Great Theoretical Ideas In Computer Science

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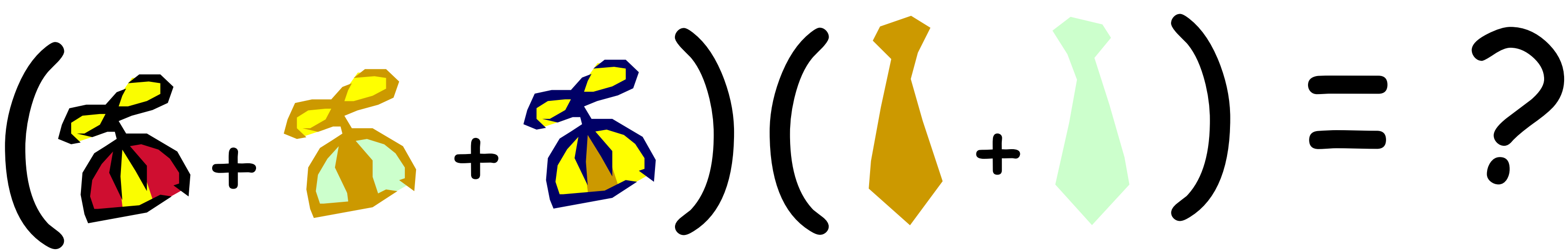
Counting II: Pascal, Binomials, and Other Tricks



Permutations vs. Combinations

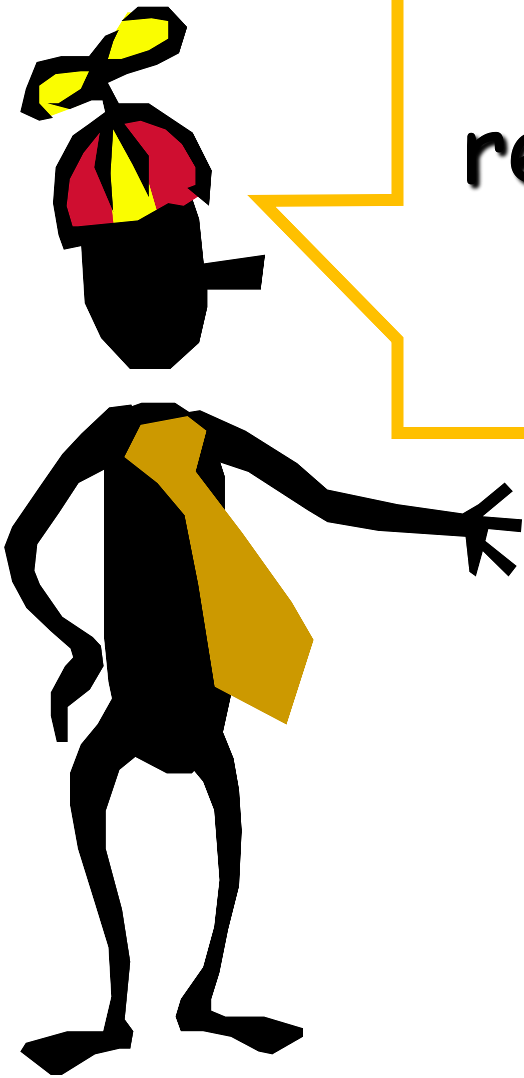
Subsets of r out of n distinct objects

$$\frac{n!}{(n-r)!}$$

Ordered

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Unordered



How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

__,__,__,__,__,__,__

7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced

7 X 6 X 5 X 4 = 840

SYSTEMS

Let's pretend that the S's are distinct:

$$S_1 Y S_2 TEMS_3$$

There are 7! permutations of $S_1 Y S_2 TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1 S_2 S_3$

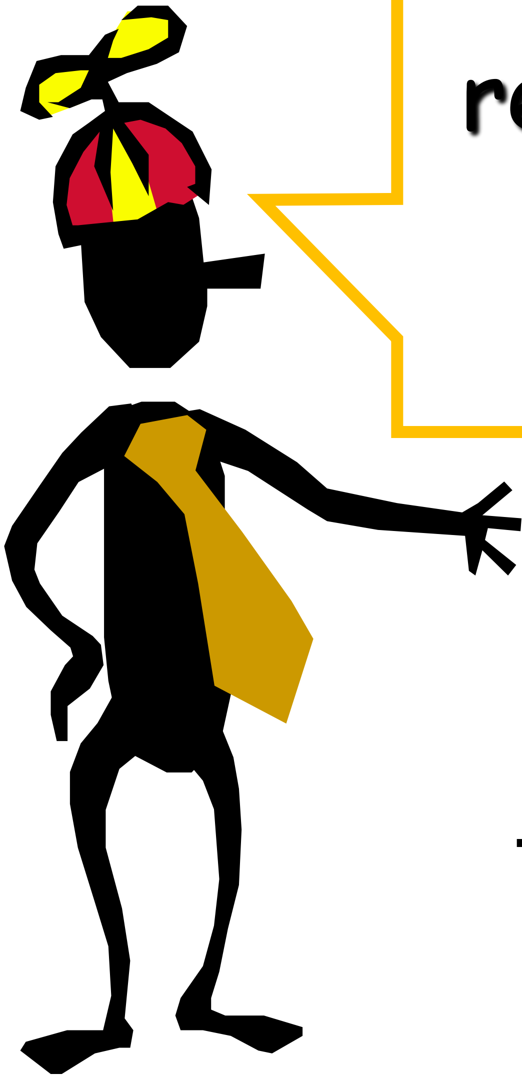
$$\frac{7!}{3!} = 840$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, ..., r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\,\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\cdots$$

$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$



How many ways to rearrange the letters in the word "CARNEGIEMELLON"?

$$\frac{14!}{2!3!2!} = 3,632,428,800$$

Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} = \begin{cases} 0, \text{if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

Four ways of choosing

We will choose 2-letters word from the alphabet (L,U,C,K,Y}

1) C(5,2) no repetitions,

the order is NOT important

LU = UL

Four ways of choosing

We will choose 2-letters word from the alphabet (L,U,C,K,Y}

2) P(5,2) no repetitions,

the order is important

LU != UL

$$P(n,r)=n*(n-1)*...*(n-r+1)$$

Four ways of choosing

We will choose 2-letters word from the alphabet (L,U,C,K,Y}

3) $5^2 = 25$ with repetitions,
the order is important

Four ways of choosing

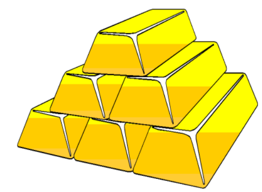
We will choose 2-letter words from the alphabet {L,U,C,K,Y}

4) ????   repetitions,

the order is NOT important

C(5,2) + {LL,UU,CC,KK,YY} = 15

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G's and 4 /'s

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates

1st pirate gets 2
2^{nd} pirate gets 1
3^{rd} and 5^{th} get nothing
4^{th} gets 17

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

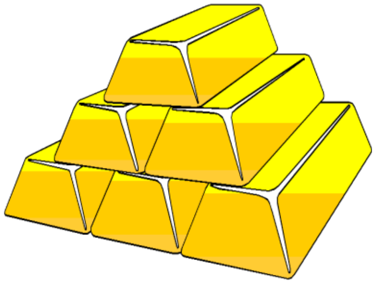
In general, the k^{th} pirate gets the number of G's after the k-1^{st} / and before the k^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

How many sequences with 20 G's and 4 /'s?

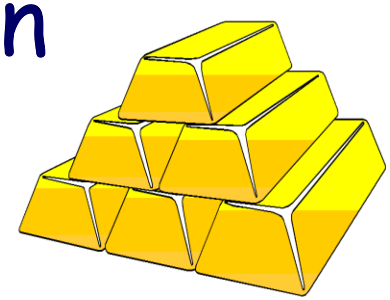
$$\binom{24}{4} = \binom{20+5-1}{5-1}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Another interpretation



How many different ways to put k indistinguishable balls into n distinguishable urns.

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Another interpretation

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\binom{24}{4}$$

Think of x_k as being the number of gold bars that are allotted to pirate k.

