Great Theoretical Ideas In Computer Science

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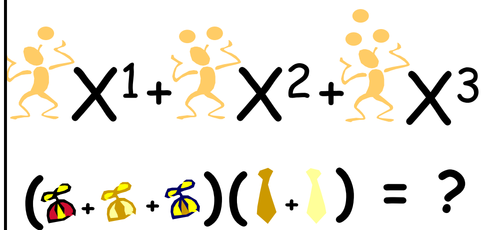
CS 15-251 Spring 2010

Lecture 6 Jan. 28, 2010 Carnegie Mellon University

Generating Functions I



Warm-Up Question

Could Your iP~~o~~ad Be Holding the Greatest Mystery in Modern Science?

B. Chazelle

http://www.cs.princeton.edu/~chazelle/pubs/algorithm.html

" Computing (*The Algorithm*) will be the most disruptive scientific paradigm since quantum mechanics."

If Google is a religion,

what is its God?

It would have to be **The Algorithm**.

Jan. 12, 2006, The Economist

Plan

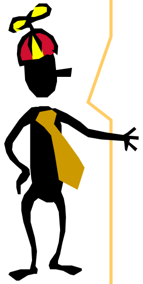
- Counting with generating functions
- Solving the Diophantine equations
- Proving combinatorial identities

Applied Combinatorics, by Alan Tucker

Generatingfunctionology, by Herbert Wilf

Counting with Generating Functions

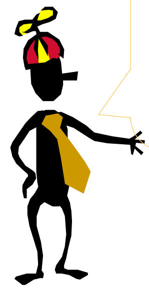
We start with George Polya's approach



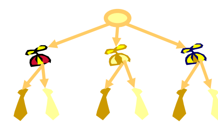
Problem

Ula is allowed to choose two items from a tray containing an apple, an orange, a pear, a banana, and a plum. In how many ways can she choose?

$\binom{5}{2}$



There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!



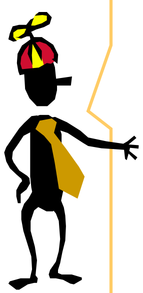
Counting with Generating Functions

(0 apple + 1 apple)(0 orange + 1 orange)
(0 pear + 1 pear)(0 banana + 1 banana)
(0 plum + 1 plum)

In this notation, apple^2 stand for choosing 2 apples., and + stands for an exclusive or.

$$(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x)$$

Take the coefficient of x^2.



Problem

Ula is allowed to choose two items from a tray containing TWO apples, an orange, a pear, a banana, and a plum. In how many ways can she choose?

The two apples are identical.

Counting with Generating Functions

(0 apple + 1 apple + + 2 apple)
(0 orange + 1 orange)(0 pear + 1 pear)
(0 banana + 1 banana)(0 plum + 1 plum)

$$(1+x+x^2)(1+x)(1+x)(1+x)(1+x)$$

Take the coefficient of x^2.

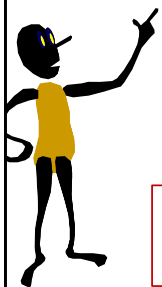
11

The function f(x) that has a polynomial expansion

$$f(x)=a_0 + a_1\, x + \ldots + a_n\, x^n$$

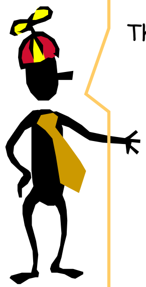
is the *generating function* for the sequence

$$a_0, a_1, \ldots, a_n$$



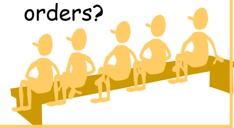
If the polynomial $(1+x+x^2)^4$ is the generating polynomial for a_k, what is the combinatorial meaning of a_k?

The number of ways to select k object from 4 types with at most 2 of each type.



The 251-staff Danish party

The 251-staff wants to order a Danish pastry from *La Prima*. Unfortunately, the shop only has 2 apple, 3 cheese, and 4 raspberry pastries left. What the number of possible orders?



Counting with Generating Functions

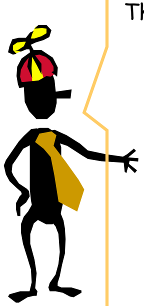
$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x+x^2+x^3+x^4)$

apple *cheese* *raspberry*

$= 1+3x+6x^2+9x^3+11x^4+11x^5+9x^6+6x^7+3x^8+x^9$

The coefficient by x^8 shows that there are only 3 orders for 8 pastries.

