15-251
Some ~~Great~~ Theoretical Ideas ~~in~~ for Computer Science

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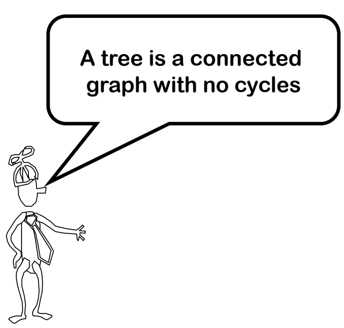
Graphs
Lecture 18 (March 23, 2010)

Graph – informal definitions
- Set of vertices (or nodes)
- Set of edges, a pair of vertices
- A self-loop is an edge that connects to the same vertex twice
- A multi-edge is a set of two or more edges that have the same two vertices
- A graph is simple if it has no multi-edges or self-loops.

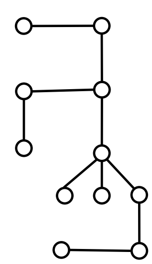
More terms

- Cycles
- Paths
- The degree of a vertex
- Directed versus undirected (this lecture is all undirected)

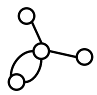
What's a tree?



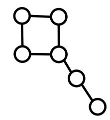
Tree



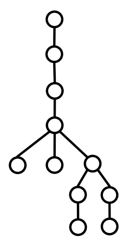
Not Tree



Not Tree

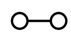


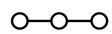
Tree

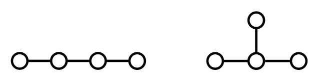


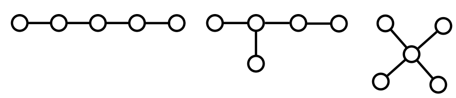
How Many n-Node Trees?

1: 

2: 

3: 

4: 

5: 

Notation

In this lecture:

n will denote the number of nodes in a graph

e will denote the number of edges in a graph

Theorem: Let G be a graph with n nodes and e edges

The following are equivalent:

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and n = e + 1
4. G is acyclic and n = e + 1
5. G is acyclic and if any two non-adjacent nodes are joined by an edge, the resulting graph has exactly one cycle

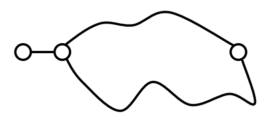
To prove this, it suffices to show
$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 2$ 1. G is a tree (connected, acyclic)

2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:



Then there exists a cycle!

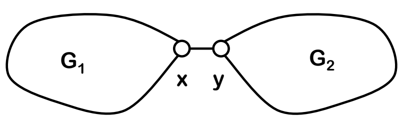
$2 \Rightarrow 3$ 2. Every two nodes of G are joined by a unique path

3. G is connected and n = e + 1

Proof: (by induction)

Assume true for every graph with < n nodes

Let G have n nodes and let x and y be adjacent



Let n_1,e_1 be number of nodes and edges in G_1

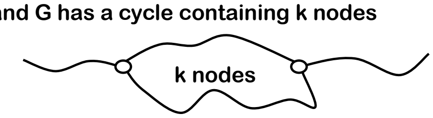
Then $n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1$

$3 \Rightarrow 4$ 3. G is connected and n = e + 1

4. G is acyclic and n = e + 1

Proof: (by contradiction)

Assume G is connected with n = e + 1, and G has a cycle containing k nodes



Note that the cycle has k nodes and k edges

Start adding nodes and edges until you cover the whole graph

Number of edges in the graph will be at least n

4→5 and 5→1 are left to the reader.

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

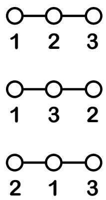
Proof (by contradiction):

Assume all but one of the points in the tree have degree at least 2

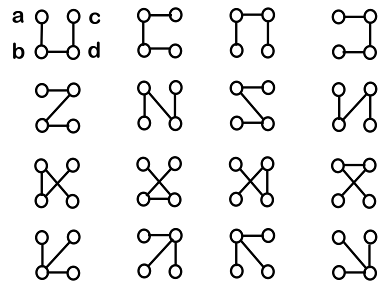
In any graph, sum of the degrees = 2e

Then the total number of edges in the tree is at least (2n-1)/2 = n - 1/2 > n - 1

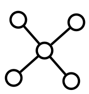
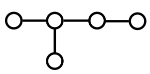
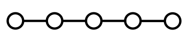
How many labeled trees are there with three nodes?



How many labeled trees are there with four nodes?



How many labeled trees are there with five nodes?

5 labelings

5 x 4 x 3 labelings

5!/ 2 labelings

125 labeled trees

How many labeled trees are there with n nodes?

3 labeled trees with 3 nodes

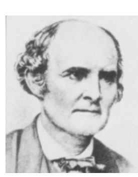
16 labeled trees with 4 nodes

125 labeled trees with 5 nodes

n^{n-2} labeled trees with n nodes

Cayley's Formula

The number of labeled trees on n nodes is n^{n-2}



The proof will use the correspondence principle

Each labeled tree on n nodes

corresponds to

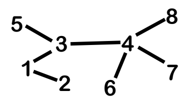
A sequence in $\{1,2,\ldots,n\}^{n-2}$ (that is, n-2 numbers, each in the range [1..n])

How to make a sequence from a tree?

Loop through i from 1 to n-2

- Let L be the degree-1 node with the lowest label
- Define the i^{th} element of the sequence as the label of the node adjacent to L
- Delete the node L from the tree

Example:



1 3 3 4 4 4

Lemma: The node labels occurring in a sequence are precisely those with degree at least 2.

Proof: Every time a label is output, that node's degree decreases by 1. At the end there are two nodes of degree 1. Therefore all the degree ≥ 2 are eventually output. □

Therefore, we can, by looking at the the sequence, identify the nodes of degree 1. Among those, the one deleted first is the lowest. This leads to the following…..