How many integer nonnegative solutions to the following equations?

$$x_1 + x_2 + \ldots + x_n = k$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n > 0$$

Think of x_k -> y_k+1

bijection with solutions to

$$y_1 + y_2 + y_3 + \ldots + y_n = k\text{-}n$$

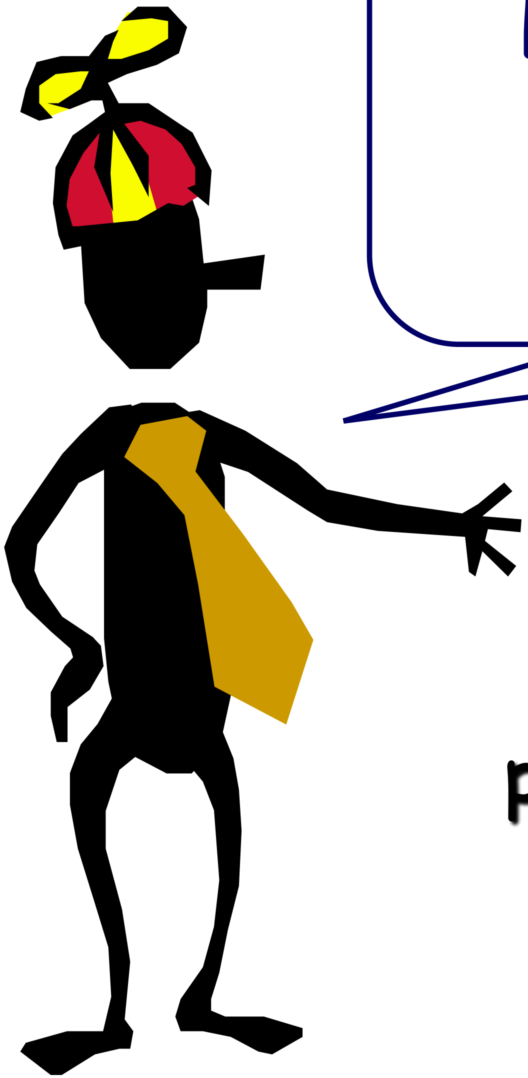
$$y_1, y_2, y_3, \ldots, y_n \geq 0$$

Remember to distinguish between Identical / Distinct Objects

If we are putting k objects into
n <u>distinct</u> bins.

k objects are distinguishable	n^k
k objects are indistinguishable	$\binom{n+k-1}{k}$

Pigeonhole Principle



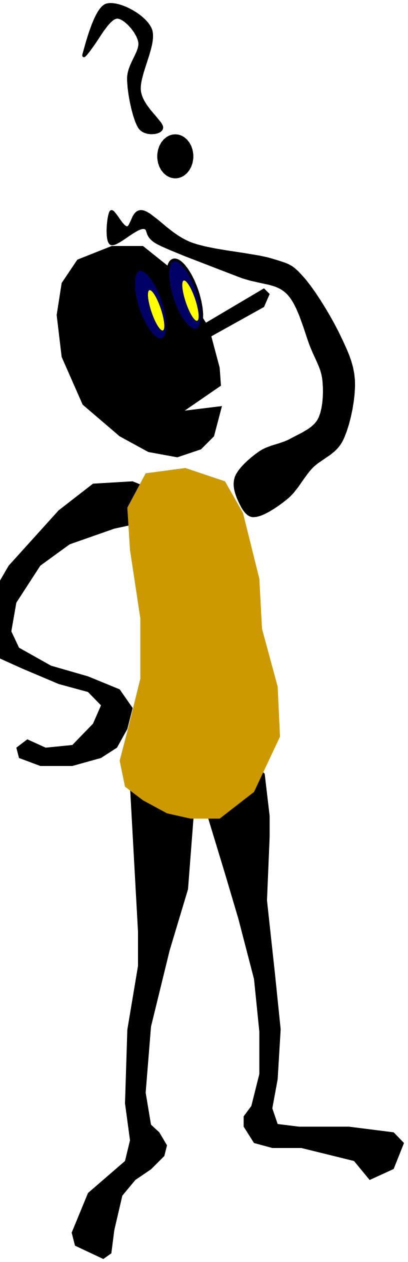
If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Example:

two people in Pittsburgh must have the same number of hairs on their heads



Pigeonhole Principle

Problem:

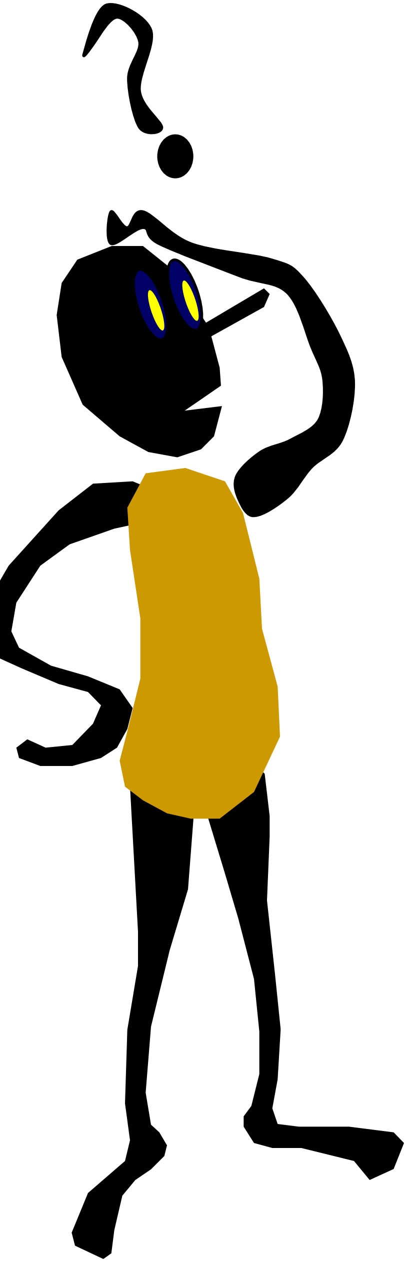
among any n integer numbers, there are some whose sum is divisible by n.

Among any n integer numbers, there are some whose sum is divisible by n.

Consider $s_i = x_1 + \ldots + x_i$ modulo n. How many s_i?

Remainders are {0, 1, 2, ..., n-1}.

Exist $s_i = s_k \pmod{n}$. Take $s_i - s_k$



Pigeonhole Principle

Problem:

The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

What are pigeons?
And what are pigeonholes?

The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17

Let $S_1=a_1+a_2+a_3$, … $S_{10}=a_{10}+a_1+a_2$
There are 10 pigeonholes.

Pigeons: $S_1+ .. + S_{10}= 3\ (a_1+a_2+a_{10}) = 3*55 = 165$

Since 165 > 10 *16, at least one pigeon-hole has at least 16 + 1 pigeons

Pigeonhole Principle

Problem:

Show that for some integer $k > 1$, 3^k ends with 0001 (in its decimal representation).

What are pigeons?
And what are pigeonholes?

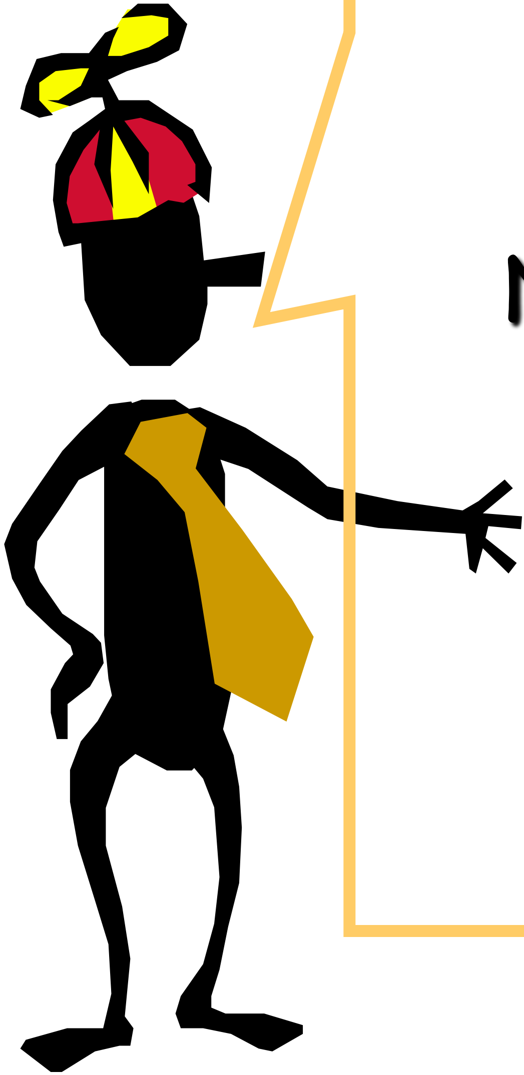
Show that for some integer $k > 1$, 3^k ends with 0001

