Great Theoretical Ideas In Computer Science

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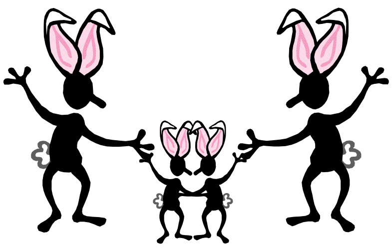
CS 15-251 Spring 2010

Lecture 8 Feb 04, 2010 Carnegie Mellon University

Recurrences and Continued Fractions



Solve in integers

$$x_1 + x_2 + \ldots + x_5 = 40$$
$$x_k \geq k;$$

$$y_k = x_k - k$$
$$y_1 + y_2 + \ldots + y_5 = 25$$
$$y_k \geq 0;$$

$$\binom{29}{4}$$

Solve in integers

$$x + y + z = 11$$
$$x \geq 0;\ y \leq 3;\ z \geq 0$$

$$[X^{11}]\ \frac{1+x+x^2+x^3}{(1-x)^2}$$

Partitions

Find the number of ways to partition the integer n

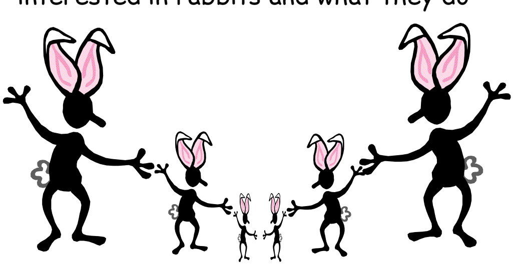
$$3 = 1+1+1 = 1+2$$
$$4 = 1+1+1+1 = 1+1+2 = 1+3 = 2+2$$

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n = n$$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}$$

Leonardo Fibonacci

In 1202, Fibonacci has become interested in rabbits and what they do



The rabbit reproduction model

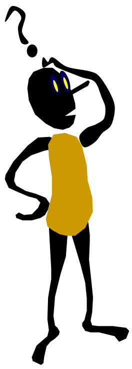
- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

F_n= # of rabbit pairs at the beginning of the n^{th} month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

F_0=0, F_1=1,

$F_n=F_{n-1}+F_{n-2}$ for $n \geq 2$



What is a closed form formula for F_n?

WARNING!!!!

This lecture has explicit mathematical content that can be shocking to some students.

Characteristic Equation

$F_n=F_{n-1}+F_{n-2}$

Consider solutions of the form:

$F_n = \lambda^n$

for some (unknown) constant $\lambda \neq 0$

λ must satisfy:

$\lambda^n - \lambda^{n-1} - \lambda^{n-2} = 0$

$\lambda^n - \lambda^{n-1} - \lambda^{n-2} = 0$

iff $\lambda^{n-2}(\lambda^2 - \lambda - 1) = 0$

iff $\boxed{\lambda^2 - \lambda - 1 = 0}$

Characteristic equation

$$\phi = \frac{1+\sqrt{5}}{2}$$

$\lambda = \phi$, or $\lambda = -1/\phi$

ϕ ("phi") is the golden ratio

$\lambda = \phi$, or $\lambda = -(1/\phi)$

So for all these values of λ the inductive condition is satisfied:

$\lambda^2 - \lambda - \mathbf{1} = 0$

Do any of them happen to satisfy the base condition as well?

F_0=0, F_1=1

$\forall a,b \quad a\,\phi^n + b\,(-1/\phi)^n$ satisfies the inductive condition

Adjust a and b to fit the base conditions.

n=0: a + b = 0

n=1: $a\,\phi^1 + b\,(-1/\,\phi)^1 = 1$

$a= 1/\sqrt{5}$ $\qquad$ $b= -1/\sqrt{5}$

Leonhard Euler (1765)

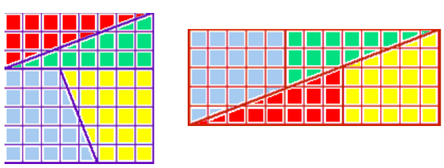
$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}$$

Fibonacci Power Series

$$\sum_{k=0}^{\infty} F_k x^k = ??$$

$$\frac{x}{1-x-x^2}$$

Fibonacci Bamboozlement



Cassini's Identity

$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$

We dissect $F_n x F_n$ square and rearrange pieces into $F_{n+1} x F_{n-1}$ square