Note, we solve the whole parameterized family of problems!!!



The 251-staff wants to order a Danish pastry from *La Prima*. The shop only has 2 apple, 3 cheese and 4 raspberry pastries left. What the number of possible orders? Raspberry pastries come in multiples of two.

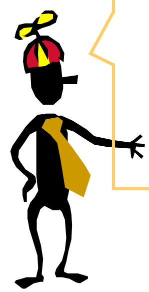
Counting with Generating Functions

$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x^2+x^4)$

apple *cheese* *raspberry*

$= 1+2x+4x^2+5x^3+6x^4+6x^5+5x^6+4x^7+2x^8+x^9$

The coefficient by x^8 shows that there are only 2 possible orders.



Problem

Find the number of ways to select R balls from a pile of 2 red, 2 green and 2 blue balls

Coefficient by x^R in $(1+x+x^2)^3$

Coefficient by x^R in $(1+x+x^2)^3$

$(1+x+x^2)^3 =$

$(x^0+x^1+x^2)(x^0+x^1+x^2)(x^0+x^1+x^2)$

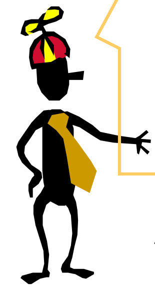
$$x^{e_1}x^{e_2}x^{e_3}$$

Find the number of ways to select R balls from a pile of 2 red, 2 green and 2 blue balls

$$x^{e_1}x^{e_2}x^{e_3}$$

$$e_1 + e_2 + e_3 = R$$

$$0 \leq e_k \leq 2$$

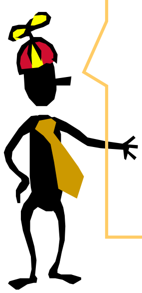


Exercise

Find the number of integer solutions to

$x_1+x_2+x_3+x_4=21$
$0 \leq x_k \leq 7$

Take the coefficient by x^{21} in $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^4$



Exercise

Find the number of integer solutions to

$x_1+x_2+x_3+x_4=21$

$0<x_k<7$

$\mathbf{x_1+x_2+x_3+x_4 = 21}$
$\mathbf{0 < x_k < 7}$

Observe

$0 < x_k < 7$
$-1 < x_k-1 < 6$
$0 \leq x_k-1 \leq 5$

Than

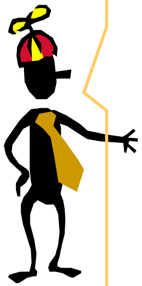
$(x_1-1)+(x_2-1)+(x_3-1)+(x_4-1) = 21-4$

$\mathbf{x_1+x_2+x_3+x_4=21}$
$\mathbf{0<x_k<7}$

The problem is reduced to solving

$y_1 + y_2 + y_3 + y_4 = 17, \qquad 0 \leq y_k \leq 5$

Solution: take the coefficient by x^{17} in
$(1+x+x^2+x^3+x^4+x^5)^4$
which is 20



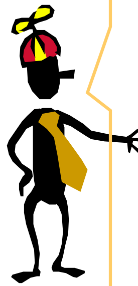
Problem

$x_1+x_2+x_3=9$

$1 \le x_1 \le 2$

$2 \le x_2 \le 4$

$1 \le x_3 \le 4$



Solution

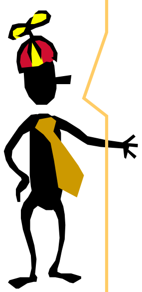
$x_1+x_2+x_3=9$	
$1 \le x_1 \le 2$	$(x+x^2)$
$2 \le x_2 \le 4$	$(x^2+x^3+x^4)$
$1 \le x_3 \le 4$	$(x+x^2+x^3+x^4)$

Take the coefficient by x^9 in the product of generating functions.

Two observations

1. A generating function approach is designed to model a selection of all possible numbers of objects.

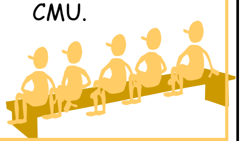
2. It can be used not only for counting but for solving the linear Diophantine equations

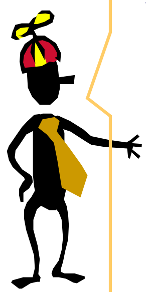


The 251-staff Danish party

Adam pulls a few strings…

and a large apple Danish factory is built next to the CMU.