Choose 10001 numbers: $3^1, 3^2, \ldots, 3^{10001}$

$3^k = 3^m \bmod (10000)$, $m < k$

$3^{k-m} = 1 \bmod (10000)$

$3^{k-m} = q*10000 + 1$ ends with 0001

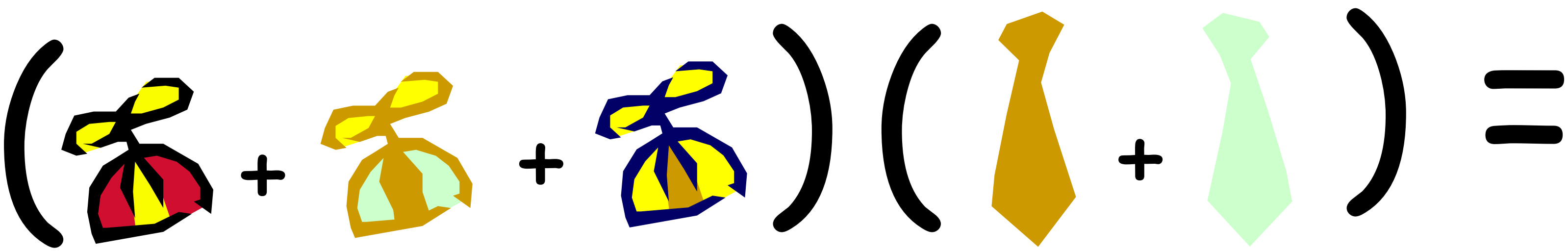


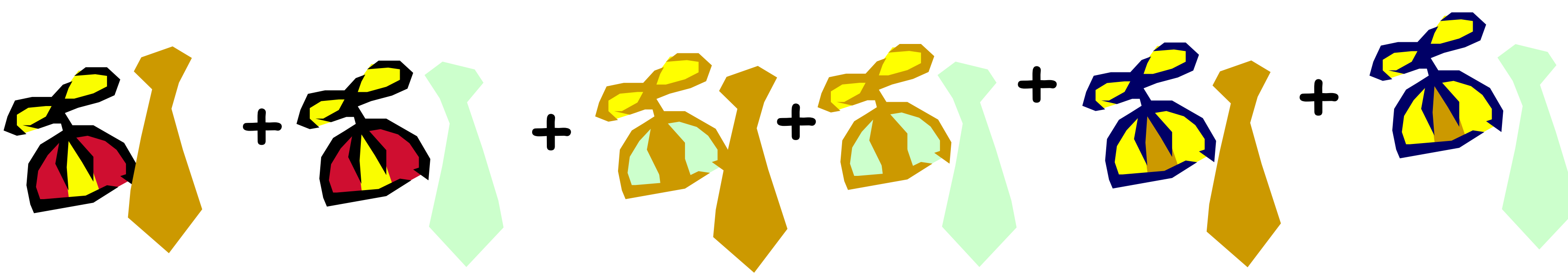
Now, something completely different…

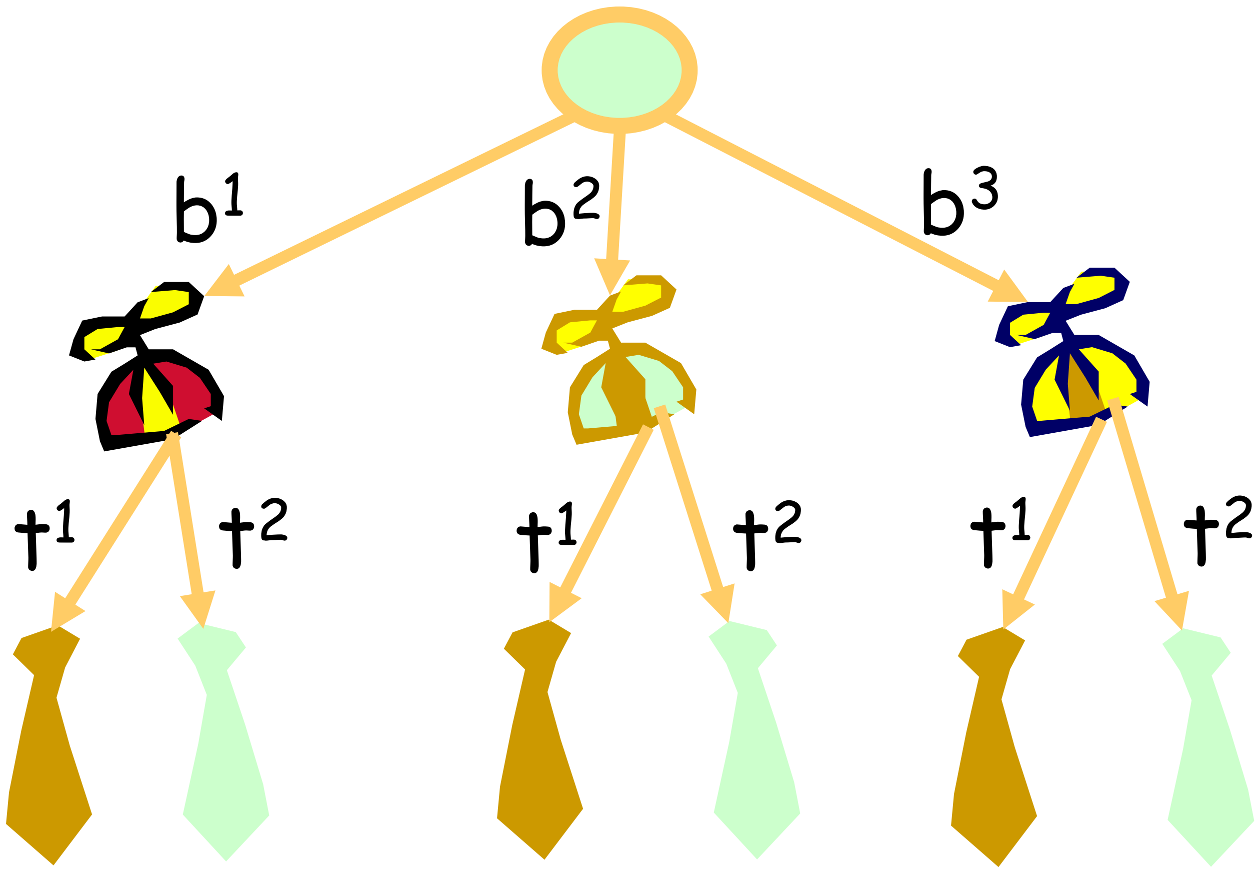
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

