

Counting I: Choice Trees and Correspondences



Not everything that can be
counted counts, and not everything
that counts can be counted.

A. Einstein

In the next few lectures we will learn some fundamental counting methods.

Why counting is important?

1. Solving optimization problems
2. Analysis of algorithms (average case)

In the next few lectures we will learn some fundamental counting methods.

- Addition and Product Rules
- Choice Tree
- Permutations and Combinations
- The Binomial Theorem
- The Inclusion-Exclusion Principal
- The Pigeonhole Principal
- Diophantine Equations
- Generating Functions

Addition Rule

Let A and B be two disjoint finite sets.

The size of $A \cup B$ is the sum of
the size of A and the size of B .

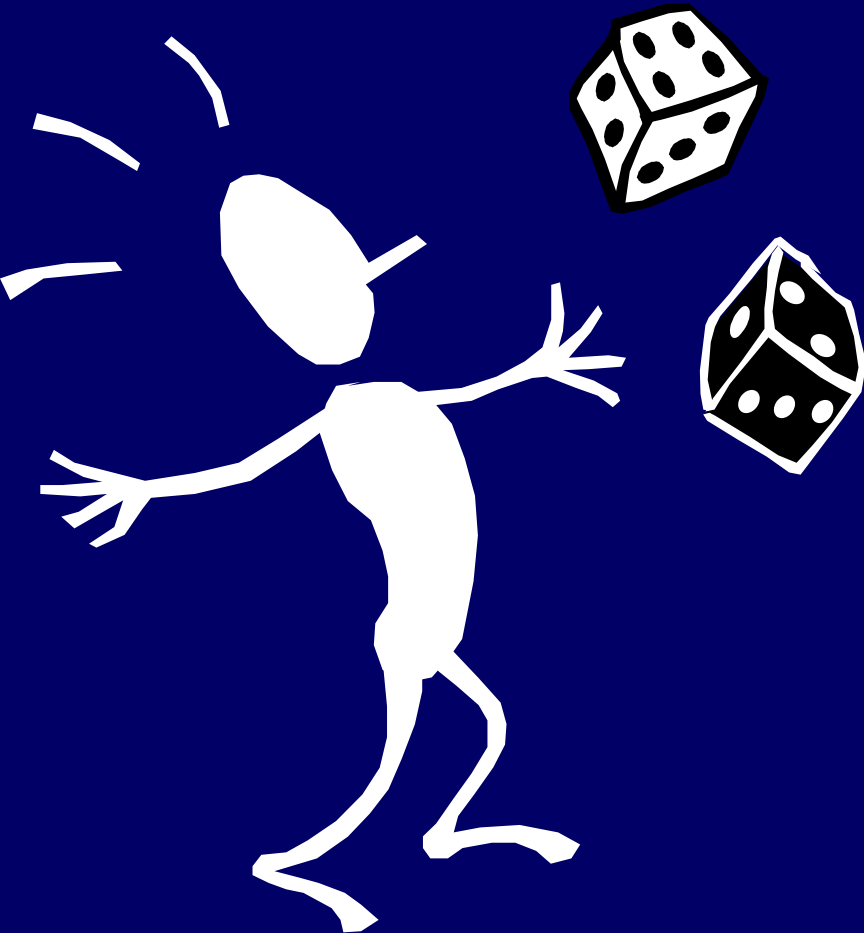
$$|A \cup B| = |A| + |B|$$

Addition Rule

Let $A_1, A_2, A_3, \dots, A_n$ be disjoint, finite sets.

$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{k=1}^n |A_k|$$

Example



Suppose I roll a white die and a black die.

What is the number of outcomes where the dice show different values?

Let S be a set of all outcomes where the dice show different values.

Let T be a set of outcomes where the two dice agree.

$$|S| = ?$$

$$|S \cup T| = 36$$

$$|S| + |T| = 36$$

$$|T| = 6$$

$$|S| = 30$$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$S = 5 + 4 + 3 + 2 + 0 = 15$$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

Another way of counting...

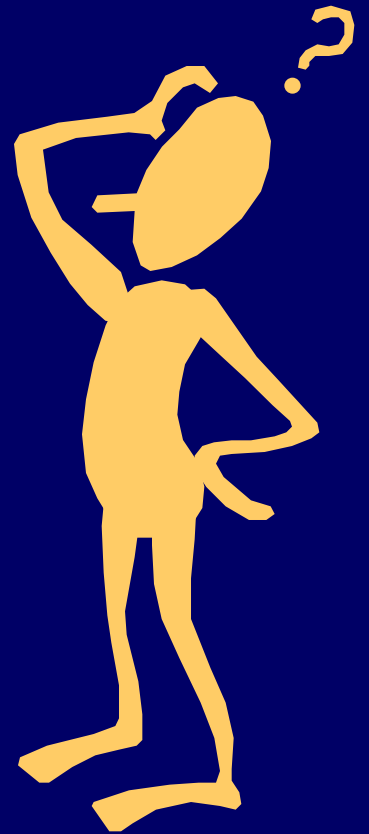
$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

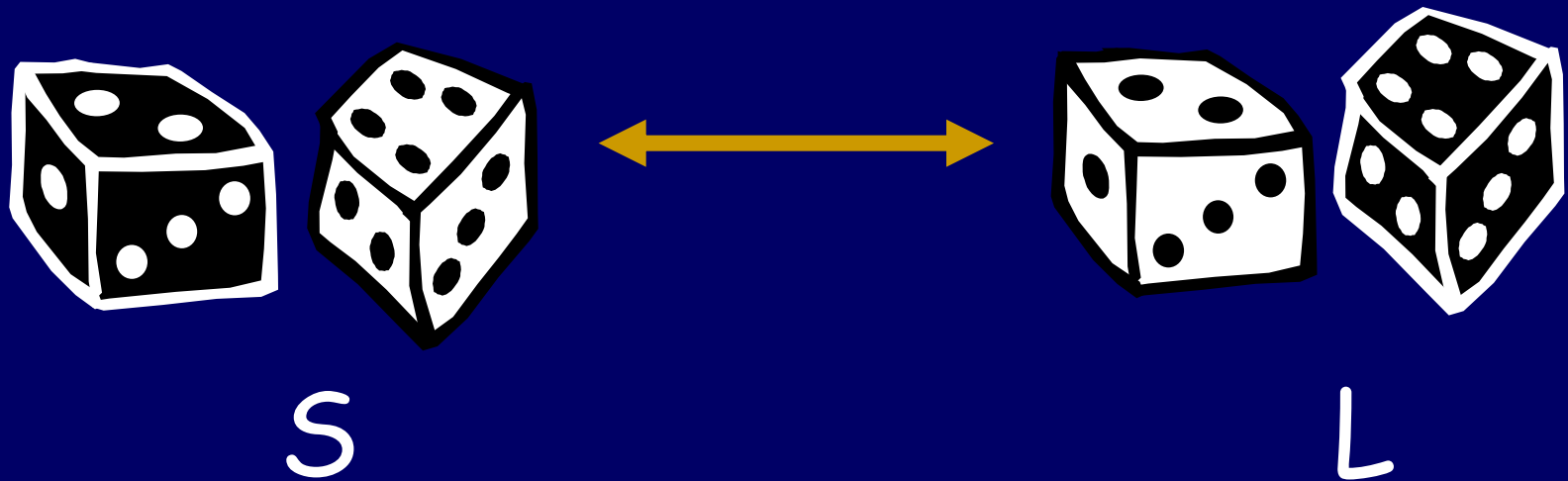
It is clear by symmetry that $|S| = |L|$.
Therefore, $|S| = 15$

It is clear by symmetry that $|S| = |L|$.