The 251-staff Danish party.

Unfortunately, still only 3 cheese and 4 raspberry pastries are available. Raspberry pastries come in multiples of two. What the number of possible orders?

Counting with Generating Functions

$(1+x+x^2+x^3)\ (1+x^2+x^4)\ (1+x+x^2+x^3+\ \ldots)$

cheese *raspberry* *apple*

There is a problem (…) with the above expression.

Counting with Generating Functions

$(1+x+x^2+x^3)\ (1+x^2+x^4)\ (1+x+x^2+x^3+\ \dots)=$

cheese *raspberry* *apple*

$(1+x+2x^2+2x^3+2x^4+2x^5+x^6+x^7)\ (1+x+x^2+x^3+\ \dots)$

$=1+2x+4x^2+6x^3+8x^4+\dots$

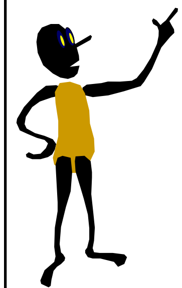
The power series

$a_0 + a_1\,x + a_2\,x^2 + \dots$

is the generating function for the *infinite* sequence

$a_0, a_1, a_2, \dots,$

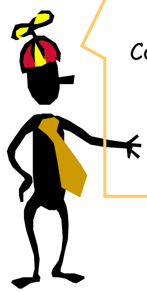
$$\sum_{k=0}^{\infty} a_k x^k$$



A famous generating function from calculus:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We can use the generating function technique to solve almost all the mathematical problems you have met in your life so far.



Exercise

Count the number of N-letter combinations of MATH in which M and A appear with repetitions and T and H only once.

Coefficient by x^N in

$(1+x+x^2+\dots)^2(1+x)^2$

What is the coefficient of X^k in the expansion of:

$(1 + X + X^2 + X^3 + X^4 + \dots)^n$?

A solution to:

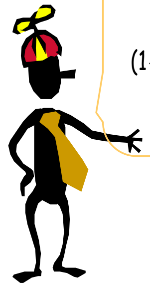
$e_1 + e_2 + \dots + e_n = k$

$e_k \geq 0$

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^n$$

$$\binom{n+k-1}{k}$$



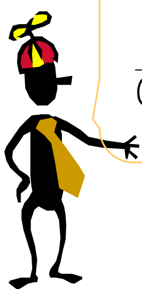
$$(1+x+x^2+\ldots)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

But what is the LHS?

$$1+x+x^2+\ldots = \frac{1}{1-x}$$

$$y = 1 + x + x^2 + \ldots$$
$$x\,y = \quad x + x^2 + \ldots$$

$$y - x\,y = 1$$
$$y(1-x) = 1$$
$$\frac{1}{1-x} = y = 1 + x + x^2 + \ldots$$



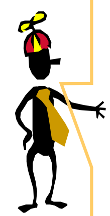
$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$1+x+x^2+\ldots+x^{n-1} = \frac{x^n-1}{x-1}$$

(when $x \neq 1$)

$$1+x+x^2+\ldots+x^{n-1}+x^n+\ldots = \frac{1}{1-x}$$

(when $|x| < 1$)

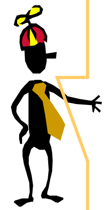


We will be dealing with formal power series (also called generating functions)

$$\sum_{k=0}^{\infty} a_k x^k$$

The coefficients can be any complex numbers, though we will be mainly working over the integers.

$$\sum_{k=0}^{\infty} a_k x^k$$



Power series have no analytic interpretation.

$$\sum_{k=0}^{\infty} a_k x^k$$

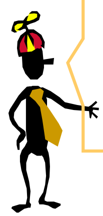
FORMAL MANIPULATION:

We deal with power series as we deal we numbers!

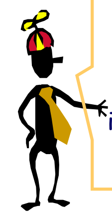
FORMAL MANIPULATION:

1/P(X) is defined to be the polynomial Q(X) such that Q(X) P(X) = 1.

Th: P(X) will have a reciprocal if and only if $a_0 \neq 0$. In this case, the reciprocal is unique.



Power series make a commutative ring!