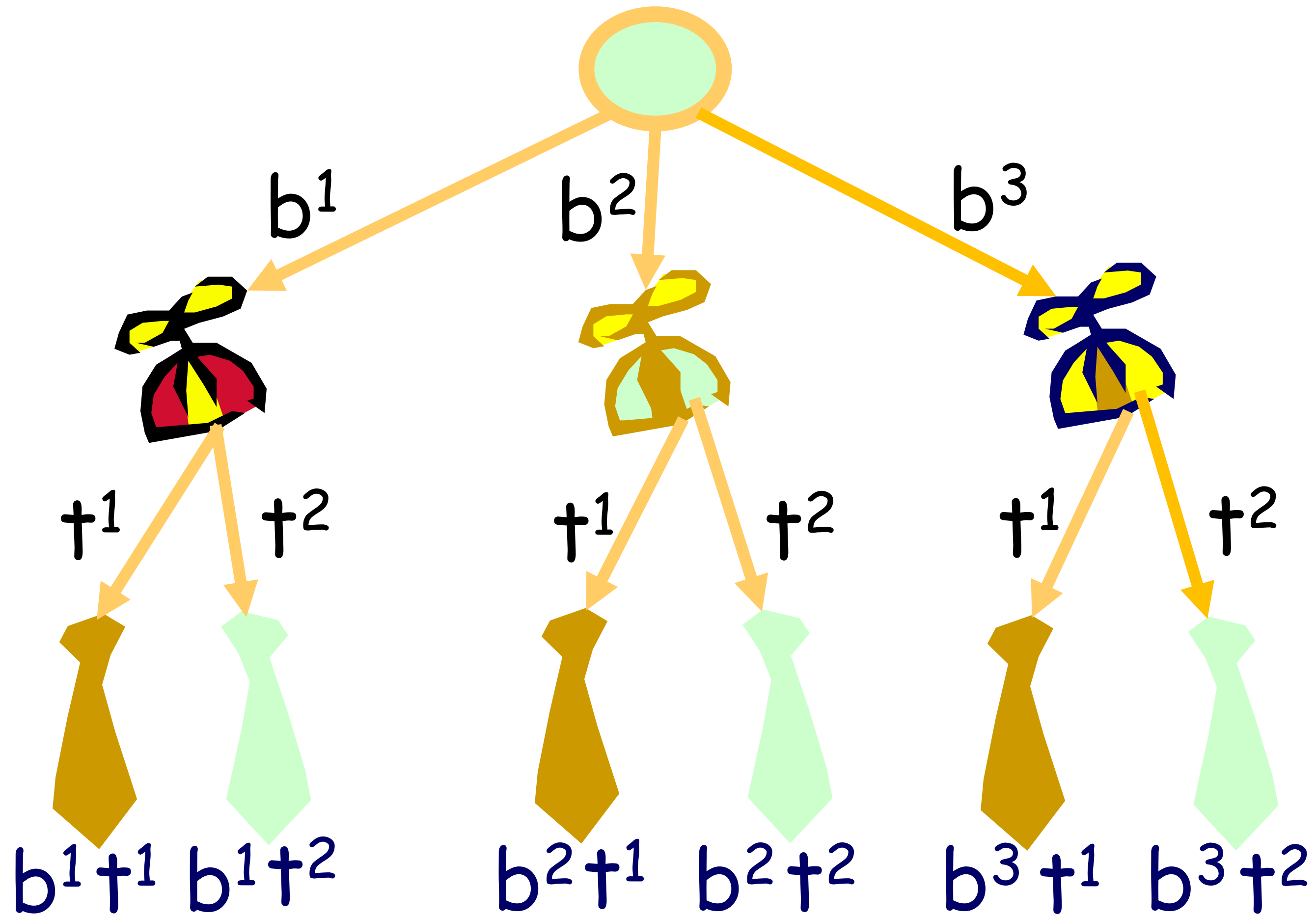
Products of Sum = Sums of Products



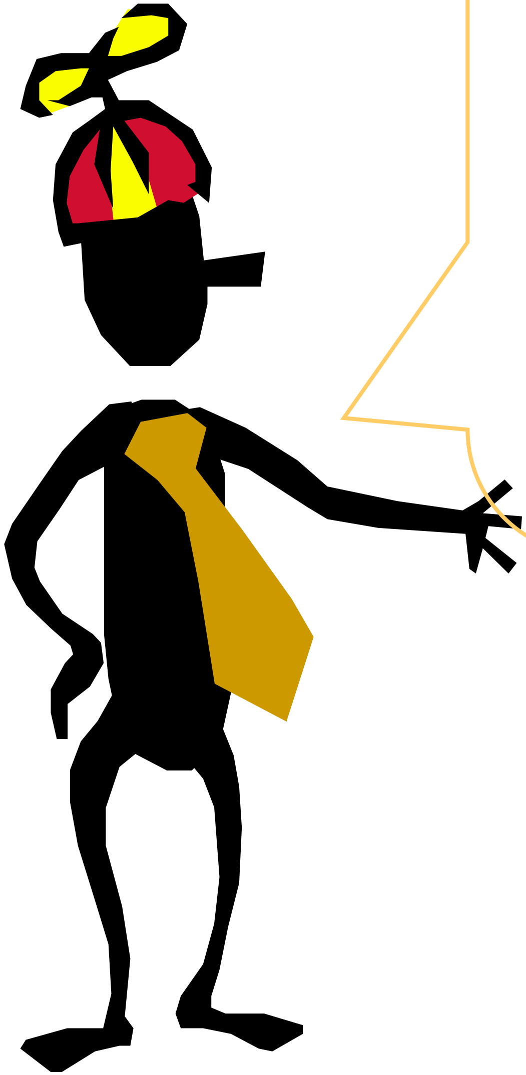




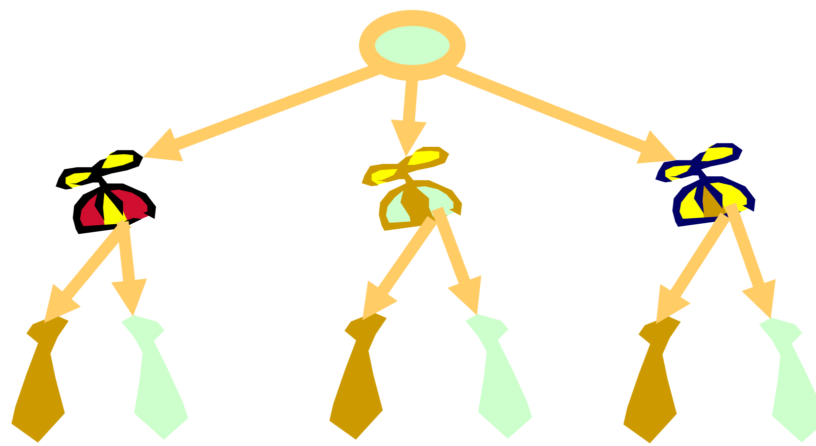
b1
b2
b3
t1
t2
t1
t2
t1
t2



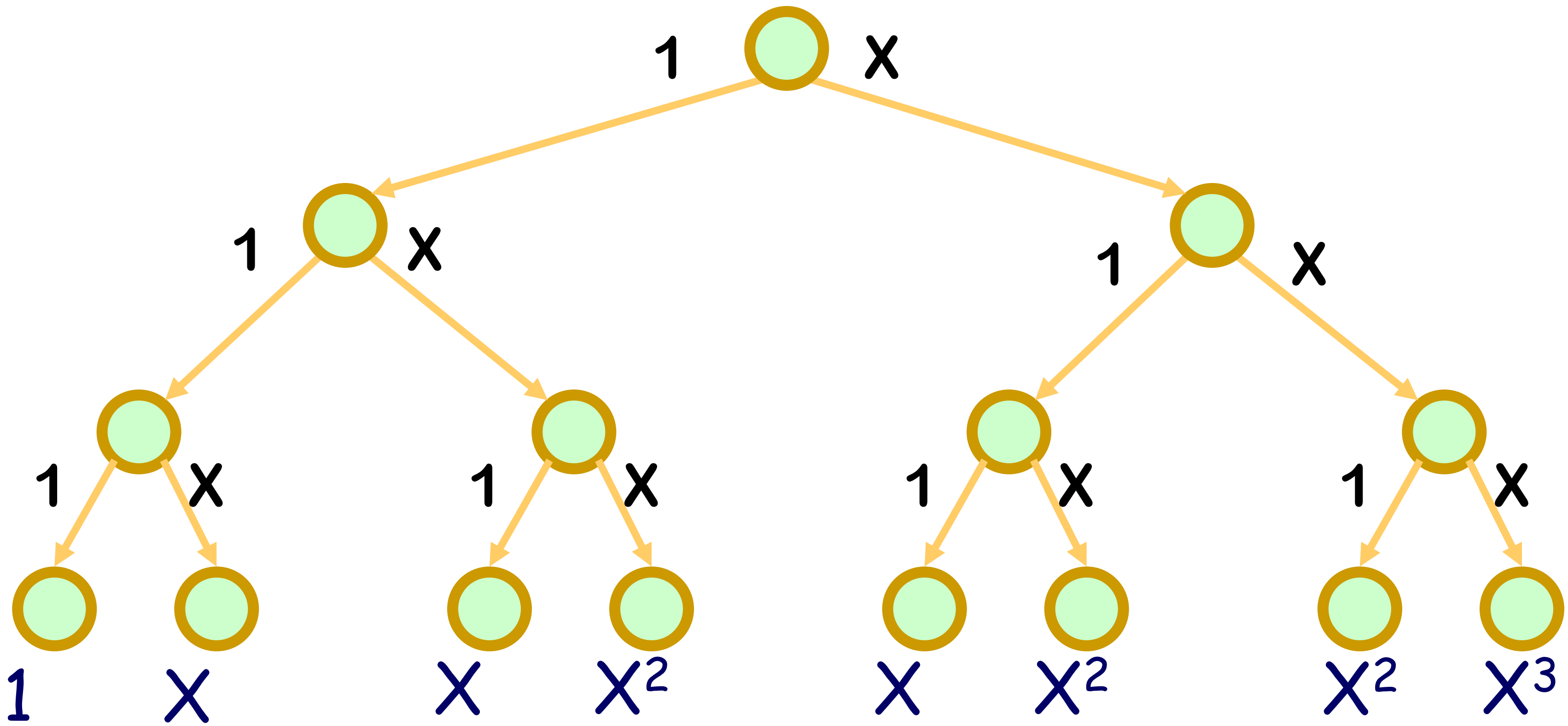
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$



There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!



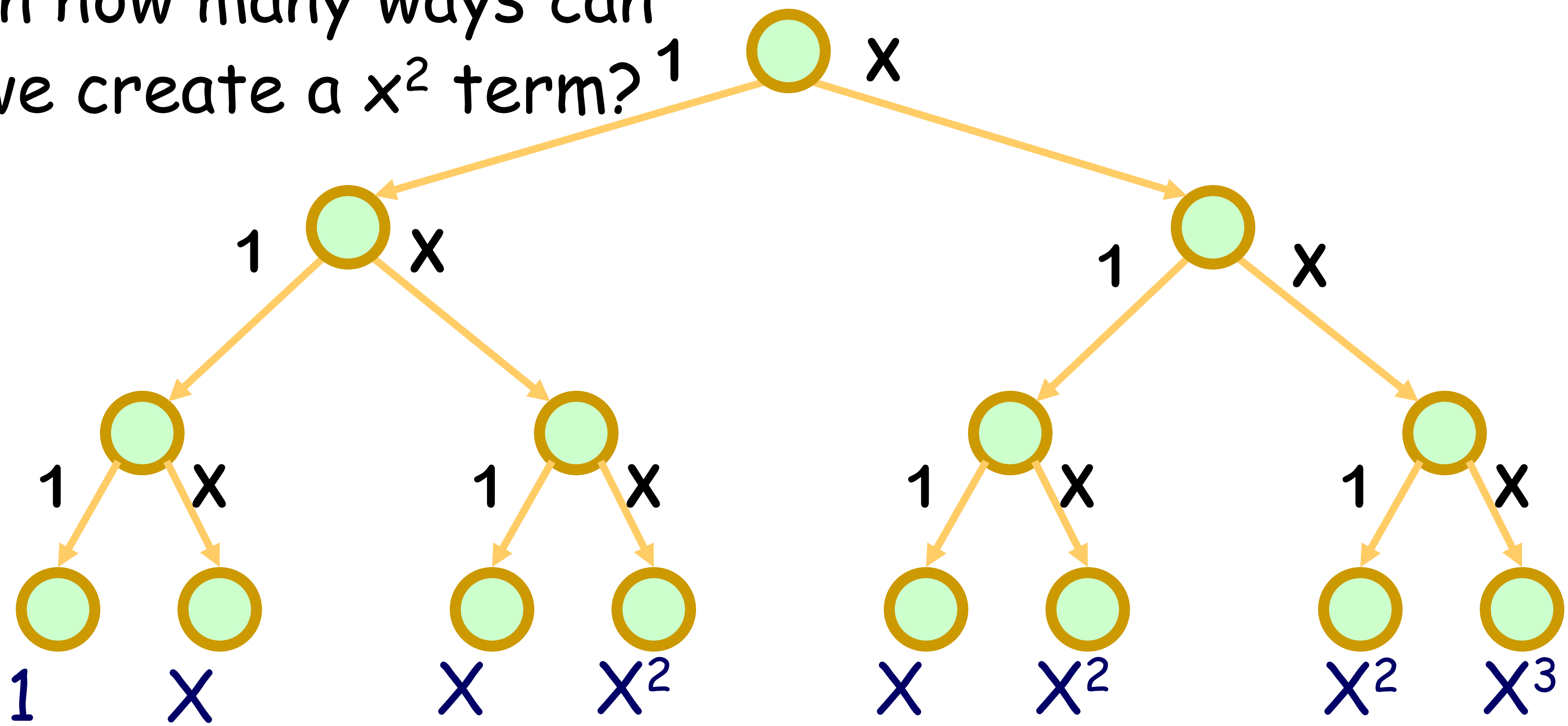
Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

$$(1+X)^3 = 1 + 3X + 3X^2 + X^3$$

In how many ways can we create a x^2 term?