Pinning down the idea of symmetry by exhibiting a correspondence.

Put each outcome in S in correspondence with an outcome in L by **swapping** the color of the dice.



Let $f:A \rightarrow B$
be a function from a set A to a set B .

f is **1-1** (or **injective**) if and only if

$$\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$$

f is **onto** (or **surjective**) if and only if

$$\forall z \in B \exists x \in A f(x) = z$$



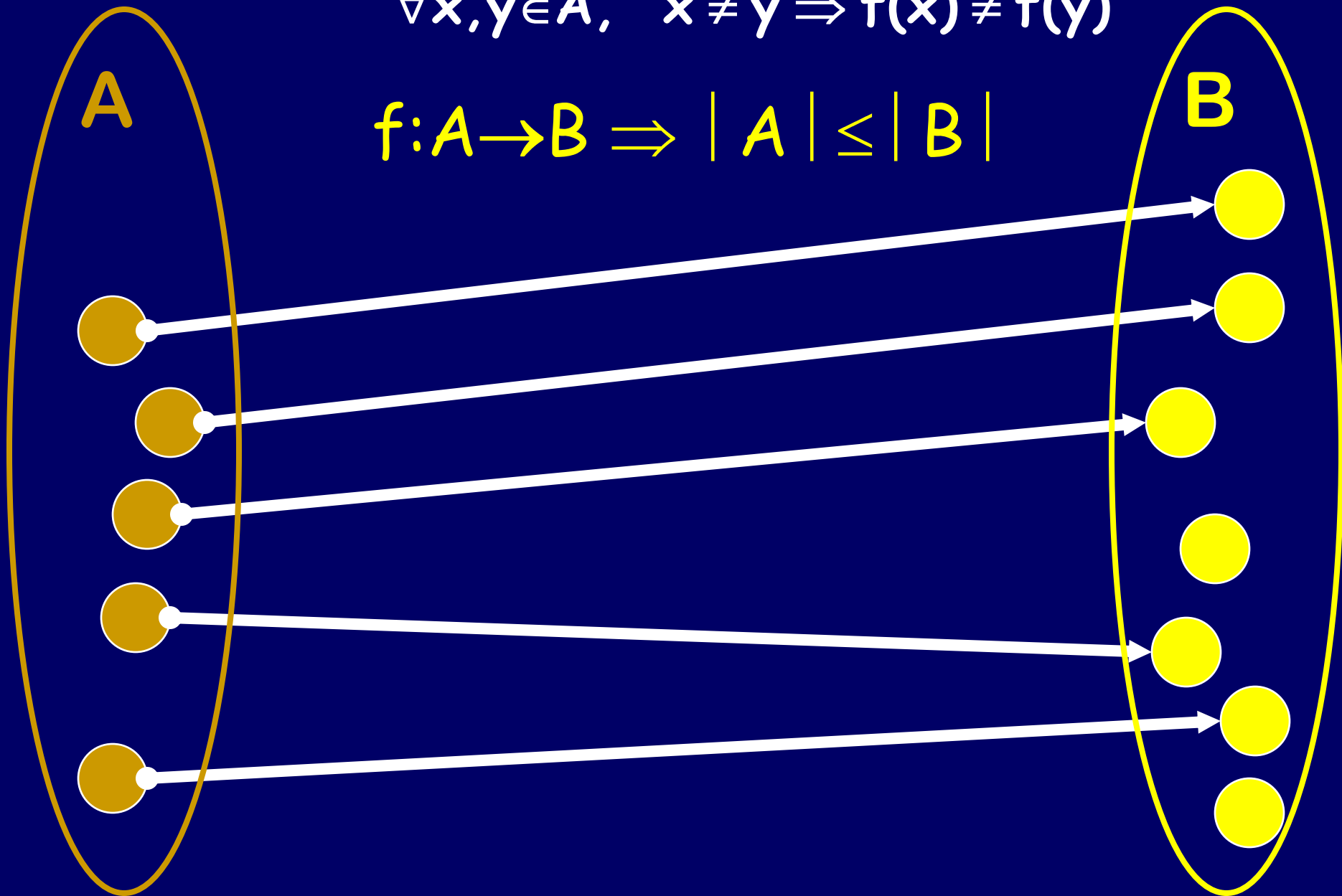
For Every

There
Exists

f is **1-1** (or **injective**) if and only if

$$\forall x, y \in A, \quad x \neq y \Rightarrow f(x) \neq f(y)$$

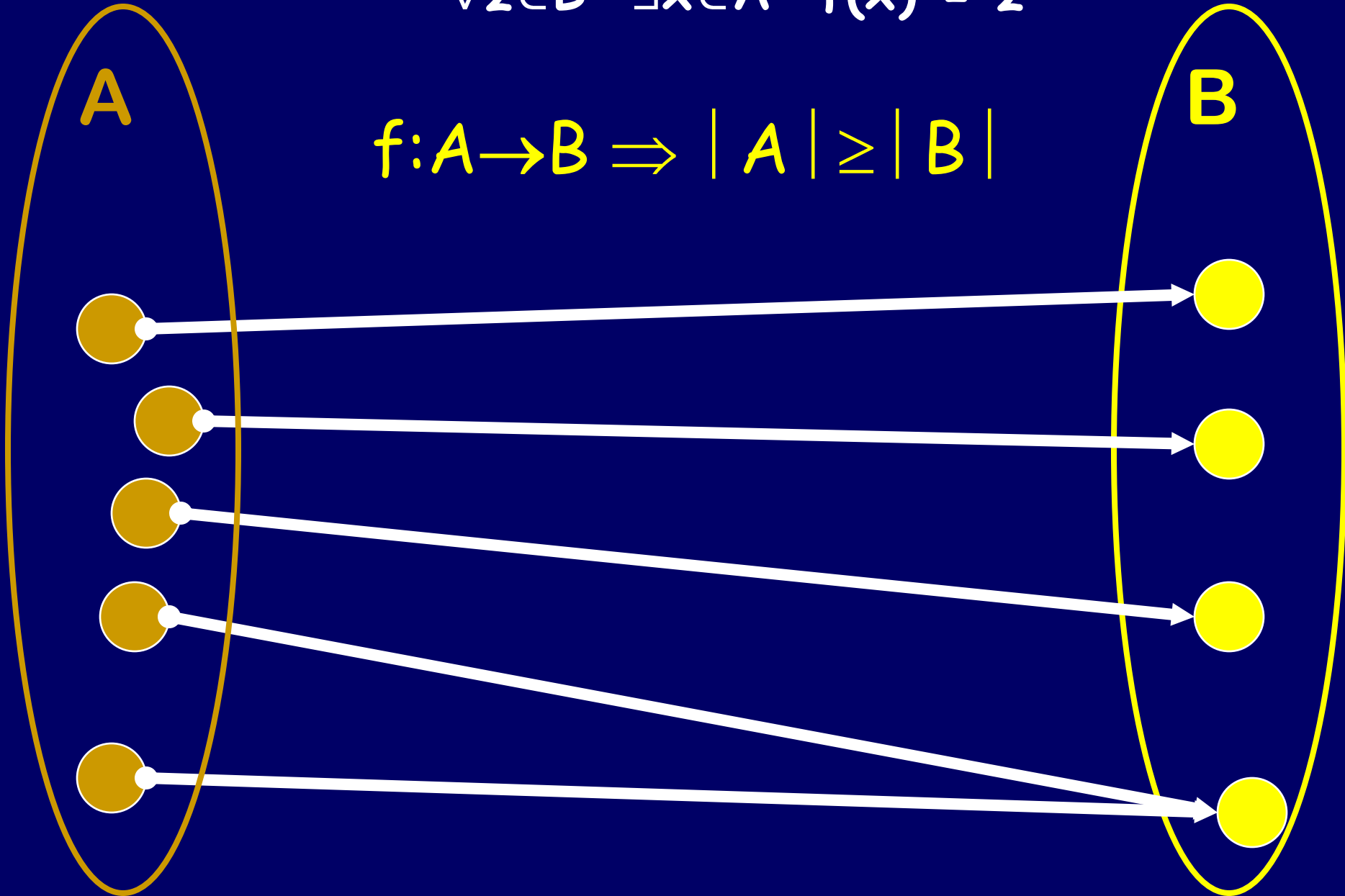
$$f: A \rightarrow B \Rightarrow |A| \leq |B|$$



