

15-251

Some **AWESOME**
~~Great Theoretical Ideas~~
~~in Computer Science~~
about ~~Generating Functions~~
Probability

Probability II

Lecture 12 (February 18, 2010)

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Expectation, Random Walks, Bounds, and the
Probabilistic Method

Expectation

The expectation, or expected value of a random variable X is written as $E(X)$ or $E[X]$, and is

$$E[X] = \sum_{x \in \Omega} x m(x)$$

Ω is the sample space (the set of possible outcomes)

$m(x)$ is the distribution function (the probability of any of the outcomes occurring)

Linearity of Expectation

$$E[X_1 + X_2 + \dots + X_n] =$$

$$E[X_1] + E[X_2] + \dots + E[X_n]$$

The expectation of the sum is the sum of the expectations

Linearity of Expectation

$$E[X_1 + X_2 + \dots + X_n] = \\ E[X_1] + E[X_2] + \dots + E[X_n]$$

Question 1: If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

Could try $\sum_k k \Pr(\text{exactly } k \text{ letters end up in correct envelopes}) \dots$

But what is $\Pr(\text{exactly } k \text{ letters end up in correct envelopes})$?

Linearity of Expectation

Let A_i be the event the i^{th} letter ends up in its correct envelope

Let X_i be a indicator R.V. for A_i

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Let } Z = X_1 + \dots + X_{100}$$

We are asking for $E[Z]$

$$E[X_i] = \Pr(A_i) = 1/100$$

$$\text{So } E[Z] = 100E[X_i] = 1$$

Linearity of Expectation

So, in expectation, 1 letter will be in the same correct envelope

Pretty neat: it doesn't depend on how many letters!

Question: were the A_i independent?

No! E.g., think of $n=2$. $P(A_1|A_2) = 1$ but $P(A_2) = 1/2$

Linearity of Expectation gets its power from the fact that independence is *not* required.

Linearity of Expectation

$$np = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

Bernoulli Trials: Event with two possible outcomes, one of which always occurs with same probability p . Repeat the same event n times.

RHS: If we view the outcomes as a string of W and L, There are n choose k ways to get k Ws.

For the probability of any of these string occurring we need each bit in the string to come out right.

Need k Ws (probability p) and $(n - k)$ Ls (probability $1 - p$)

Linearity of Expectation

$$np = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

LHS: Let X = number of Ws when n independent trials are run

Break X into n simpler RVs:

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ trial is W} \\ 0 & \text{if the } i^{\text{th}} \text{ trial is L} \end{cases}$$

$$E[X] = E[\sum_i X_i] = nE[X_i] = np$$

Linearity of Expectation

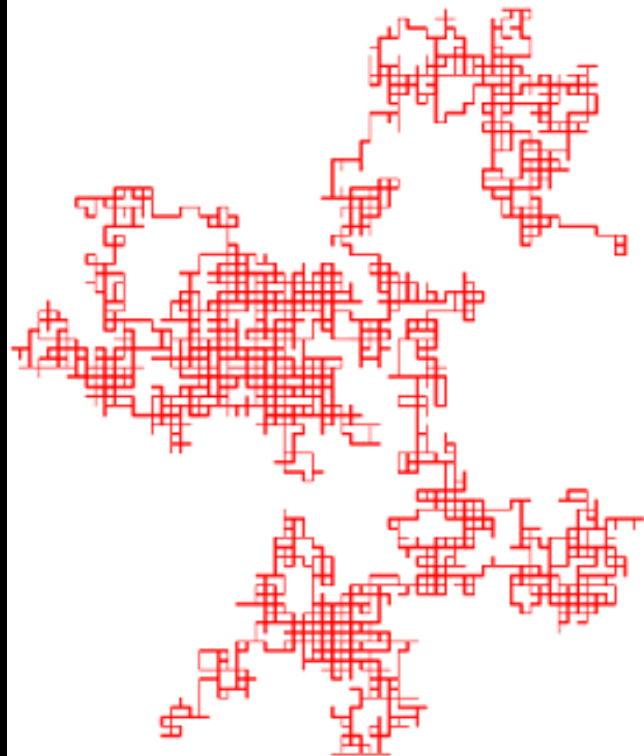
General approach:

View thing you care about as
expected value of some RV

Write this RV as sum of simpler
RVs (typically indicator RVs)

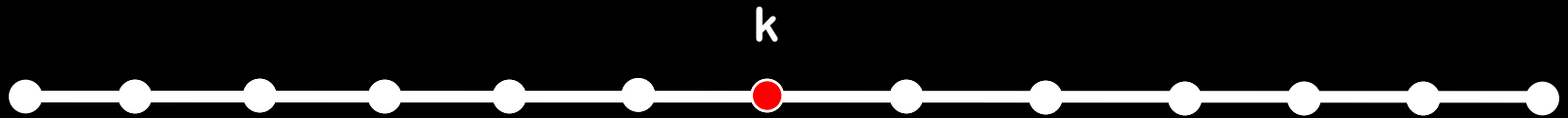
Solve for their expectations
and add them up!

Random Walks



Random Walk on a Line

Say we start at a point k on the number line. At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).

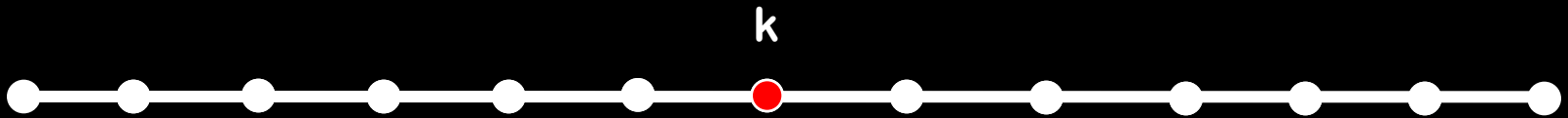


Question 1: What is your expected location at time t ?

Let X_t be a random variable for the location at time t

Random Walk on a Line

Say we start at a point k on the number line.
At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).



$$X_t = k + \delta_1 + \delta_2 + \dots + \delta_t,$$

(δ_i is a RV for change in location at time i)

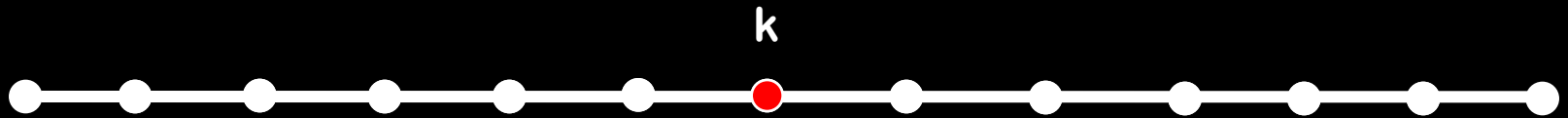
$$E[\delta_i] = 0$$

(1/2 the time it increases, 1/2 it decreases)

$$\text{So } E[X_t] = E[k + \delta_1 + \delta_2 + \dots + \delta_t] = E[k] + tE[\delta_i] = k$$

Random Walk on a Line

Say we start at a point k on the number line. At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).



Question 2: What is your expected square of the distance from k at time t ?

Let X_t be a random variable for the square of the distance from k at time t

Random Walk on a Line

Question 2: What is your expected distance from k at time t ?

$$X_t = (k - (k + \delta_1 + \delta_2 + \dots + \delta_t))^2$$

(δ_i is a RV for change in your distance at time i)

$$E[\delta_i \delta_i] = (1/2)1 + (1/2)1 = 1$$

(δ_i is either -1 or 1. So $\delta_i \delta_i$ is always 1)

$$E[\delta_i \delta_j] = (1/4)1 + (1/4)1 + (1/4)-1 + (1/4)-1 = 0$$

(1/2 of the time they share a sign and is 1, 1/2 of the time they don't and is -1)

Random Walk on a Line

Question 2: What is your expected distance from k at time t ?

