Great Theoretical Ideas In Computer Science
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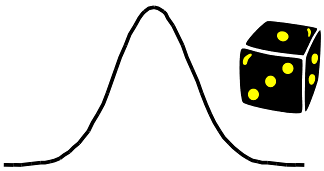
CS 15-251 Spring 2010
Lecture 11 Feb. 16, 2010 Carnegie Mellon University

Probability Theory I



We will consider chance experiments with a finite number of possible outcomes $w_1, w_2, \ldots, w_n$

The sample space Ω of the experiment is the set of all possible outcomes.

The roll of a die: $\Omega = \{1,2,3,4,5,6\}$

Each subset of a sample space is defined to be an event.

The event: $E = \{2,4,6\}$

Probability of a event

Let X be a random variable which denotes the value of the outcome of a certain experiment.

We will assign probabilities to the possible outcomes of an experiment.

We do this by assigning to each outcome w_j a nonnegative number $m(w_j)$ in such a way that

$$m(w_1) + \ldots + m(w_n)=1$$

The function $m(w_j)$ is called the distribution function of the random variable X.

Probabilities

For any subset E of Ω, we define the probability of E to be the number P(E) given by

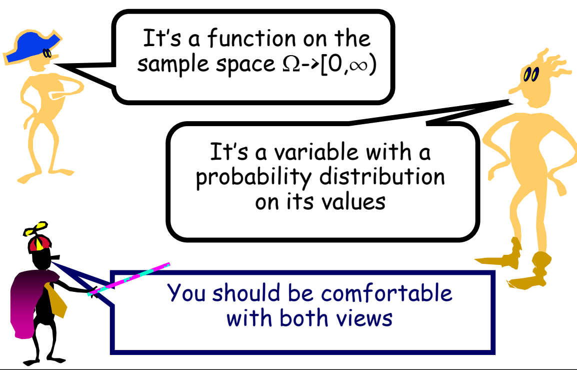
$$\underbrace{P(E)}_{\text{event}} = \sum_{w\in E} \underbrace{m(w)}_{\text{distribution function}}$$

From Random Variables to Events

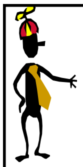
For any random variable X and value a, we can define the event E that X = a

$$P(E) = P(X=a) = P(\{t \in \Omega \mid X(t)=a\})$$

From Random Variables to Events



Example 1



Consider an experiment in which a coin is tossed twice.

Ω= {HH,TT,HT,TH}
m(HH)=m(TT)=m(HT)=m(TH)=1/4

Let E ={HH,HT,TH} be the event that at least one head comes up. Then probability of E is

P(E) = m(HH)+m(HT)+m(TH) =3/4

Notice that it is an immediate consequence
P({w}) = m(w)

Example 2

Three people, A, B, and C, are running for the same office, and we assume that one and only one of them wins. Suppose that A and B have the same chance of winning, but that C has only 1/2 the chance of A or B.

Let E be the event that either A or C wins.

P(E) = m(A)+m(C) = 2/5+1/5=3/5

Theorem

The probabilities satisfy the following properties:

$P(\Omega) = 1$

$P(E) \geq 0$

$P(A \cup B) = P(A) + P(B)$, for disjoint A and B

$P(\overline{A}) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B)$, for disjoint A and B

Proof.

$$P(A \cup B) = \sum_{w \in A \cup B} m(w) =$$

$$\sum_{w \in A} m(w) + \sum_{w \in B} m(w) = P(A) + P(B)$$

More Theorems

For any events A and B

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

For any events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Uniform Distribution

When a coin is tossed and the die is rolled, we assign an equal probability to each outcome.

The uniform distribution on a sample space containing n elements is the function m defined by

$$m(w) = 1/n$$

for every $w \in \Omega$

Example

Consider the experiment that consists of rolling a pair of dice.
What is the probability of getting a sum of 7 or a sum of 11?

We assume that each of 36 outcomes is equally likely.

Example

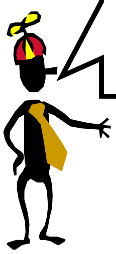
E

S = { (1,1), (1,2), (1,3), (1,4), (1,5), $\boxed{(1,6)}$
(2,1), (2,2), (2,3), (2,4), $\boxed{(2,5)}$, (2,6),
(3,1), (3,2), (3,3), $\boxed{(3,4)}$, (3,5), (3,6),
(4,1), (4,2), $\boxed{(4,3)}$, (4,4), (4,5), (4,6),
(5,1), $\boxed{(5,2)}$, (5,3), (5,4), (5,5), $\boxed{(5,6)}$
$\boxed{(6,1)}$, (6,2), (6,3), (6,4), $\boxed{(6,5)}$, (6,6) }

F

P(E)= 6 * 1/36   P(F)= 2 * 1/36

P(E $\cup$ F)= 8/36



A fair coin is tossed 100 times in a row

What is the probability that we get exactly half heads?