From the previous lecture

$$d_n = d_{n-1} + 2d_{n-2}$$
$$d_0 = 0, \quad d_1 = 1$$

Characteristic Equation

$$d_n = d_{n-1} + 2d_{n-2}$$
$$d_0 = 0, \quad d_1 = 1$$

$$\lambda^2 - \lambda - 2 = 0 \qquad \lambda_1 = -1, \lambda_2 = 2$$

$$d_n = a\,(-1)^n + b\,2^n$$

$$d_n = a(-1)^n + b\,2^n$$

$$d_0 = 0 = a(-1)^0 + b\,2^0 = a + b$$

$$d_1 = 1 = a(-1)^1 + b\,2^1 = -a + 2b$$

$$b = \frac{1}{3}, a = -\frac{1}{3}$$

Characteristic Equation

$$x_n = 2x_{n-1} - x_{n-2}$$
$$x_0 = 1, \quad x_1 = 2$$

Characteristic Equation

$$x_n = 2x_{n-1} - x_{n-2}$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_1 = \lambda_2 = 1$$

Characteristic Equation

Theorem. Let λ be a root of multiplicity p of the characteristic equation. Then

$$\lambda^n, n\,\lambda^n, n^2\lambda^n, \ldots, n^{p-1}\lambda^n$$

are all solutions.

$$x_n = 2x_{n-1} - x_{n-2}$$

The theorem says that $x_n=n$ is a solution.

This can be easily verified: n = 2 n - n

From the previous lecture: Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$
$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Characteristic equation:

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$$

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$
$$a_0 = 0,\quad a_1 = 1,\quad a_2 = 4$$

General solution: $a_n = a + b\,2^n + c\,n\,2^n$

$$a_0 = 0 = a + b \qquad a = b = 0,$$
$$a_1 = 1 = a + 2b + 2c \qquad c = \frac{1}{2}$$
$$a_2 = 4 = a + 4b + 8c$$
$$\boxed{a_n = n\,2^{n-1}}$$

The Golden Ratio

"Some other majors have their mystery numbers, like π and e.
In Computer Science the mystery number is ϕ, the Golden Ratio." - S.Rudich

Golden Ratio -Divine Proportion

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

$$\phi^2 = \frac{AC}{AB} \times \frac{AB}{BC} = \frac{AC}{BC}$$
$$\boxed{\phi^2 - \phi - 1 = 0}$$
$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{AC - AB}{BC} = 1$$

Golden Ratio - the divine proportion

$$\phi^2 - \phi - 1 = 0 \qquad \phi = \frac{1+\sqrt{5}}{2}$$

ϕ = 1.6180339887498948482045...

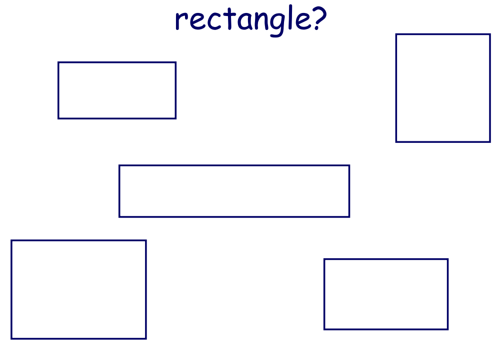
"Phi" is named after the Greek sculptor Phidias

Aesthetics

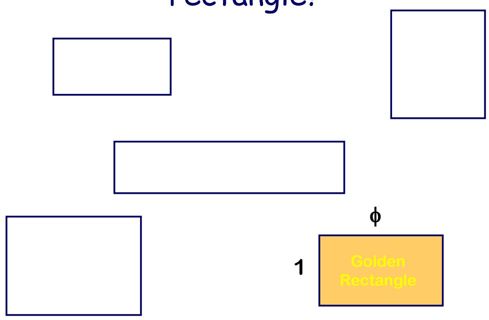
ϕ plays a central role in renaissance art and architecture.

After measuring the dimensions of pictures, cards, books, snuff boxes, writing paper, windows, and such, psychologist Gustav Fechner claimed that the preferred rectangle had sides in the golden ratio (1871).

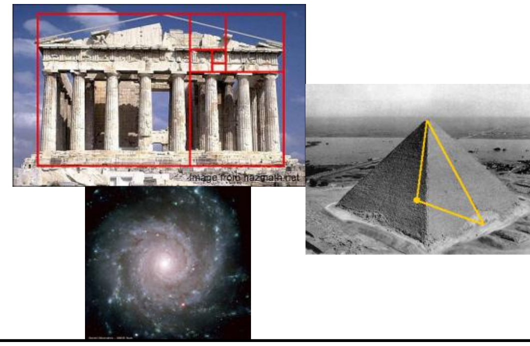
Which is the most attractive rectangle?



Which is the most attractive rectangle?



The Golden Ratio



Divina Proportione
Luca Pacioli (1509)

Pacioli devoted an entire book to the marvelous properties of ϕ. The book was illustrated by a friend of his named:

Leonardo Da Vinci

Table of contents

- The first considerable effect
- The second essential effect
- The third singular effect
- The fourth ineffable effect
- The fifth admirable effect
- The sixth inexpressible effect
- The seventh inestimable effect
- *The ninth most excellent effect*
- The twelfth incomparable effect
- The thirteenth most distinguished effect

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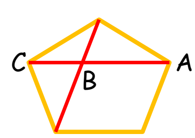
"For the sake of salvation, the list must end here" – Luca Pacioli

Divina Proportione
Luca Pacioli (1509)

"Ninth Most Excellent Effect"

two diagonals of a regular pentagon divide each other in the Divine Proportion.