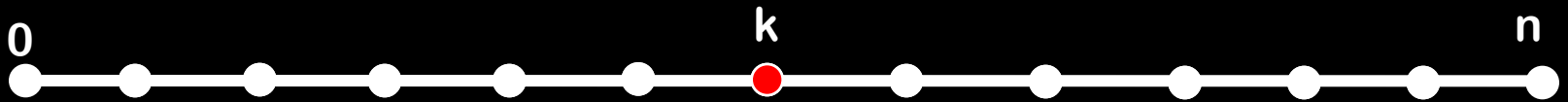
$$E[X_t] = E[(k - (k + \delta_1 + \delta_2 + \dots + \delta_t))^2]$$

$$\begin{aligned} E[X_t] &= E[\delta_1 \delta_1 + \delta_2 \delta_2 + \dots + \delta_t \delta_t + (\delta_1 \delta_2 + \delta_1 \delta_3 + \dots \\ &+ \delta_{t-1} \delta_t)] = E[\delta_1 \delta_1] + E[\delta_2 \delta_2] + \dots + E[\delta_t \delta_t] + (E[\delta_1 \delta_2] \\ &+ E[\delta_1 \delta_3] + \dots + E[\delta_{t-1} \delta_t]) = 1 + 1 + \dots + 1 + (0 + 0 + \dots + 0) = t \end{aligned}$$

So if the expected square of the distance after t steps is t . (In class we incorrectly claimed this made the expected distance is $\sqrt{t}$. Convince yourself this is false)

Random Walk on a Line

Say we start at a point k on the number line. At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).



Question 3: What probability you reach n before you reach 0 ?

Random Walk on a Line

Question 3: What is the probability that you reach n before you reach 0 ?

$E[X_t] = k$ (If we've stopped at 0 or n , δ_i still is 0)

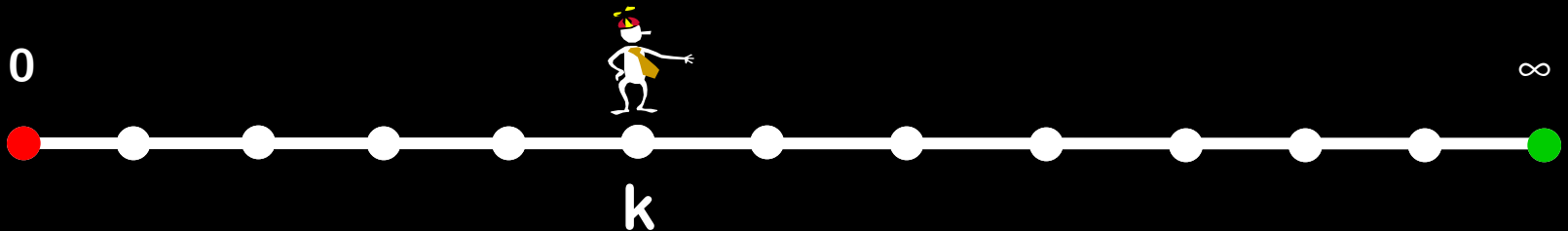
$$\begin{aligned} E[X_t] &= E[X_t | X_t = 0] \times \Pr(X_t = 0) \\ &\quad + E[X_t | X_t = n] \times \Pr(X_t = n) \\ &\quad + E[X_t | \text{neither}] \times \Pr(\text{neither}) \end{aligned}$$

$$\begin{aligned} k &= n \times \Pr(X_t = n) \\ &\quad + E[X_t | \text{neither}] \times \Pr(\text{neither}) \end{aligned}$$

As $t \rightarrow \infty$, $\Pr(\text{neither}) \rightarrow 0$, also $E[X_t | \text{neither}] < n$
Hence $\Pr(X_t = n) \rightarrow k/n$

Random Walk on a Line

Say we start at a point k on the number line. At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).