f is **onto** (or **surjective**) if and only if

$$\forall z \in B \quad \exists x \in A \quad f(x) = z$$

$$f: A \rightarrow B \Rightarrow |A| \geq |B|$$

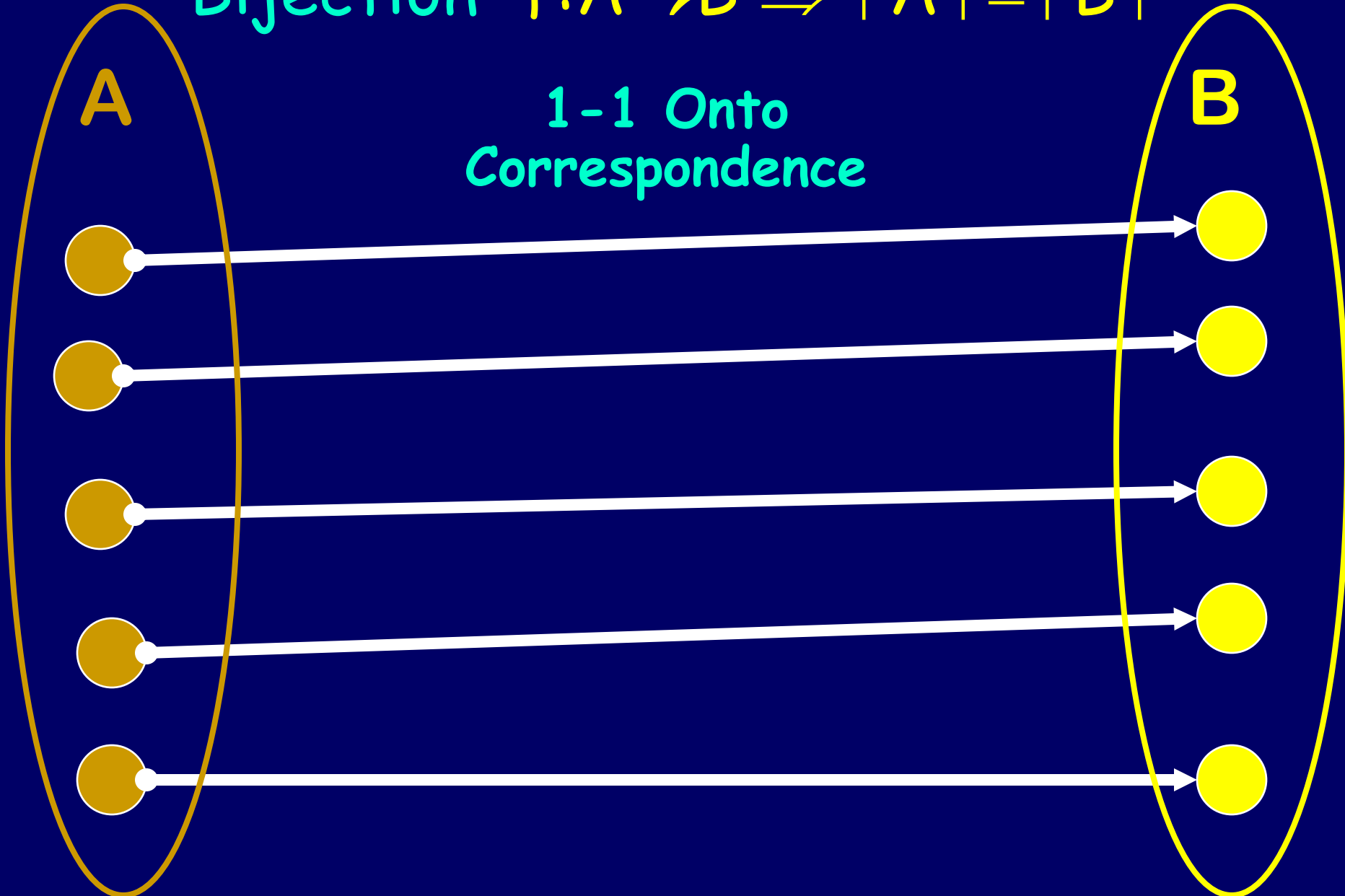


Bijection $f:A \rightarrow B \Rightarrow |A| = |B|$

A

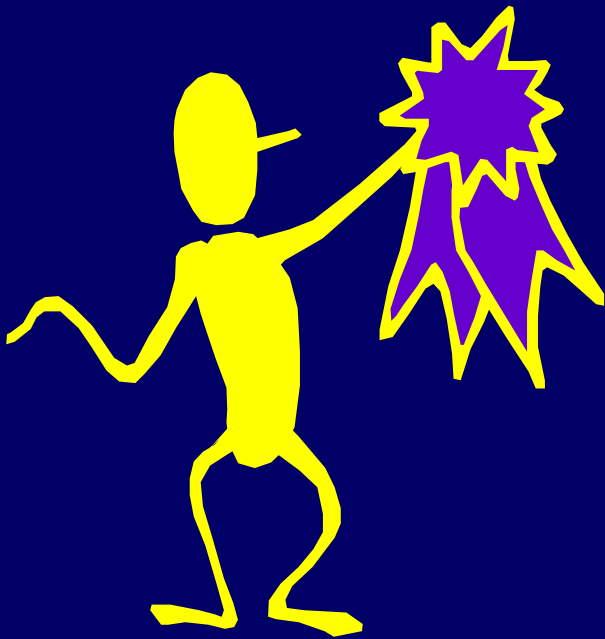
B

1-1 Onto
Correspondence



Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.



**It's one of the
most important
mathematical
ideas of all time!**

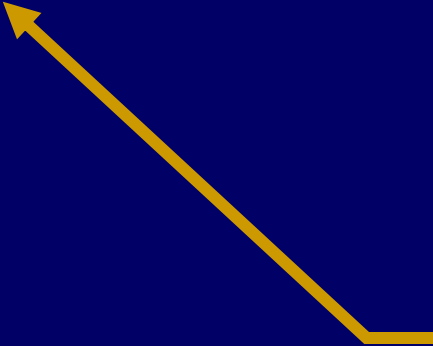
Question: how many n-bit binary sequences are there?

000000	$\leftrightarrow$	0
000001	$\leftrightarrow$	1
000010	$\leftrightarrow$	2
000011	$\leftrightarrow$	3
...		
1...1111	$\leftrightarrow$	$2^n - 1$

2^n sequences

$S = \{a,b,c,d,e\}$ has many subsets.