FORMAL MANIPULATION:

The derivative of $\sum_{i=0}^{i=\infty} a_i X^i$

is defined by: $\sum_{i=1}^{i=\infty} i a_i X^{i-1}$

Power Series = Generating Function

Given a sequence of integers $a_0, a_1, \ldots, a_n$

We will associate with it a function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

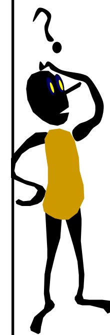
The On-Line Encyclopedia of Integer Sequences

Generating Functions

Sequence: 1, 1, 1, 1, …

Generating function:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$



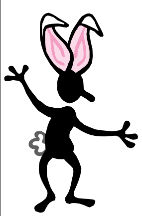
Find a generating function f(x)

1, 2, 3, 4, 5, …

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$f(x) = \frac{1}{(1-x)^2}$$

Generating Functions



$$f(x) = \sum_{k=0}^{\infty} F_k x^k$$

$F_0=0$, $F_1=1$,

$F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$

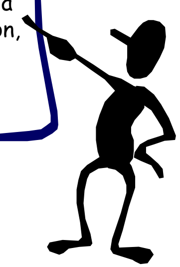
$$f(x) = \sum_{k=0}^{\infty} F_k x^k$$

$(x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+\ldots)(1-x-x^2)$

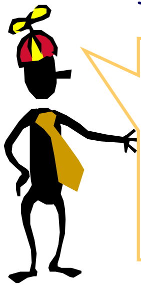
$$\begin{aligned}
&= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \ldots \\
&\quad\;\; - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 8x^7 - \ldots \\
&\quad\;\; - x^3 - x^4 - 2x^5 - 3x^6 - 5x^7 - \ldots \\
\hline
&= x
\end{aligned}$$

$$f(x) = \sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}$$

If a sequence satisfies a linear recurrence relation, we can always find its generating function.



Proving Identities with GF



Use generating functions to show that

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

Prove by induction

Prove algebraically

15-355 Computer Algebra

Prove combinatorially

Prove by Manhattan walk (see previous lecture)

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

Consider the right hand side:

$$\binom{2n}{n}=[x^n](1+x)^{2n}$$

Coefficient by x^n

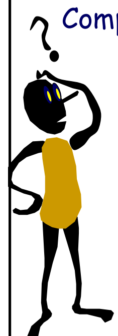
$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

Use the binomial theorem to obtain

$$(1+x)^{2n}=(\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)^2$$

Compute a coefficient of x^k

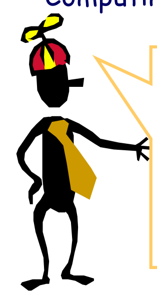
Computing Combinatorial Sums



Find a closed form

$$\sum_{k=0}^{n}\binom{k}{n-k}=??$$

Computing Combinatorial Sums



Don't try to evaluate the sum that you're looking at. Instead, find the generating function for it, then read off the coefficients.

$$\sum_{k=0}^{n}\binom{k}{n-k} = ??$$

Multiply it by x^n and sum it over n

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{k}{n-k}$$

Interchange the sums

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{k}{n-k} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} x^n \binom{k}{n-k}$$

Take $r = n - k$ as the new dummy variable of inner summation

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} x^n \binom{k}{n-k} = \sum_{k=0}^{\infty}\sum_{r=0}^{k} x^{r+k}\binom{k}{r}$$

We recognize the inner sum as $x^k (1 + x)^k$

$$\sum_{k=0}^{\infty} x^k (1+x)^k$$

This is a geometric series

$$\sum_{k=0}^{\infty} x^k (1+x)^k = \frac{1}{1-x(1+x)} = \frac{1}{1-x-x^2}$$

The RHS is our old friend …

What did we find?

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{k}{n-k} = \frac{1}{1-x-x^2}$$

$$\sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}$$

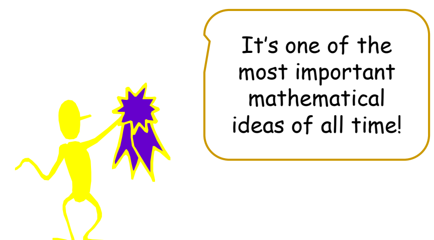
$$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{1}{1-x-x^2}$$

What did we find?

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{k}{n-k} = \sum_{n=0}^{\infty} F_{n+1} x^n$$

$$\sum_{k=0}^{n}\binom{k}{n-k} = F_{n+1}$$

Generating Functions



Study Bee

- Counting with generating functions
- Solving the Diophantine equations
- Proving combinatorial identities