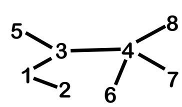
How to reconstruct the unique tree from a sequence S:

Let I = {1, 2, 3, …, n}

Loop until S is empty

- Let i = smallest # in I but not in S
- Let s = first label in sequence S
  - Add edge {i, s} to the tree
  - Delete i from I
  - Delete s from S

Add edge {a,b}, where I = {a,b}



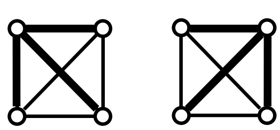
1 3 3 4 4 4

For any sequence this algorithm always generates a labeled tree that inverts the encoding algorithm.

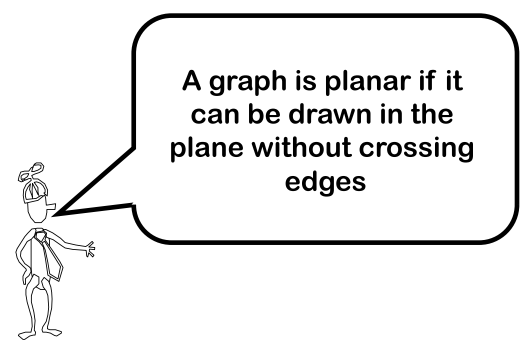
The invariant that is preserved as the algorithm runs is that the set of available labels (I) always contains all the labels remaining in the sequence.

Spanning Trees

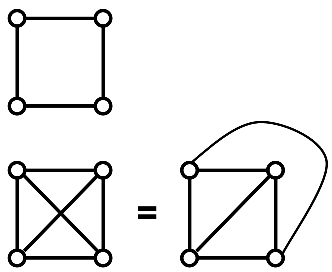
A spanning tree of a graph G is a tree that touches every node of G and uses only edges from G



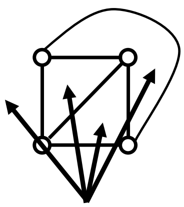
Every connected graph has a spanning tree



Examples of Planar Graphs



http://www.planarity.net



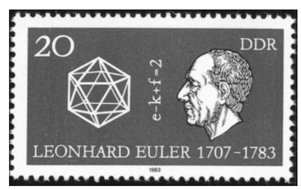
4 faces

Faces

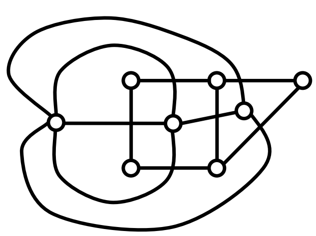
A planar graph splits the plane into disjoint faces

Euler's Formula

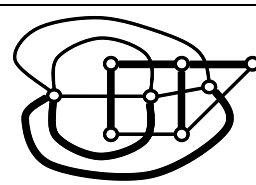
If G is a connected planar graph with n vertices, e edges and f faces, then n – e + f = 2



Rather than using induction, we'll use the important notion of the dual graph



Dual = put a node in every face, and an edge for each edge joining two adjacent faces



Let G* be the dual graph of G

Let T be a spanning tree of G

Let T* be the graph where there is an edge in dual graph for each edge in G – T

Then T* is a spanning tree for G*

$n = e_T + 1$ $\quad$ $n + f = e_T + e_{T^*} + 2$

$f = e_{T^*} + 1$ $\quad$ $= e + 2$

Corollary: Let G be a simple planar graph with n > 2 vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most 3n – 6 edges

Proof of 1:

In any graph, (sum of degrees) = 2e

Assume all vertices have degree ≥ 6

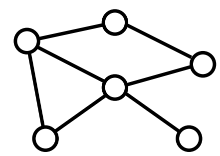
Then $3n \leq e$

Furthermore, since G is simple, $3f \leq 2e$

So $3n + 3f \leq 3e \Rightarrow 3(n-e+f) \leq 0$, contradict.

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color



Graph Coloring

Arises surprisingly often in CS

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register

Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored

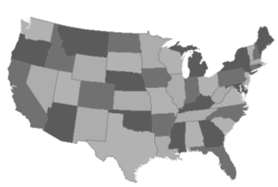
Assume G has n vertices

Since G is planar, it has some node v with degree at most 5

Remove v and color by Induction Hypothesis

Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree ≤ 5

4-color theorem was finally proven in the 1970s by Appel and Haken using computer assistance.



Implementing Graphs

Adjacency Matrix

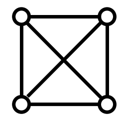
Suppose we have a graph G with n vertices. The adjacency matrix is the n x n matrix $A=[a_{ij}]$ with:

$a_{ij} = 1$ if (i,j) is an edge

$a_{ij} = 0$ if (i,j) is not an edge

Good for dense graphs!

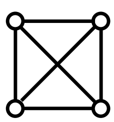
Example



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Counting Paths

The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k



$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

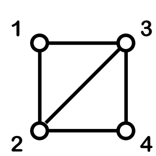
$$= \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

Adjacency List

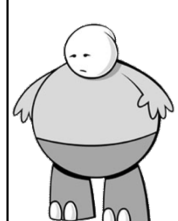
Suppose we have a graph G with n vertices. The adjacency list is the list that contains all the nodes that each node is adjacent to

Good for sparse graphs!

Example



1: 2,3
2: 1,3,4
3: 1,2,4
4: 2,3



Here's What You Need to Know…

Trees

- Counting Trees
- Different Characterizations

Planar Graphs

- Definition
- Euler's Theorem
- Coloring Planar Graphs

Adjacency Matrix and List

- Definition
- Useful for counting