$\emptyset, \{a\}, \{a,b\}, \{a,d,e\}, \{a,b,c,d,e\}, \dots$



The empty set is a set with all the rights and privileges pertaining thereto.

Question: how many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

{b c e}

1 means "TAKE IT"
0 means "LEAVE IT"

Question: How many subsets can be formed from the elements of a 5-element set?

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

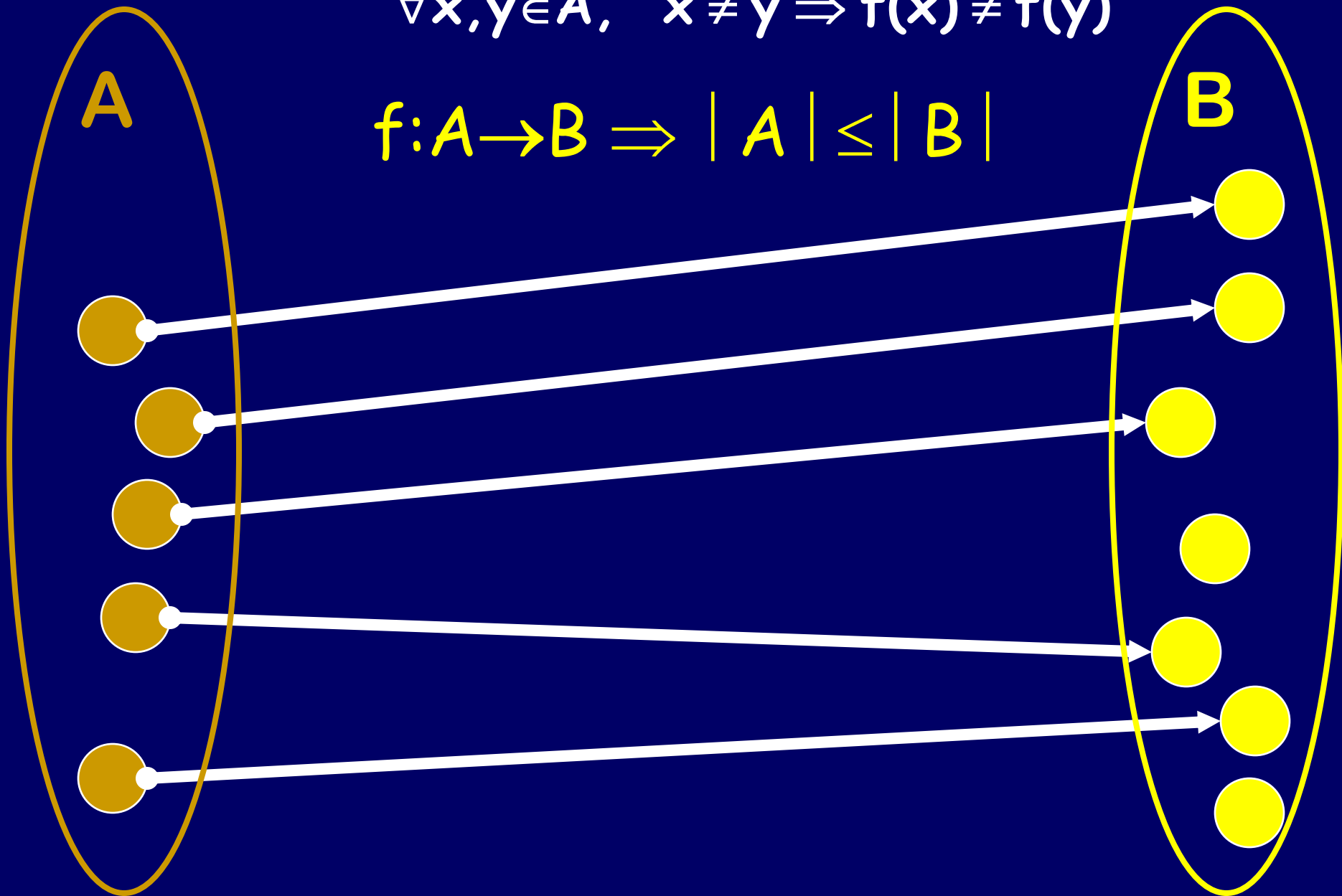
$f: \text{bit strings} \rightarrow \text{subsets}$

How do we prove bijection?

f is **1-1** (or **injective**) if and only if

$$\forall x, y \in A, \quad x \neq y \Rightarrow f(x) \neq f(y)$$

$$f: A \rightarrow B \Rightarrow |A| \leq |B|$$



$S = \{a_1, a_2, a_3, \dots, a_n\}$, $T =$ all subsets of S
 $B =$ set of all n -bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1b_2b_3\dots b_n$, let $f(b) = \{a_i \mid b_i=1\}$

Claim: f is injective

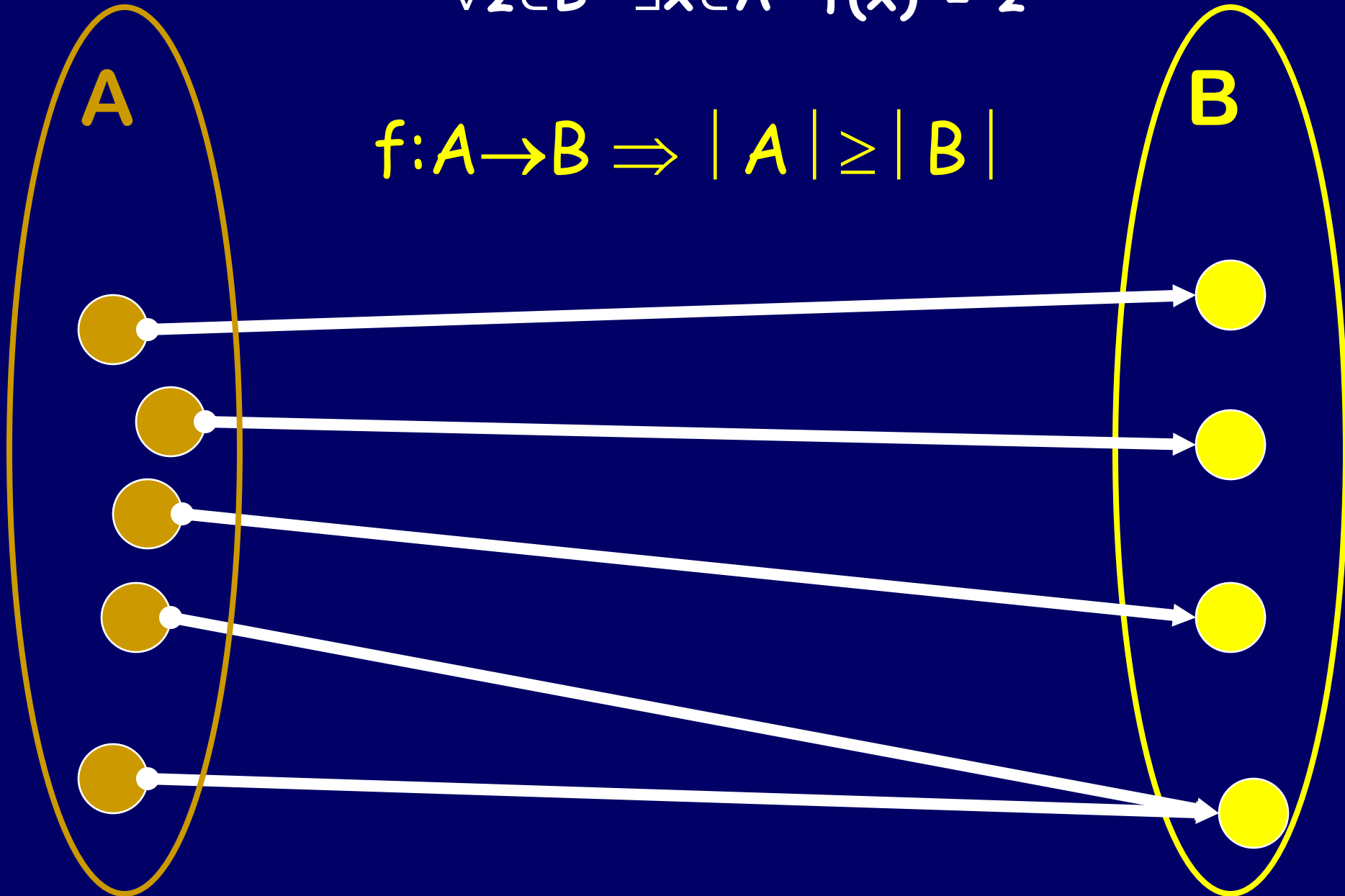
Any two distinct binary sequences b and b'
have a position **i** at which they differ