Question 4: Will you always return to zero?

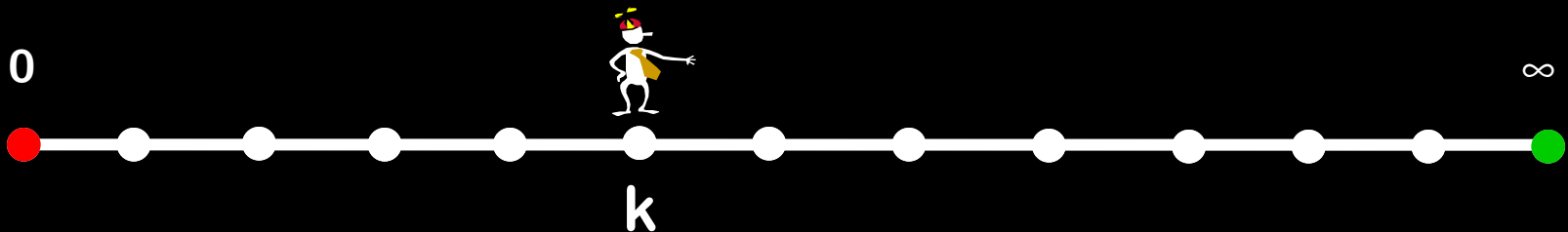
Random Walk on a Line

Question 4: Will you always return to zero?

YES! Can think of the problem as question 3.
As $n \rightarrow \infty$, $k/n \rightarrow 0$. So the probability you reach zero before you reach $n \rightarrow 1$

Random Walk on a Line

Say we start at a point k on the number line. At each step we randomly move one step left or right (to $k - 1$ or $k + 1$).



Question 5: If you start at zero, what is the probability you are again at zero after exactly t steps?

Random Walk on a Line

Question 5: If you start at zero, what is the probability you are again at zero after exactly t steps?

If t is odd, we will be at an odd location so we cannot be at zero after t steps.

If $t = 2n$ is even, we need half the steps to be to the right and half the steps to the left. There are $\binom{2n}{n}$ ways to pick which steps go right, and the probability of getting any specified sequence of steps is 2^{-2n} so the total probability is $\binom{2n}{n} 2^{-2n}$

Random Walk on a Plane

Say we start at a point (a, b) on the plane. At each step we randomly move in one direction. (to $(a + 1, b)$, $(a - 1, b)$, $(a, b + 1)$, or $(a, b - 1)$)

Question 1: What is your expected location at time t ?

Same as a the line! The expected change in both x and y direction is zero!

Random Walk on a Plane

Say we start at a point (a, b) on the plane. At each step we randomly move in one direction. (to $(a + 1, b)$, $(a - 1, b)$, $(a, b + 1)$, or $(a, b - 1)$)

Question 2: What is your expected square of the distance from (a, b) at time t ?

Let X_t be a random variable for the square of the distance from (a, b) at time t

Random Walk on a Plane

Question 2: What is your expected distance from k at time t ?

$$E[X_t] = E[(a - (a + \delta_1 + \delta_2 + \dots + \delta_t))^2 + (b - (b + \zeta_1 + \zeta_2 + \dots + \zeta_t))^2]$$

(δ_i is RV for change in x distance from a at time i)

(ζ_i is RV for change in y distance from b at time i)

$$E[\delta_i \delta_i] = E[\zeta_i \zeta_i] = 1 \text{ or } 0$$

(δ_i is either -1, 1 or 0. So $\delta_i \delta_i$ is always 1 or 0)

$$E[\delta_i \delta_j] = E[\zeta_i \zeta_j] = 0$$

(1/8 of the time they share a sign and is 1, 1/8 of the time they don't and is -1 and otherwise one of them is already zero)