The sample space Ω is the set of all outcomes (sequences) $\{H,T\}^{100}$

Each sequence in Ω is equally likely, and hence has probability

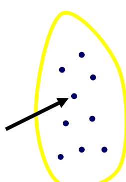
$1/|\Omega|=1/2^{100}$

Visually

Ω = all sequences of 100 tosses

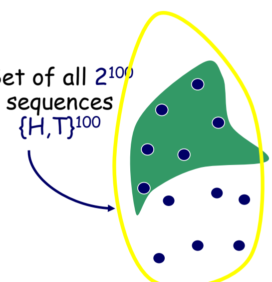
t = HHTTT……TH

P(t) = 1/|S|



Event E = Set of sequences with 50 H's and 50 T's

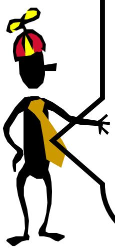
Set of all 2^{100} sequences $\{H,T\}^{100}$



Probability of event E = proportion of E in S

$\binom{100}{50} / 2^{100}$

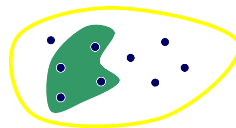
Birthday Paradox



How many people do we need to have in a room to make it a favorable bet (probability of success greater than 1/2) that two people in the room will have the same birthday?

And The Same Methods Again!

Sample space $\Omega = 365^x$



We must find sequences that have no duplication of birthdays.

Event E = { $w \in \Omega$ | two numbers not the same }

$$P(E) = \frac{365 \times 364 \times \ldots \times (365 - x + 1)}{365^x}$$

Birthday Paradox

21	0.556
22	0.524
23	**0.492**
24	0.461

Infinite Sample Spaces

A coin is tossed until the first time that a head turns up.

$\Omega = \{1, 2, 3, 4, \ldots\}$

A distribution function: $m(n) = 2^{-n}$.

$$P = \sum_w m(w) = \frac{1}{2} + \frac{1}{4} + \ldots = 1$$

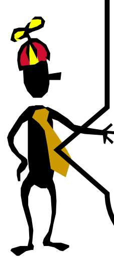
Infinite Sample Spaces

Let E be the event that the first time a head turns up is after an even number of tosses.

$E = \{2, 4, 6, \ldots\}$

$$P(E) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}$$

Conditional Probability



Consider our voting example: three candidates A, B, and C are running for office. We decided that A and B have an equal chance of winning and C is only 1/2 as likely to win as A.

Suppose that before the election is held, A drops out of the race. What are new probabilities to the events B and C

$P(B|A)=2/3$ $\quad$ $P(C|A)=1/3$

Conditional Probability

Let $\Omega = \{w_1, w_2, \ldots, w_r\}$ be the original sample space with distribution function $m(w_k)$. Suppose we learn that the event E has occurred. We want to assign a new distribution function $m(w_k \mid E)$ to reflect this fact.

It is reasonable to assume that the probabilities for w_k in E should have the same relative magnitudes that they had before we learned that E had occurred: $m(w_k \mid E) = c\, m(w_k)$, where c is some constant.

Also, if w_k is not in E, we want $m(w_k \mid E) = 0$

Conditional Probability

By the probability law:

$$\sum_{\Omega} m(w_k \mid E) = \sum_{E} m(w_k \mid E) = c \sum_{E} m(w_k) = 1$$

From here,

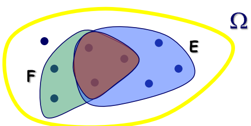
$$c = \frac{1}{\sum_{E} m(w_k)} = \frac{1}{P(E)}$$

$$m(w_k \mid E) = \frac{m(w_k)}{P(E)}$$

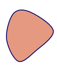
Conditional Probability

The probability of event F given event E is written P(F | E) and is defined by

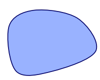
$$P(F \mid E) = \sum_{F \cap E} m(w_k \mid E) = \sum_{F \cap E} \frac{m(w_k)}{P(E)} = \frac{P(F \cap E)}{P(E)}$$

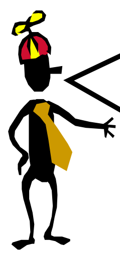


proportion of $F \cap E$



to E





Suppose we roll a white die and black die

What is the probability that the white is 1 given that the total is 7?

event A = {white die = 1}

event B = {total = 7}

S = { (1,1), (1,2), (1,3), (1,4), (1,5), $\boxed{(1,6)}$,
(2,1), (2,2), (2,3), (2,4), $\boxed{(2,5)}$, (2,6),
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$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = \frac{1}{6}$$

event A = {white die = 1} event B = {total = 7}

Independence!