Hence, $f(b)$ is not equal to $f(b')$ because
they disagree on element **a_i**

f is **onto** (or **surjective**) if and only if

$$\forall z \in B \quad \exists x \in A \quad f(x) = z$$

$$f: A \rightarrow B \Rightarrow |A| \geq |B|$$



$S = \{a_1, a_2, a_3, \dots, a_n\}$, $T =$ all subsets of S
 $B =$ set of all n -bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1b_2b_3\dots b_n$, let $f(b) = \{a_i \mid b_i=1\}$

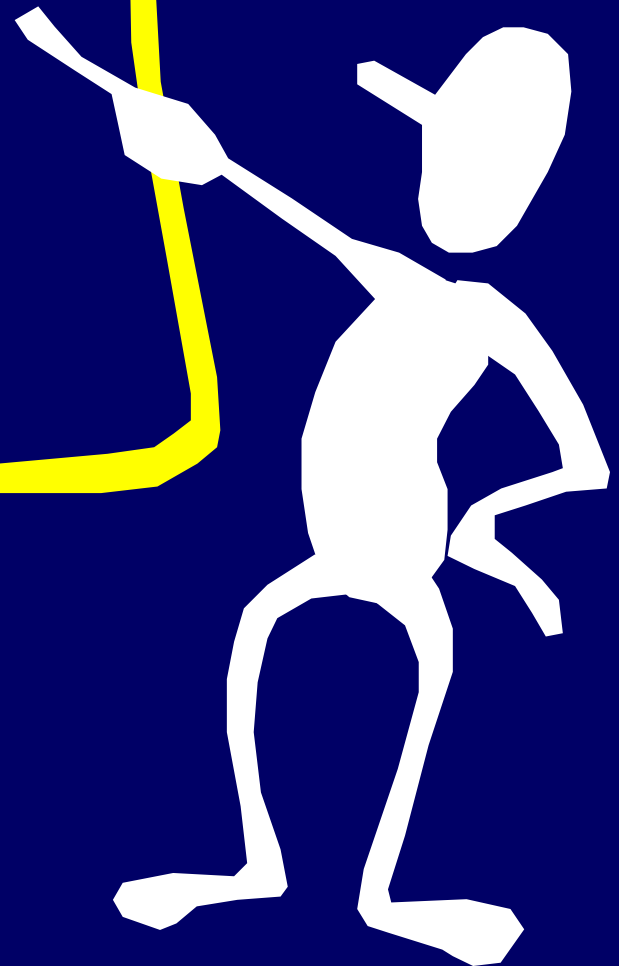
Claim: f is surjective

Let X be a subset of $\{a_1, \dots, a_n\}$.

Define $b_k = 1$ if a_k in X and $b_k = 0$ otherwise.

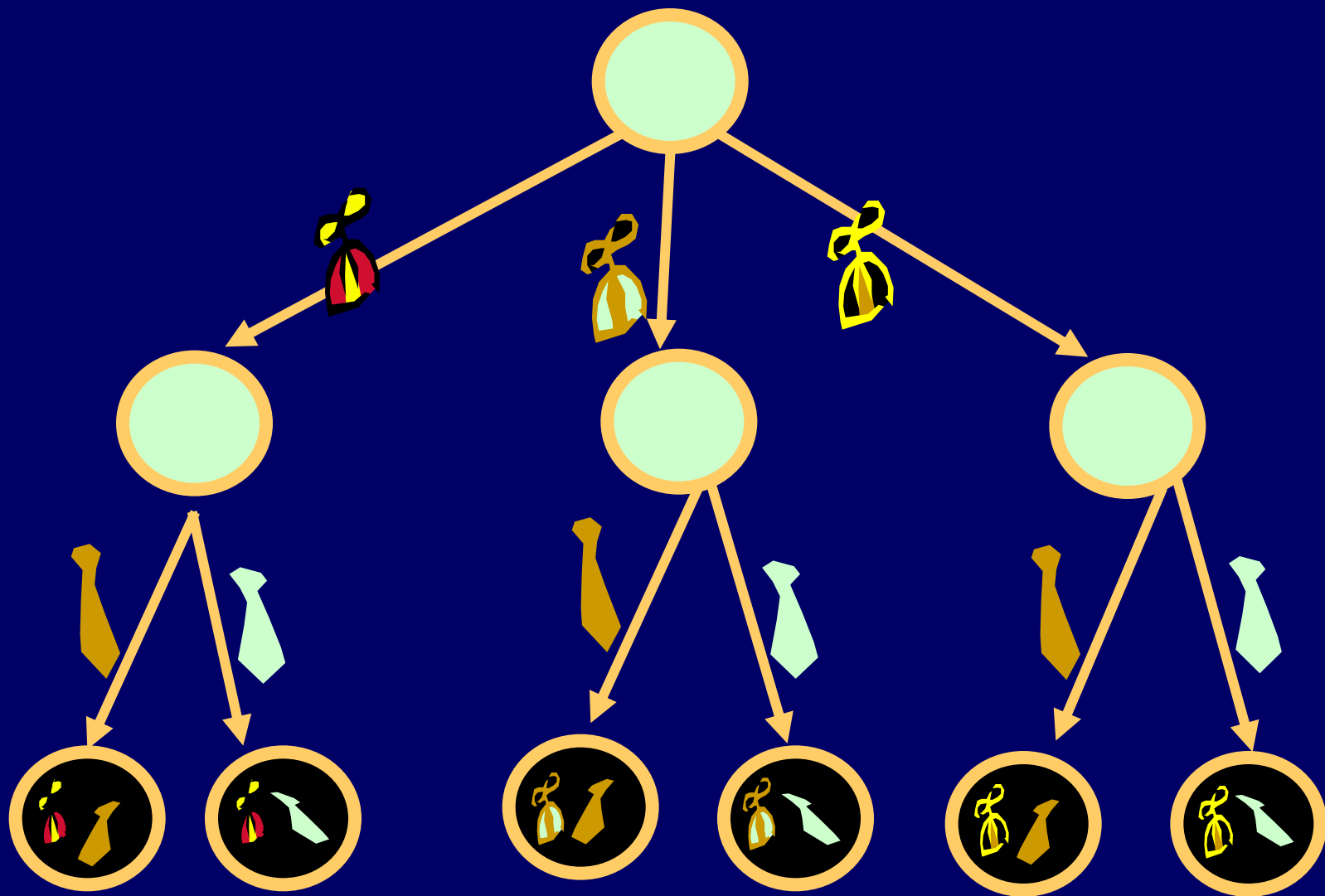
Note that $f(b_1b_2\dots b_n) = X$.