Random Walk on a Plane

Question 2: What is your expected distance from k at time t ?

$$E[X_t] = E[(a - (a + \delta_1 + \delta_2 + \dots + \delta_t))^2 + (b - (b + \zeta_1 + \zeta_2 + \dots + \zeta_t))^2]$$

$$E[X_t] = E[\delta_1\delta_1] + E[\zeta_1\zeta_1] + \dots + E[\delta_t\delta_t] + E[\zeta_t\zeta_t]$$

(All other terms will be zero)

If $\delta_i = 0$ then the move was in the y direction and $\zeta_i\zeta_i = 1$. If $\zeta_i = 0$ then $\delta_i\delta_i = 1$ for the same reason.

So $E[\delta_i\delta_i] + E[\zeta_i\zeta_i] = 1$ and thus $E[X_t] = t$

Random Walk on a Plane

Say we start at a point $(0, 0)$ on the plane. At each step we randomly move in one direction.

Question 3: Will you always return to $(0, 0)$?

Amazing, yes! George Pólya showed you'll always return. Even more interesting, he showed the same property does not hold for random walks on the cube or higher dimensions

Random Walk on a n-Cube

For a random walk in d dimensions, the probability p of returning home is:

$$d = 1 = 2, p = 1$$

$$d = 3, p = 0.340537$$

$$d = 4, p = 0.193206$$

$$d = 5, p = 0.135178$$

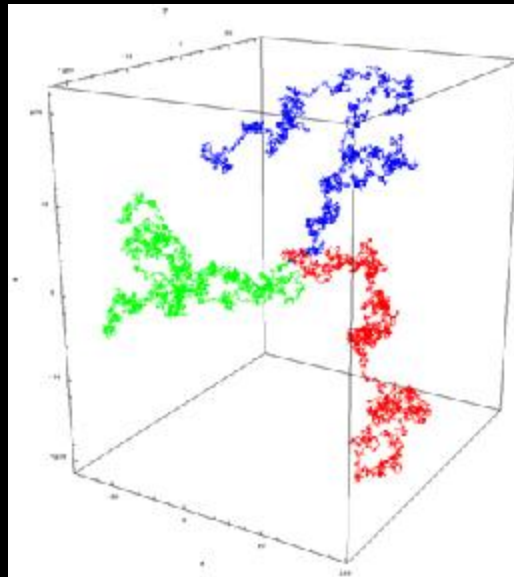
$$d = 6, p = 0.104715$$

$$d = 7, p = 0.0858449$$

$$d = 8, p = 0.0729126$$

Random Walks

“A drunk man will find his way home, but a drunk bird may get lost forever.”
(don't drink and fly)



Probabilistic Method



(Paul Erdős, the guy who invented this stuff)

Probabilistic Method

11 People are at a party. Does there always exist some group of five people who either all know each other or none of whom know each other.

How should we approach this problem?

- Brute Force?

- Number of possible parties is $2^{\binom{11}{2}}$
- Even with a computer, that would take a while
- Could do some clever tricks, but would still take time

- Pigeonhole Principle?

- Not clear what are the pigeons and holes
- Isn't a magic wand for solving any type of problem

- Probabilistic Method?

- Sounds interesting, but what is that?

Probabilistic Method

$$E[X] = \sum_{x \in \Omega} x m(x)$$

If $E[X] < k$ there some outcome $x < k$

Otherwise $E[X] = \sum x m(x) \geq \sum m(x) k = k > E[X]$

This means if X is non-negative integer RV and $E[X] < 1$ there must be some outcome $x = 0$

Probabilistic Method

11 People are at a party. Does there always exist some group of five people who either all know each other or none of whom know each other.

