for at most p(|x|) time
and answers question "is x in L?" correctly.

can think of A as "proving" that x is in L

NP

A set $L \in NP$

if there exists an algorithm A and a polynomial p( )

For all $x \in L$	For all $x' \notin L$
there exists a y with $\|y\| \leq p(\|x\|)$	For all y′ with $\|y'\| \leq p(\|x'\|)$
such that A(x,y) = YES	Such that A(x′,y′) = NO
in p(|x|) time	in p(|x|) time

Example: HAM $\in$ NP

Let A(x,y) be a program that takes two strings x and y, and returns YES if the following conditions hold otherwise it returns NO.

- y is a representation of a labeled graph
- x is a representation of a cycle with the same labeled vertices as y
- every edge of the cycle x is in the graph y

(All of these conditions can be easily checked in linear time)

By our definition, this proves HAM $\in$ NP

The Class NP

The class of sets L for which there exist "short" proofs of membership
(of polynomial length)
that can be "quickly" verified
(in polynomial time).

Recall: A doesn't have to find these proofs y; it just needs to be able to verify that y is a "correct" proof.

P $\subseteq$ NP

For any L in P, we can just take y to be the empty string and satisfy the requirements.

Hence, every language in P is also in NP.

Languages/Functions in NP?

G $\in$ HAM? (Yes, already saw)

G $\in$ BI-MATCH? (is in P)

G $\in$ SAT? (Yes. explain it)

G $\in$ CO-HAM? (not clear)

Proof that something is in NP is often trivial.

Summary: P versus NP

Set L is in P if membership in L can be decided in poly-time.

Set L is in NP if each x in L has a short "proof of membership" that can be verified in poly-time.

Fact: $P \subseteq NP$

Question: Does $NP \subseteq P$?

Why Care?

NP Contains Lots of Problems We Don't Know to be in P

Classroom Scheduling
Packing objects into bins
Scheduling jobs on machines
Finding cheap tours visiting a subset of cities
Allocating variables to registers
Finding good packet routings in networks
Decryption
…

OK, OK, I care…

But where do I begin if I want to reason about the P=NP problem?

How can we prove that $NP \subseteq P$?

I would have to show that every set in NP has a polynomial time algorithm…

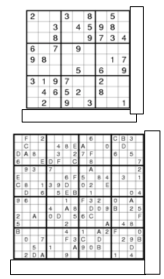
How do I do that?
It may take a long time!
Also, what if I forgot one of the sets in NP?

We can describe just one problem L in NP, such that if this problem L is in P, then $NP \subseteq P$.

It is a problem that can capture all other problems in NP.

The “Hardest” Set in NP

Sudoku



n x n x n

Sudoku has a polynomial time algorithm

if and only if

P = NP

The “Hardest” Sets in NP

Sudoku   Clique

SAT   Independent-Set

3-Colorability   HAM

These problems are all “polynomial-time equivalent”.

I.e., each of these can be reduced to any of the others in poly-time

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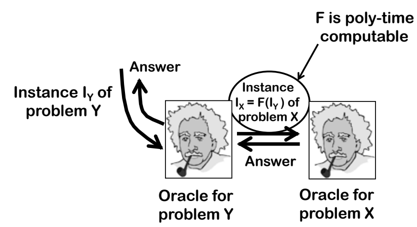
“Poly-time reducible to each other”

Reducing problem Y to problem X in poly-time



How do you prove these are the hardest?

Theorem [Cook/Levin]:

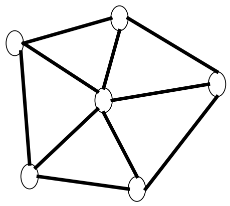
SAT is one language in NP, such that if we can show SAT is in P, then we have shown NP $\subseteq$ P.

SAT is a language in NP that can capture all other languages in NP.

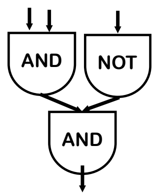
We say SAT is NP-complete.

Last lecture…

3-colorability



Circuit Satisfiability

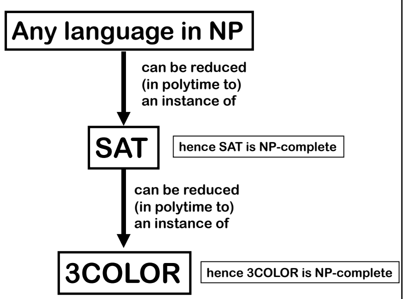


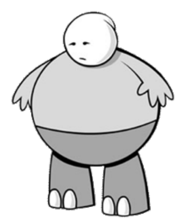
Last lecture…

SAT and 3COLOR: Two problems that seem quite different, but are substantially the same.

Also substantially the same as CLIQUE and INDEPENDENT SET.

If you get a polynomial-time algorithm for one, you get a polynomial-time algorithm for ALL.





Here's What You Need to Know…

Definition of P and NP

Definition of problems
  SAT, 3-COLOR, HAM, SUDOKU, BI-MATCH

SAT, 3-COLOR, HAM, SUDOKU all essentially equivalent.

Solve any one in poly-time
  ⇒ solve all of them in poly-time