What is the combinatorial meaning of those coefficients?

What is a closed form expression for c_k?

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

In how many ways can we create a x^2 term?

What is a closed form expression for c_n?

$$(1+X)^n$$

$$= \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{\text{n times}}$$

After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with exactly k X's.

$$c_k = \binom{n}{k}$$

The Binomial Theorem

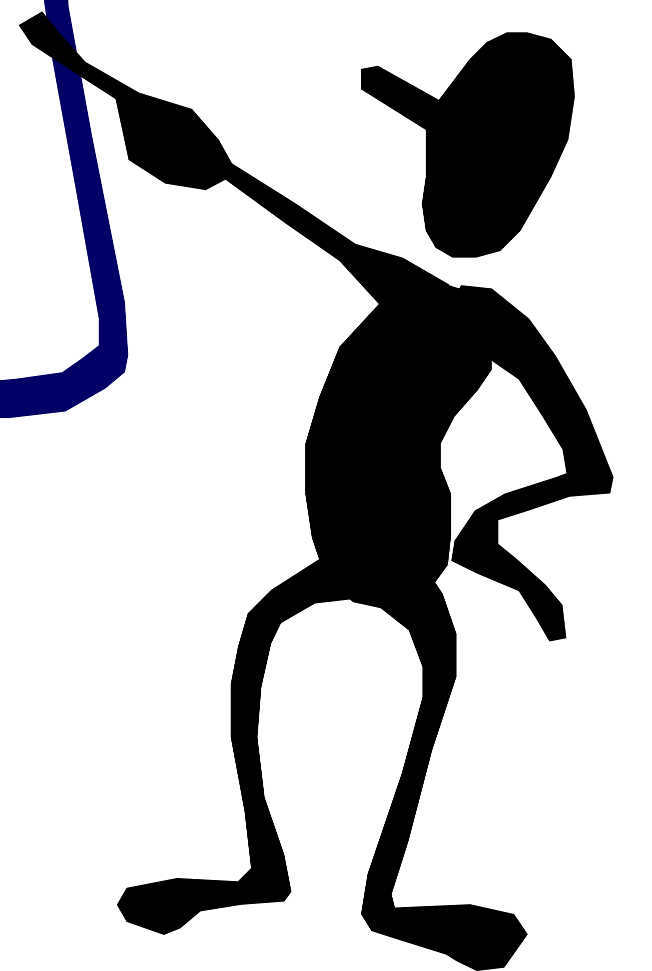
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

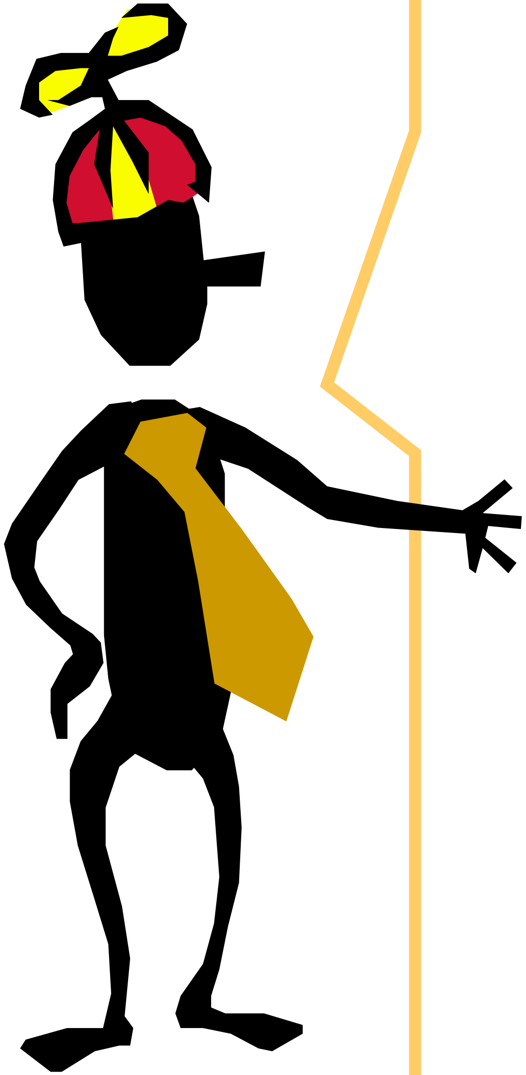
Binomial Coefficients

binomial expression

The Binomial Formula

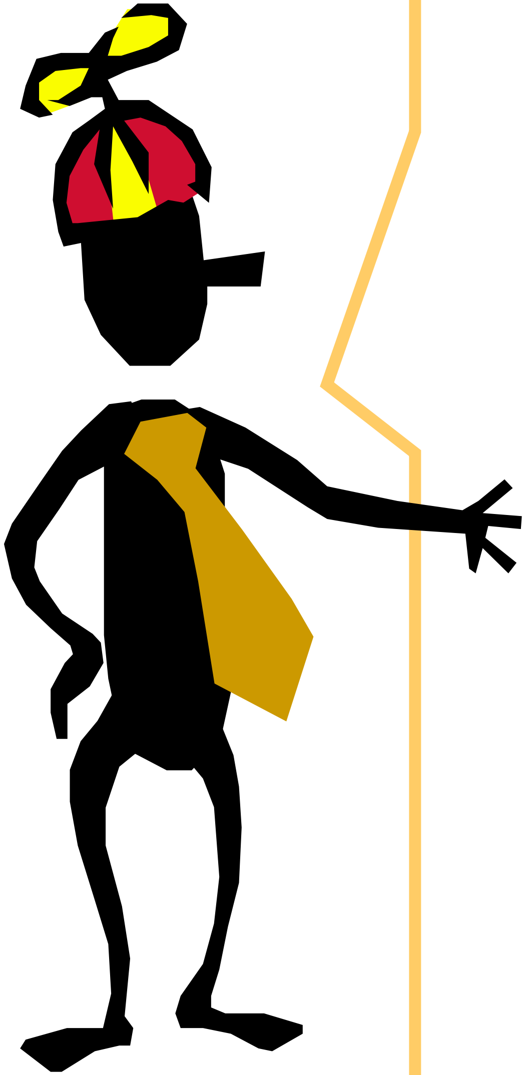
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$





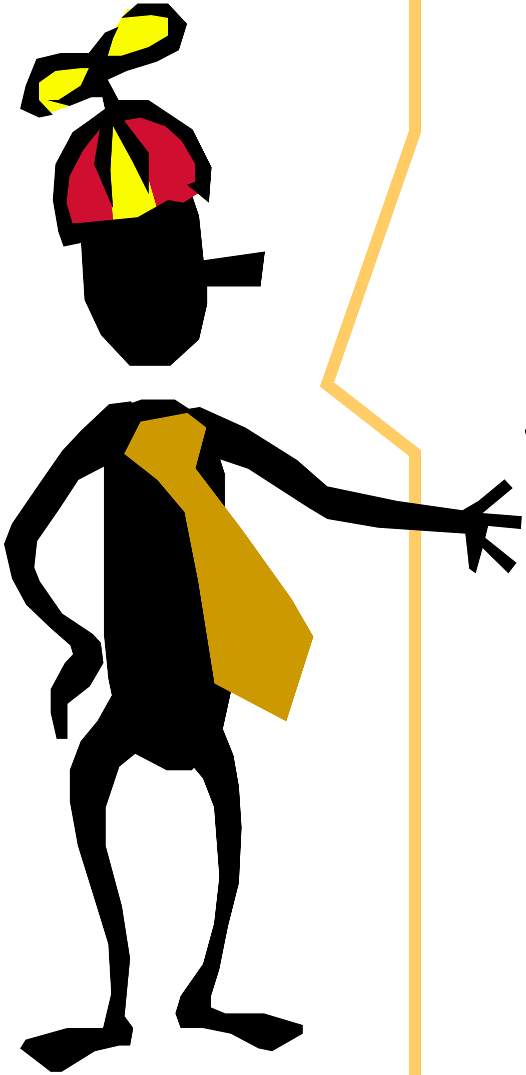
What is the coefficient of EMPTY in the expansion of $(E + M + P + T + Y)^5$?

5!