Suppose we randomly decide who knows who at the party

If we take a random group of five people, the probability they all know each other is $(1/2)^{10}$

On the other hand, the probability they all don't know each other is $(1/2)^{10}$

So the probability of either of the above being true is $2(1/2)^{10} = (1/2)^9$

There are 11 choose 5 groups of five people. So the expected number of groups that satisfy the problem is...

Probabilistic Method

11 People are at a party. Does there always exist some group of five people who either all know each other or none of whom know each other.

$$E[X] = \binom{11}{5} \left(1 \left(\frac{1}{2}\right)^9 + 0 \left(1 - \left(\frac{1}{2}\right)^9\right)\right) = \binom{11}{5} \left(\frac{1}{2}\right)^9$$

$$\binom{11}{5} \left(\frac{1}{2}\right)^9 = \frac{462}{512} < 1$$

“If X is non-negative integer RV and $E[X] < 1$ there must be some outcome $x = 0$ ”

Our sample space is how the 11 people might know each other. $x = 0$ means there is a party for which no such group of five people exists

Probabilistic Method

12 People are at a party. Does there always exist some group of five people who either all know each other or none of whom know each other.

$$\binom{12}{5} \left(\frac{1}{2}\right)^9 = \frac{792}{512} > 1$$

$E[X] > 1$ so we cannot conclude it isn't always possible

Turns out the lowest number of people for which is always possible is between 42 and 49

The exact value is still unknown. (Finding it would make John Mackey very happy)

Probabilistic Method

10% of the surface of a sphere is colored green, and the rest is colored blue. Show that no matter how the colors are arranged, it is possible to inscribe a cube in the sphere so that all of its vertices are blue.



Probabilistic Method

Pick a random cube. (Note: any particular vertex is uniformly distributed over surface of sphere).

Let $X_i = 1$ if i^{th} vertex is blue, 0 otherwise

$$\text{Let } X = X_1 + X_2 + \dots + X_8$$

$$E[X_i] = P(X_i=1) = 9/10$$

$$E[X] = 8 \cdot 9/10 = 7.2 > 7$$

So, must have some cubes where $X = 8$.

Probabilistic Method

“If $E[X] < k$ there some outcome $x < k$ ”

- Good for showing existence. Doesn't work for proving something is always true
- Non-constructive. Doesn't tell you how to get a solution, just that one exists.
- Can save you from having to perform a lot of annoying case work or other tedious analysis

Probabilistic Bounds

There are n boys and n girls in a class (it clearly isn't at CMU). Each boy has a crush on exactly k of the girls and visa versa.

Question 1: What is the probability that some girl has no potential date (a boy who she has a crush on who also has a crush on her)

Pick a boy b . The probability that a boy doesn't have a crush on her is $(n - k)/n$. There are k boys, all of whom cannot have a crush on her. The boys crushes are independent of each other, so the final probability is $\left(\frac{n - k}{n}\right)^k$

Probabilistic Bounds

Question 2: What is the probability that *every* girl has a potential date?

We'd like to claim "The probability of a girl having a potential date is $1 - ((n - k)/n)^k$ so the probability of every girl having a potential date is $(1 - ((n - k)/n)^k)^n$ " Can we do this?

Suppose $k = 1$ and $n = 2$ and the first girl has a potential date (event g_1). In order for the second girl to also have a potential date (event g_2), she must be into the other boy and visa versa. So $P(g_2 | g_1) = 1/4 \neq P(g_2) = 1/2$

So g_1 and g_2 are not independent!

Probabilistic Bounds

Question 2: What is the probability that *every* girl has a potential date?

We'd like to claim "The probability of a girl having a potential date is $1 - ((n - k)/n)^k$ so the probability of every girl having a potential date is $(1 - ((n - k)/n)^k)^n$ " Can we do this?

NO! The probabilities of each girl having an potential date are not independent! We need independence to multiply them!

So what should we do?

A Simple Calculation

True or False:

If the average income of people is \$100 then more than 50% of the people can be earning more than \$200 each

False! Otherwise the average would be higher!