The number
of subsets of
an n -element
set is 2^n .

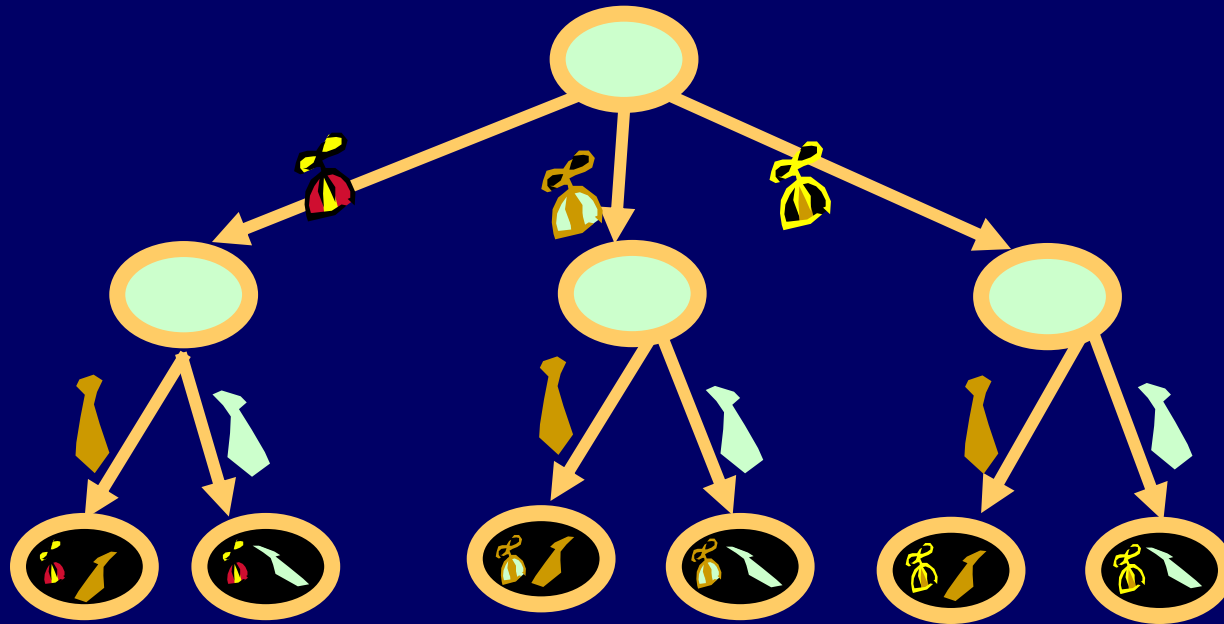




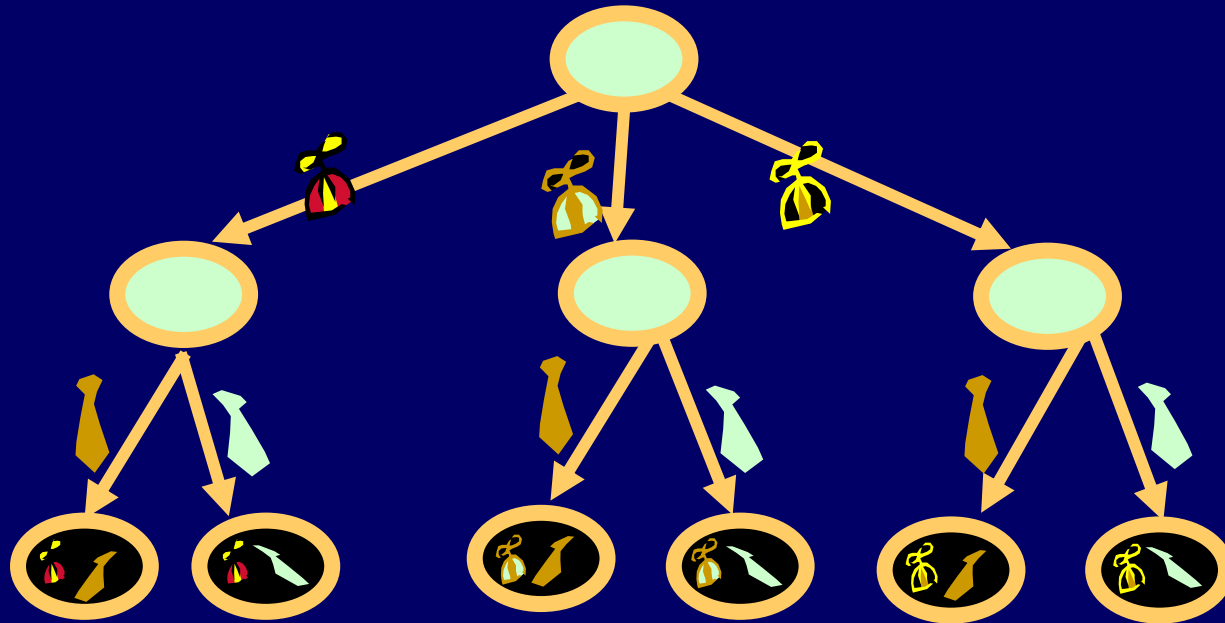
I own 3 beanies and 2 ties.
How many different ways can I
dress up in a beanie and a tie?