What is the coefficient of EMP^3TY in the expansion of $(E + M + P + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS



What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA

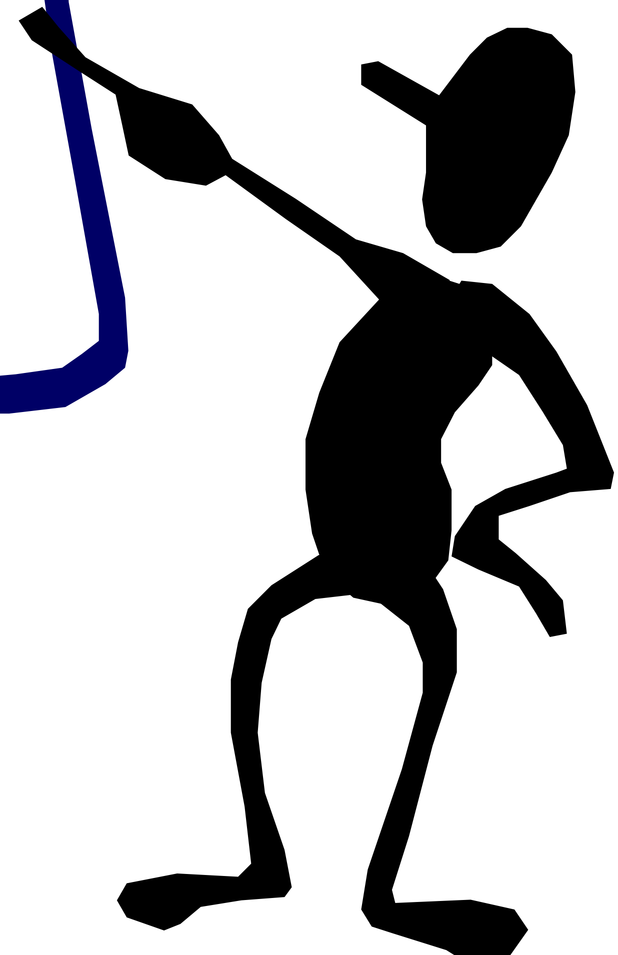
What is the coefficient of

$$X_1^{r_1}X_2^{r_2}X_3^{r_3}\dots X_k^{r_k}$$

in the expansion of

$$(X_1+X_2+\dots+X_k)^n?$$

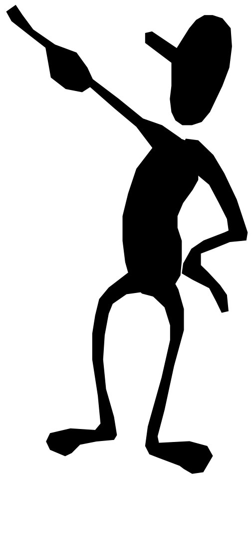
$$\boxed{\frac{n!}{r_1!\,r_2!\,r_3!\dots r_k!}}$$



Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} = \begin{cases} 0, \text{if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

The Multinomial Formula



$$X_1+X_2+\dots+X_k \quad {}^{n}$$

$$= \sum_{\substack{r_1,r_2,\dots,r_k \\ \sum r_i = n}} \binom{n}{r_1;r_2;\dots;r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \dots X_k^{r_k}$$

On to Pascal…

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

Pascal's Triangle

Blaise Pascal
1654

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1

1 + 1

1 + 2 + 1

1 + 3 + 3 + 1

1 + 4 + 6 + 4 + 1

1 + 5 + 10 + 10 + 5 + 1

1 + 6 + 15 + 20 + 15 + 6 + 1

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	=1
1 + 1	=2
1 + 2 + 1	=4
1 + 3 + 3 + 1	=8
1 + 4 + 6 + 4 + 1	=16
1 + 5 + 10 + 10 + 5 + 1	=32
1 + 6 + 15 + 20 + 15 + 6 + 1	=64

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

Summing on 1st Avenue

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Summing on k^{th} Avenue

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$$\sum_{k=1}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

Hockey-stick identity

Fibonacci Numbers

$$\sum_{k=0}^{n-2} \binom{n-k}{k} = F_{n+1}$$

1

1 1 =2

1 2 1 =3

1 3 3 1 =5

1 4 6 4 1 =8

1 5 10 10 5 1 =13

1 6 15 20 15 6 1

Sums of Squares

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

1

1 1

$\boxed{1^2 \quad 2^2 \quad 1^2}$

$\boxed{1^2 \quad 3^2 \quad 3^2 \quad 1^2}$

1 4 6 4 1

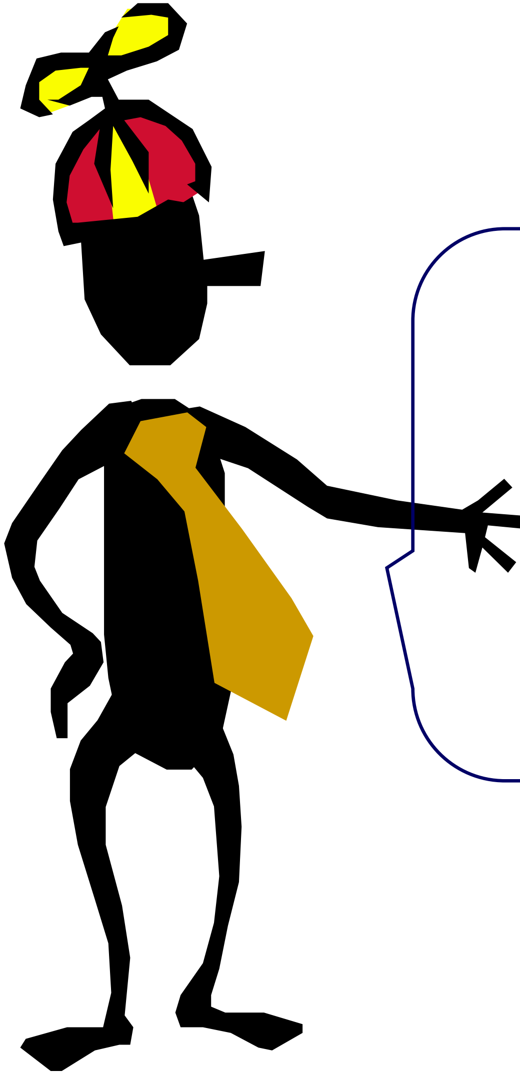
1 5 10 10 5 1

1 6 15 20 15 6 1

Al-Karaji Squares

				1			=0
			1		1		=1
		1		2 + 2*1			=4
	1		3 + 2*3		1		=9
	1	4 + 2*6	4	1			=16
1	5 + 2*10	10	5	1			=25
1	6 + 2*15	20	15	6	1		=36

The art of combinatorial proof



All these properties can be proved inductively and algebraically. We will give combinatorial proofs.

The art of combinatorial proof

$$\binom{n}{k} = \binom{n}{n-k}$$

How many ways we can create a size k committee of n people?

LHS : By definition

RHS : We choose n-k people to exclude from the committee.

The art of combinatorial proof

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

How many ways we can create a size k committee of n people?

LHS : By definition

RHS : Pick a person, say n.

There are committees that exclude n

There are committees that include n

The art of combinatorial proof

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$$

How many ways we can create an even size committee of n people?

LHS : There are so many such committees

RHS : Choose n-1 people. The fate of the nth person is completely determined.

The art of combinatorial proof

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

LHS : We create a size k committee, then we select a chairperson.

RHS : We select the chair out of n, then from the remaining n-1 choose a size k-1 committee.

The art of combinatorial proof

$$\sum_{k=0}^{n} k \binom{n}{k} = n\,2^{n-1}$$

LHS : Count committees of any size, one is a chair.

RHS : Select the chair out of n, then from the remaining n-1 choose a subset.

The art of combinatorial proof

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

LHS : m-males, n–females, choose size k.

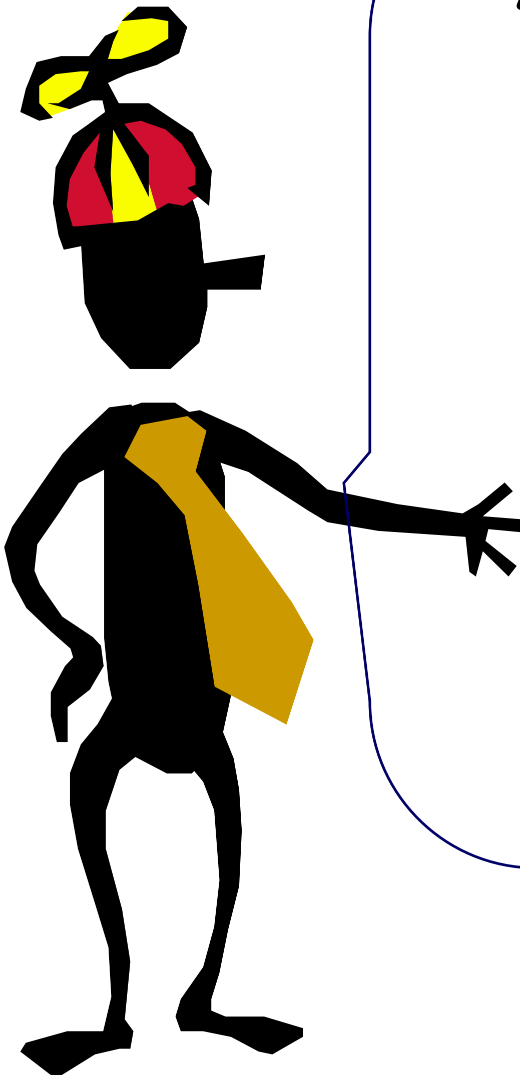
RHS : Select a committee with j men, the remaining k-j members are women.