Markov's Inequality

If X is a non-negative random variable with expectation $E[X]$, then

$$\Pr[X \geq k] \leq E[X]/k$$



Andrei Andreevich Markov (1856–1922)

Markov's Inequality

Question 2: What is the probability that *every* girl has a potential date?

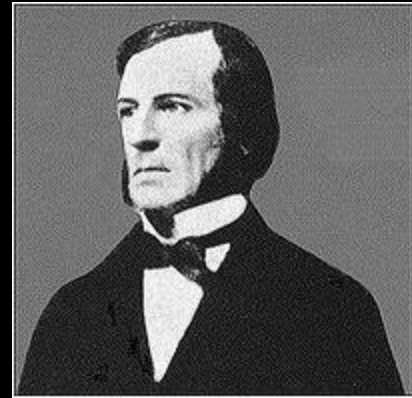
The *expected* number of girls who get a potential date, $E[X]$, is $n(1 - ((n - k)/n)^k)$

$$P(X \geq n) \leq E[X]/n = n(1 - ((n - k)/n)^k)/n$$

$$P(X = n) \leq (1 - ((n - k)/n)^k)$$

Union Bound

$$P(x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_n) \leq P(x_1) + P(x_2) + \dots + P(x_n)$$



George Boole

Union Bound

Question 2: What is the probability that *every* girl has a potential date?

Lets instead find the probability that some girl does *not* have a potential date! Call the event that girl i doesn't have a potential date g_i .

Using the union bound,

$$P(g_1 \text{ or } g_2 \text{ or } \dots \text{ or } g_n) \leq P(g_1) + P(g_2) + \dots + P(g_n)$$

$P(g_i) = ((n - k)/n)^k$ for any i and so

$$P(g_1 \text{ or } g_2 \text{ or } \dots \text{ or } g_n) \leq n((n - k)/n)^k$$

$$\text{Thus } 1 - P(g_1 \text{ or } g_2 \text{ or } \dots \text{ or } g_n) \geq 1 - n((n - k)/n)^k$$

Union Bound vs Markov

Question 2: What is the probability that *every* girl has a potential date?

UNION BOUND: $P \geq 1 - n((n - k)/n)^k$

MARKOV: $P \leq 1 - ((n - k)/n)^k$

If $k = 3$, $n = 20$ we can conclude $-11.28 \leq P \leq 0.386$

If $k = 8$, $n = 20$ we can conclude $0.664 \leq P \leq 0.983$

If $k = 10$, $n = 20$ we can conclude $0.980 \leq P \leq 0.999$

If $k = 15$, $n = 20$ we can get P within a factor of 10^{-8}

Probabilistic Bounds

Markov's Inequality

$$\Pr[X \geq k] \leq E[X]/k$$

Union Bound

$$P(x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_n) \leq P(x_1) + P(x_2) + \dots + P(x_n)$$

Use them to put bounds on problems we can't
compute exactly

Symmetric Arguments

Every passenger has an assigned seat

There are $n-1$ passengers and n seats

Before the passengers board, an air marshal sits on a random seat

When a passenger enters the plane, if their assigned seat is taken, they pick a seat at random

What is the probability that the last passenger to enter the plane sits in their assigned seat?

Symmetric Arguments

There is one ticket which no one has. Call that seat k

Call the seat the last person has j

Can any seat other than j or k be empty at the end?

No! The person who owns that seat will have filled it!

Suppose the last person had k instead of j . Probability should be the same! Any time he'd get his seat, the original case wouldn't get his seat and visa versa.

So $P(k) + P(j) = 1$ and $P(k) = P(j)$ so $P(j)$ (the probability his seat is empty) is $1/2$!



**Here's What
You Need to
Know...**

Linearity of Expectation

Probabilistic Method

Random Walk in a Line

Markov's Inequality

Union Bound