A and B are independent events if

$P(A \mid B) = P(A)$

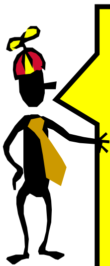
$\Leftrightarrow$

$P(A \cap B) = P(A)\,P(B)$

$\Leftrightarrow$

$P(B \mid A) = P(B)$

Silver and Gold



One bag has two silver coins, another has two gold coins, and the third has one of each

One bag is selected at random. One coin from it is selected at random. It turns out to be gold

What is the probability that the other coin is gold?

Let G_1 be the event that the first coin is gold

$P(G_1) = 1/2$

Let G_2 be the event that the second coin is gold

$P(G_2 \mid G_1) = P(G_1 \cap G_2) / P(G_1)$

$= (1/3) / (1/2)$

$= 2/3$

Note: G_1 and G_2 are not independent

Monty Hall Problem

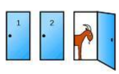
The host show hides a car behind one of 3 doors at random. Behind the other two doors are goats.

You select a door.

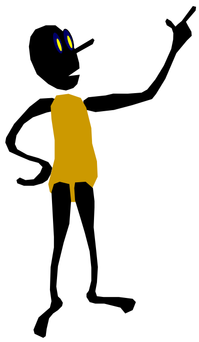
Announcer opens one of others with <u>no</u> prize.

You can decide to keep or switch.

What to do? To switch or not to switch?



This is Tricky?



We are inclined to think:

"After one door is opened, others are equally likely…"

Monty Hall Problem

Sample space = { car behind door 1, car behind door 2, car behind door 3 }

Each has probability 1/3

Staying	Switching
we win if we choose the correct door	we win if we choose the incorrect door
P(choosing correct door) = 1/3	P(choosing incorrect door) = 2/3

Monty Hall Problem

Let the doors be called X, Y and Z.

Let Cx, Cy, Cz be the events that the car is behind door X and so on.

Let Hx, Hy, Hz be the events that the host opens door X and so on.

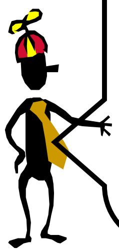
Supposing that you choose door X, the possibility that you win a car if you switch is

$P(Hz \cap Cy) + P(Hy \cap Cz) =$

$P(Hz|Cy)\,P(Cy) + P(Hy|Cz)\,P(Cz) =$

$1 \times 1/3 + 1 \times 1/3 = 2/3$

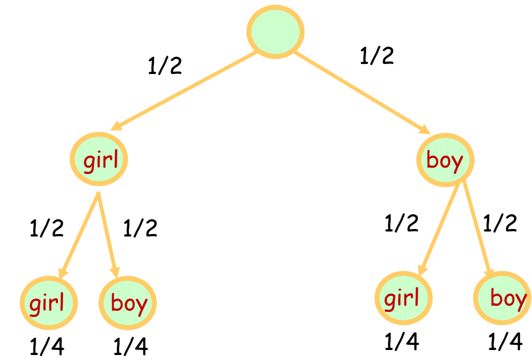
Another Paradox



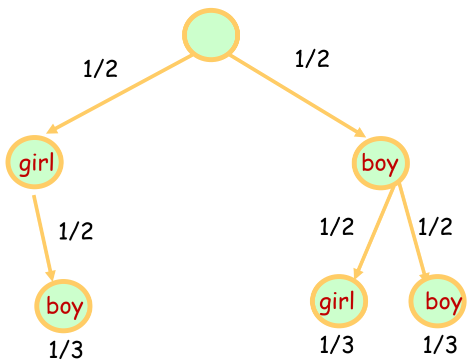
Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

1/3

Tree Diagram



Tree Diagram



Bayes' formula

Suppose we have a set of disjoint events (hypotheses)

$$\Omega = H_1 \cup H_2 \cup \ldots \cup H_k$$

We also have an event E called evidence.

We want to find the probabilities for the hypotheses given the evidence.

Posterior probability

Prior probability

$$P(H_j \mid E) = \frac{P(H_j)P(E \mid H_j)}{\sum_k P(H_k)P(E \mid H_k)}$$

Expected Value

A Quick Calculation…

I flip a coin 3 times. What is the expected (average) number of heads?

Let X denote the number of heads which appear: X = 0, 1, 2 and 3.

The corresponding probabilities are 1/8, 3/8, 3/8, and 1/8.

$E(X) = (1/8)\times 0 + (3/8)\times 1 + (3/8)\times 2 + (1/8)\times 3 = 1.5$

This is a frequency interpretation of probability

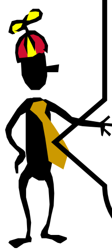
Definition: Expectation

The expectation, or expected value of a random variable X is written as E(X) or E[X], and is

$$E[X] = \sum_{x \in \Omega} x\, m(x)$$

with a sample space Ω and a distribution function m(x).

Exercise



Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses which were made. What is E[X]?

$$E[X] = \sum_{k=1}^{\infty} k \frac{1}{2^k} = 2$$

Here's What You Need to Know…

Language of Probability

Events

P(A | B)

Independence

Random Variables

Expectation