The art of combinatorial proof

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}$$

LHS : The number of (k+1)-subsets in {1,2,...,n+1}

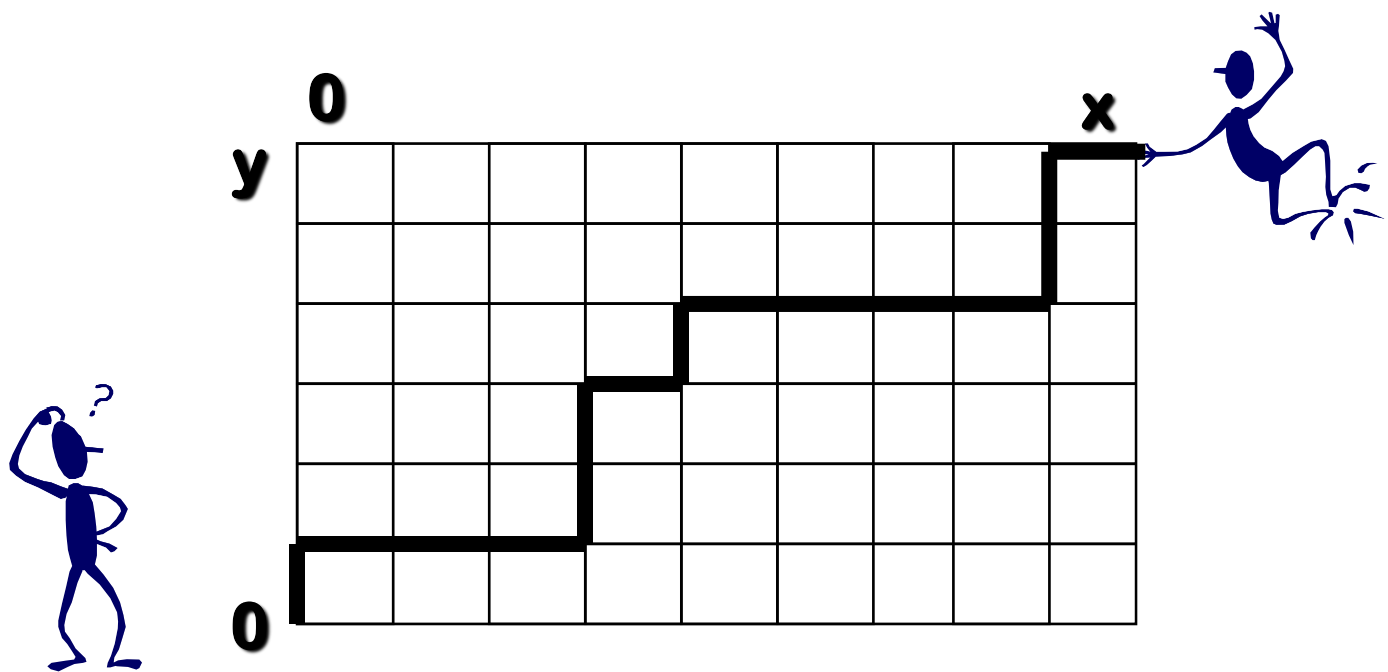
RHS : Count (k+1)-subsets with the largest element m+1, where $k \leq m \leq n$.



All these properties can be proved by using the Manhattan walking representation of binomial coefficients.

Manhattan walk

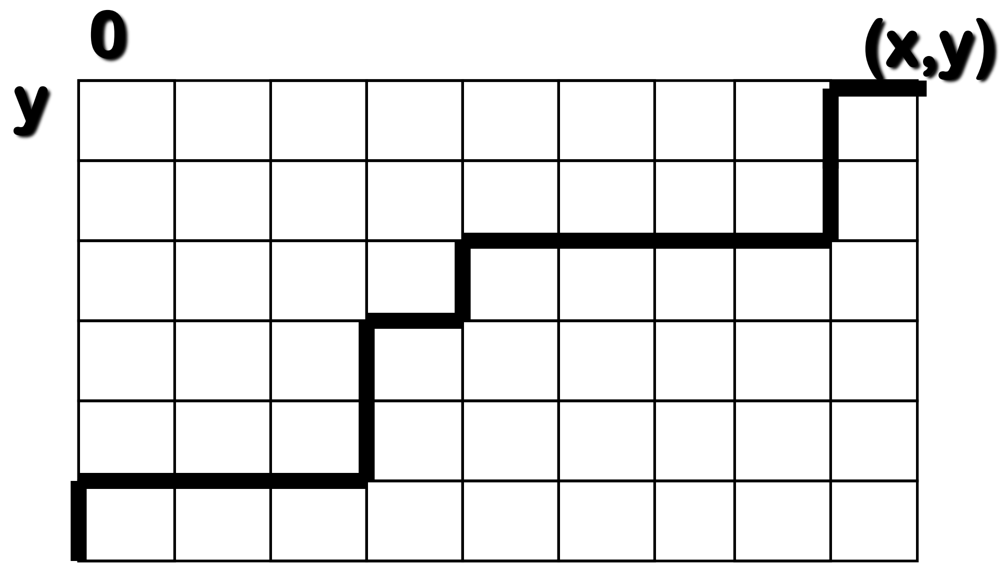
You're in a city where all the streets, numbered 0 through x, run north-south, and all the avenues, numbered 0 through y, run east-west. How many [sensible] ways are there to walk from the corner of 0th st. and 0th avenue to the opposite corner of the city?



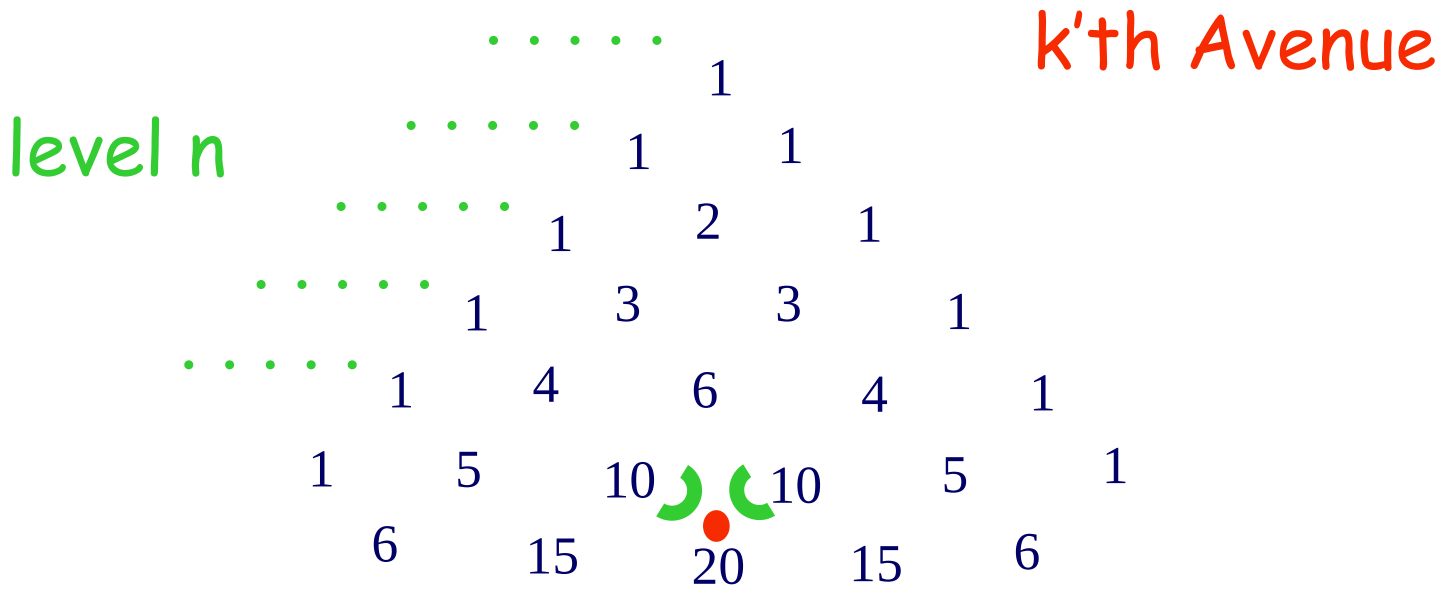
Manhattan walk

- All paths require exactly $x+y$ steps:
- x steps east, y steps north
- Counting paths is the same as counting which of the $x+y$ steps are northward steps:

$$\binom{x+y}{x}$$

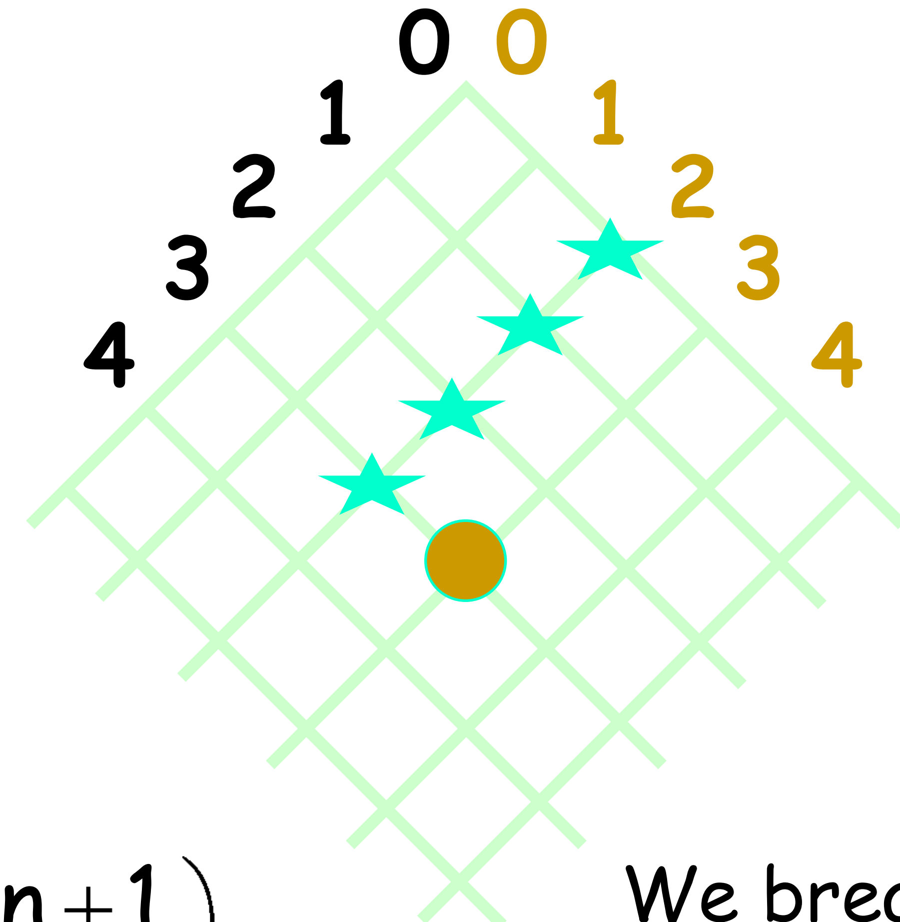


Manhattan walk



$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

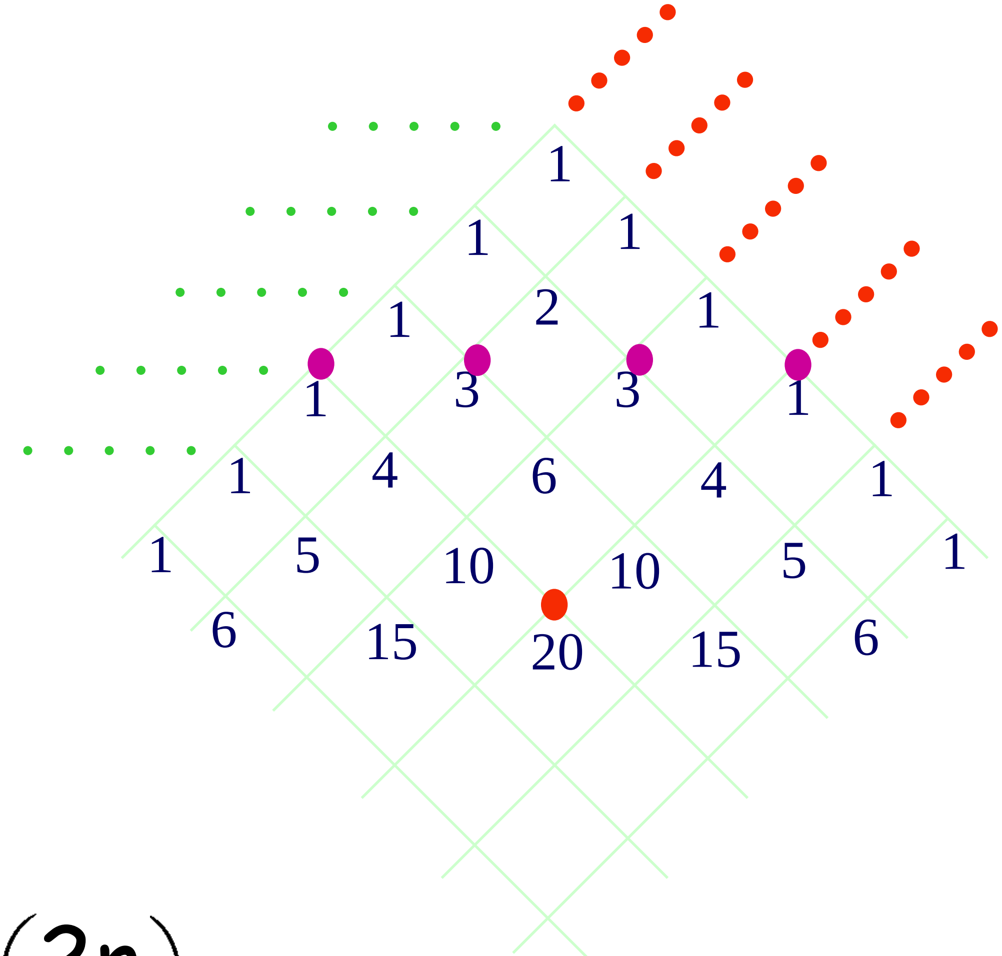
Manhattan walk



$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

We break all routes into: reach a star + 1 right and all left turns

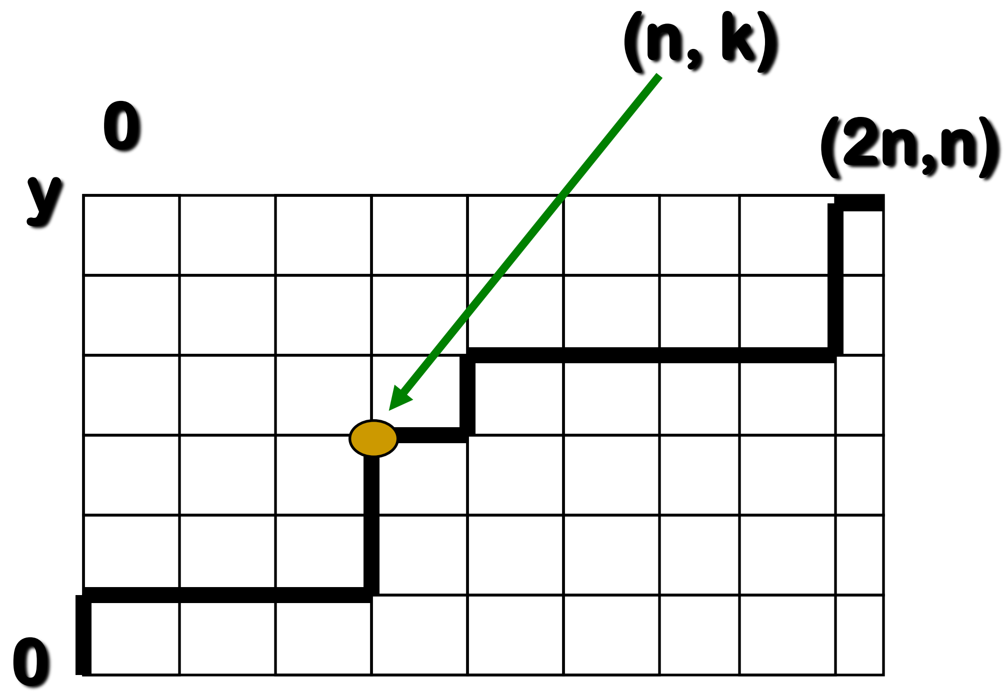
Manhattan walk



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Manhattan walk

Now, what if we add the constraint that the path must go through a certain intersection, call it (n,k) ?

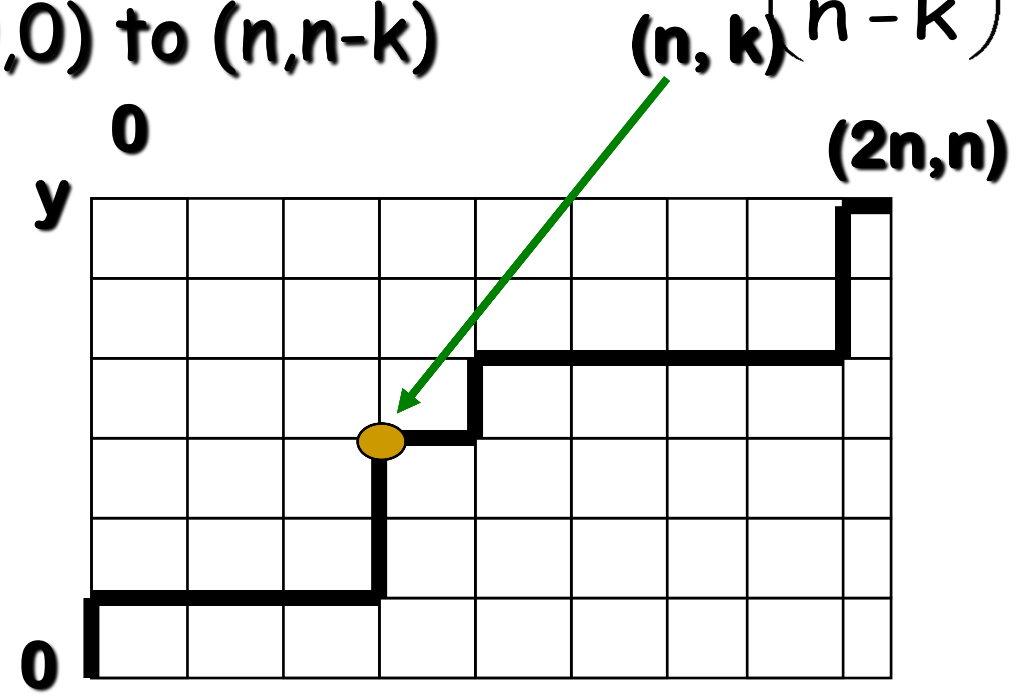


$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Manhattan walk

The number of routes from (0,0) to (n,k) is $\binom{n}{k}$

The number of routes from (n,k) to (2n,n) is the same as from(0,0) to (n,n-k) $\binom{n}{n-k}$



$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}=\binom{2n}{n}$$

Study Bee

- Binomials and Multinomials
- Pirates and Gold
- Pigeonhole principal
- Combinatorial proofs of binomial identities
- Manhattan walk