$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

Expanding Recursively

$$\phi^2 - \phi - 1 = 0$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\phi}}$$

Expanding Recursively

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\phi}}}}$$

A (Simple) Continued Fraction Is Any Expression Of The Form:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + ...}}}}}}}}}$$

where a, b, c, … are whole numbers.

A Continued Fraction can have a finite or infinite number of terms.

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + ...}}}}}}}}}$$

We also denote this fraction by [a,b,c,d,e,f,…]

Continued Fraction Representation

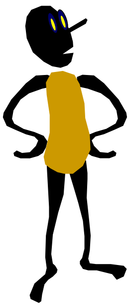
$$\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

= [1,1,1,1,1,0,0,0,…]

Recursively Defined Form For CF

$$\text{CF} = \text{whole number}$$

$$= \text{whole number} + \frac{1}{\text{CF}}$$



Proposition: Any finite continued fraction evaluates to a rational.

Converse: Any rational has a finite continued fraction representation.

Euclid's GCD = Continued Fraction

Euclid(A,B) = Euclid(B, A mod B)
Stop when B=0

$$a = b\,q_1 + r_1$$
$$b = r_1\,q_2 + r_2$$
$$r_1 = r_2\,q_3 + r_3$$
$$....$$
$$r_{k-1} = r_k\,q_{k+1} + 0$$

Euclid's GCD = Continued Fraction

$$a = b\,q_1 + r_1 \quad \longrightarrow \quad \frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{\frac{b}{r_1}}$$
$$b = r_1\,q_2 + r_2 \quad \longrightarrow \quad \frac{b}{r_1} = q_2 + \frac{r_2}{r_1} = q_2 + \frac{1}{\frac{r_1}{r_2}}$$
$$r_1 = r_2\,q_3 + r_3 \quad \longrightarrow \quad \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2} = q_3 + \frac{1}{\frac{r_2}{r_3}}$$
$$....$$
$$r_{k-1} = r_k\,q_{k+1} + 0$$

Euclid's GCD = Continued Fraction

$$a = b\,q_1 + r_1$$
$$b = r_1\,q_2 + r_2$$
$$r_1 = r_2\,q_3 + r_3$$
$$....$$
$$r_{k-1} = r_k\,q_{k+1} + 0$$

$$\frac{a}{b} = q_1 + \frac{1}{\frac{b}{r_1}}$$
$$= q_1 + \frac{1}{q_2 + \frac{1}{\frac{r_1}{r_2}}}$$
$$= q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{\frac{r_2}{r_3}}}}$$

A Pattern for ϕ

Let r_1 = [1,0,0,0,…] = 1
r_2 = [1,1,0,0,0,…] = 2/1
r_3 = [1,1,1,0,0,0…] = 3/2
r_4 = [1,1,1,1,0,0,0…] = 5/3
and so on.

$$\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}$$

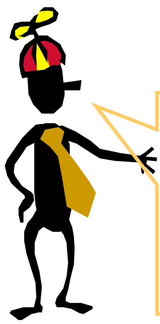
Theorem:

$\phi = F_n/F_{n-1}$ when $n \to \infty$

Divine Proportion

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{\phi^n - \left(-\frac{1}{\phi}\right)^n}{\phi^{n-1} - \left(-\frac{1}{\phi}\right)^{n-1}} = \lim_{n\to\infty} \frac{\phi^n}{\phi^{n-1}} = \phi$$

Heads-on



How to convert kilometers into miles?

Magic conversion

50 km = 34 + 13 + 3

$50 = F_9 + F_7 + F_4$

$F_8 + F_6 + F_3 = 31$ miles

Quadratic Equations

$X^2 - 3x - 1 = 0$

$X^2 = 3X + 1$

$X = 3 + 1/X$

$$X = \frac{3+\sqrt{13}}{2}$$

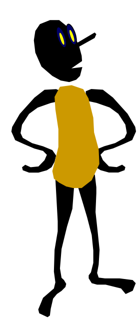
$X = 3 + 1/X = 3 + 1/[3 + 1/X] = \ldots$

A Periodic CF

$$\frac{3+\sqrt{13}}{2} = 3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\ldots}}}}}}}}}$$

A period-2 CF

$$\frac{1+\sqrt{2}}{3} = \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+\ldots}}}}}}$$



Proposition: Any quadratic solution has a periodic continued fraction.

Converse: Any periodic continued fraction is the solution of a quadratic equation

What about those non-periodic continued fractions?

What is the continued fraction expansion for π?