Choice Tree



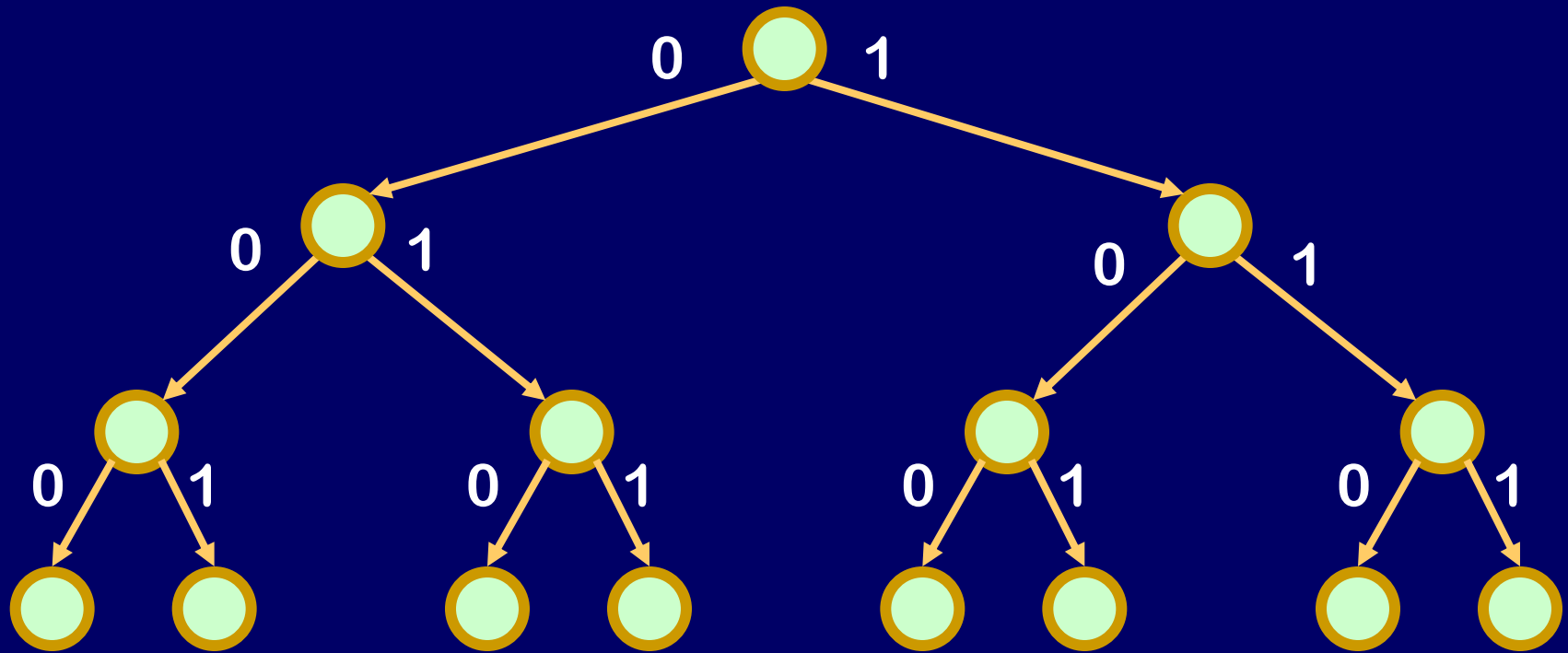
A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a choice tree representation of a set S , if

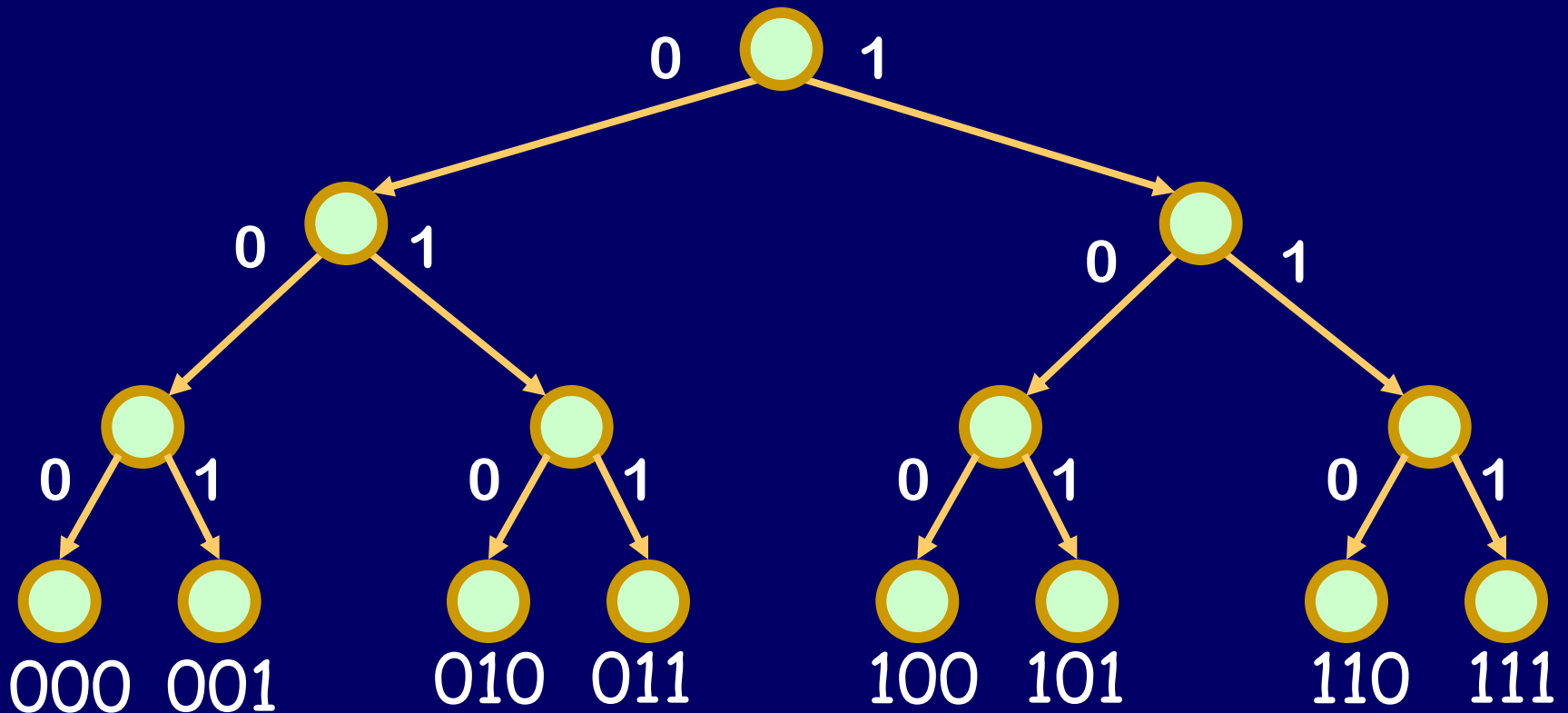
- 1) Each leaf label is in S
- 2) No two leaf labels are the same

Choice Tree for 2^n n-bit sequences



We can use a "choice tree" to represent the construction of objects of the desired type.

2^n n-bit sequences



Label each leaf with the object constructed by the choices along the path to the leaf.