Euclid's GCD = Continued Fraction

$$a = b\, q_1 + r_1$$
$$b = r_1\, q_2 + r_2$$
$$r_1 = r_2\, q_3 + r_3$$
$$....$$
$$r_{k-1} = r_k\, q_{k+1} + \boxed{0}$$

??

$$\pi = 3 \times 1 + 0.1415$$
$$1 = 7 \times 0.1415 + 0.00885$$
$$0.1415 = 15 \times 0.00885 + 0.00882$$
$$0.00885 = 1 \times 0.00882 + 0.000078$$

What is the pattern?

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + ...}}}}}}}}}$$

What is the pattern?

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + ...}}}}}}}}}$$

Every irrational number greater than 1 is the limit of a unique infinite continued fraction.

Every irrational number greater than 1 is the limit of a unique infinite continued fraction.

Suppose that $y>1$ is irrational and $y = [b_0;b_1,b_2, . . .]$.

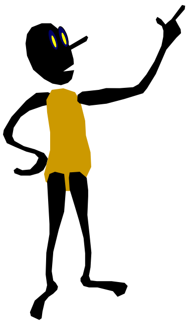
$$y = b_0 + \frac{1}{b_1 + ...} = b_0 + \frac{1}{y_1}$$

where $y_1 > 1$.

Thus,

$$b_0 = y - \frac{1}{y_1} = \lfloor y \rfloor$$

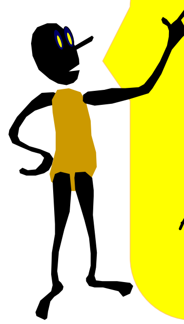
Similarly, for all b_k.



To calculate the continued fraction for a real number y, set $y_0 = y$ and then

$$b_k = \lfloor y_k \rfloor$$

$$y_{k+1} = \frac{1}{y_k - b_k}$$

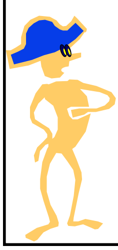


What a cool representation!

Finite CF: Rationals

Periodic CF: Quadratic roots

And some numbers reveal hidden regularity.



Let us embark now on approximations

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

CF for Approximations

$$\pi = 3$$

$$\pi = 3 + \frac{1}{7} = 3.1428$$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = 3.14151$$

We say that an positive irrational number y is approximable by rationals to order n if there exist a positive constant c and infinitely many rationals p/q with q > 0 such that

$$\left| y - \frac{p}{q} \right| \le \frac{c}{q^n}$$

We will see that algebraic numbers are not approximable to arbitrarily high order.

Positive rational numbers are approximable to order 1

Let y = a/b be a rational number, with gcd(a,b) = 1. By Euclid's algorithm we can find p_0, q_0 such that $p_0 b - q_0 a = 1$. Then

$$\left|\frac{a}{b} - \frac{p_0}{q_0}\right| \leq \frac{1}{q_0}$$

Positive rational numbers are approximable to order 1

$$p_0 b - q_0 a = 1$$

$$p_0 - q_0 \frac{a}{b} = \frac{1}{b}$$

$$\frac{p_0}{q_0} - \frac{a}{b} = \frac{1}{bq_0}$$

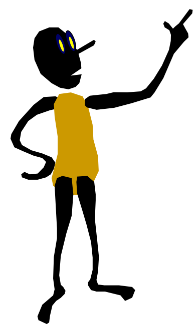
$$\left|\frac{a}{b} - \frac{p_0}{q_0}\right| = \frac{1}{bq_0} \leq \frac{1}{q_0}$$

Positive rational numbers are approximable to order 1

$$\left|\frac{a}{b} - \frac{p_0}{q_0}\right| = \frac{1}{bq_0} \leq \frac{1}{q_0}$$

Similarly,

$$\left|\frac{a}{b} - \frac{p_m}{q_m}\right| \leq \frac{1}{q_m}$$

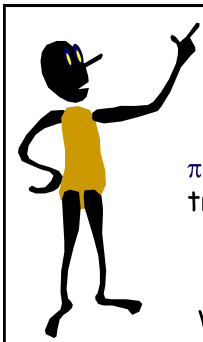


Liouville's number

$$y = \sum_{k=1}^{\infty} 10^{-k!}$$

Transcendental number

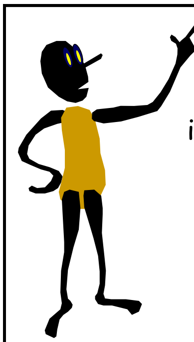
$$y = 0.110001000000000000000001\ldots$$



Liouville's number

π and e are transcendental number

What about π^e ?



Apery's constant

irrational number

Riemann Zeta function

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

transcendental number - ???

Study Bee

- Review GCD algorithm
- Recurrences, Phi and CF
- Solving Recurrences
- Liouville's number