Product Rule

If S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

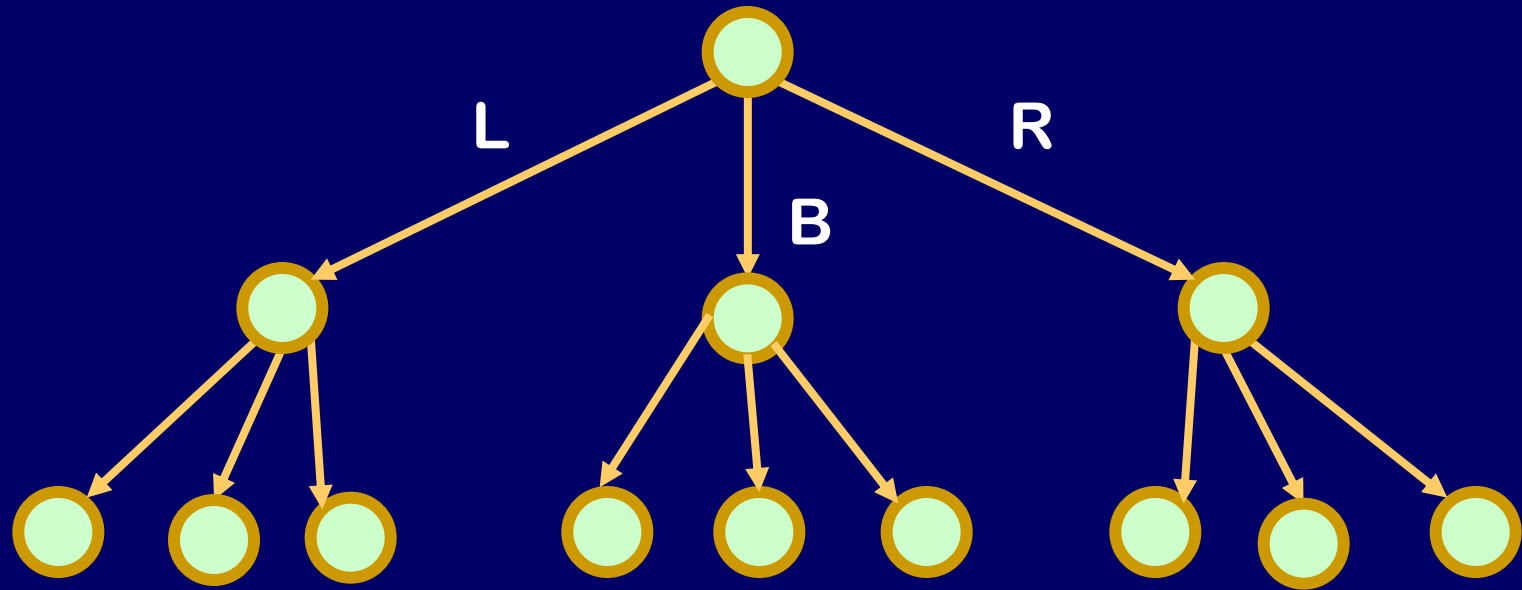
THEN

there are $P_1 P_2 P_3 \dots P_n$ objects in S



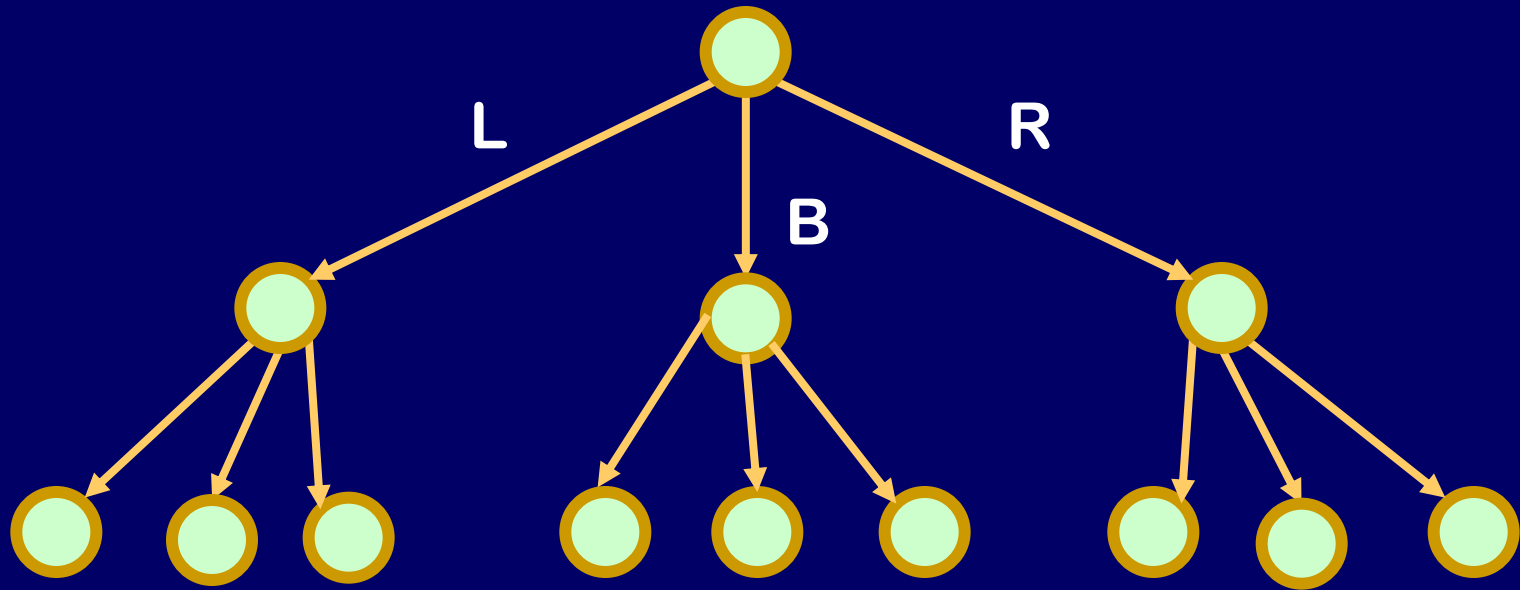
You have 13 coins and a balance scale. One coin is fake. You are asked to find that coin. What is a good **lower** bound on the number of weighings?

13 coins problem



L - left sinks, R - right sinks, B - balance

13 coins problem



In our specific problem, there are 13 coins, thus 13×2 situations.

Since $3^3=27$, we need at least 3 weighings.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \dots P_n$ objects of type S .

How many different orderings of deck with 52 cards?

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

50 possible choices for the third card;

...

1 possible choice for the 52nd card.

How many different orderings of deck with 52 cards?

By the product rule:

$$52 * 51 * 50 * \dots * 3 * 2 * 1 = 52!$$

A permutation or arrangement of n objects is an ordering of the objects.

The number of permutations of n distinct objects is $n!$



How many sequences of 7
letters are there?

$$26^7$$

26 choices for each
of the 7 positions



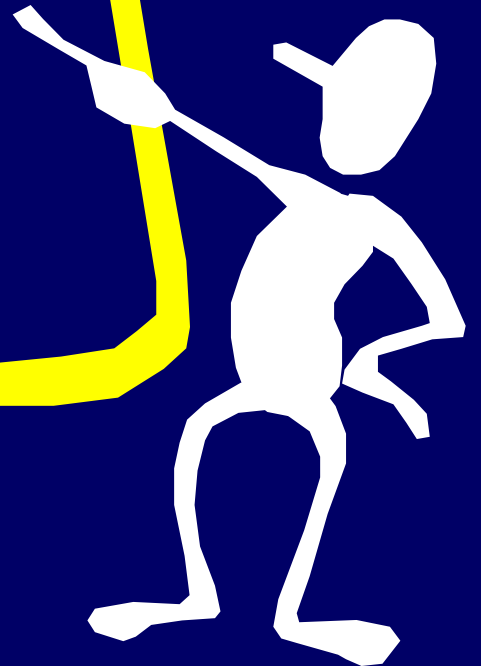
How many sequences of 7
letters contain at least two
of the same letter?

$$26^7 - 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$$



number of sequences
containing
all different letters

Sometimes it is easiest to count the number of objects with property Q , by counting the number of objects that do not have property Q .



Ordered Versus Unordered

From a set of $\{1,2,3\}$ how many **ordered** pairs can be formed?

$\{1,2\}, \{1,3\}, \{2,1\}, \{2,3\}, \{3,1\}, \{3,2\}$

How many **unordered** pairs?

3×2 divide by overcount = 3

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many **ordered** pairs can be formed?

- $52 * 51$

How many **unordered** 5 card hands?

- $52 * 51 * 50 * 49 * 48 / 5!$ ← divide by overcount

Number of ways of ordering,
permuting,
or arranging r out of n objects

n choices for the first place, $n-1$ for
the second and so on

$$n \times (n-1) \times \dots \times (n-(r-1)) = \frac{n!}{(n-r)!}$$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects.

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n choose r



Permutations vs. Combinations

$$\frac{n!}{(n-r)!}$$

Ordered

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Unordered

How many 8 bit sequences
have two 0's and six 1's?

1) Choose the set of 2 positions to put
the 0's. The 1's are forced.

$$\binom{8}{2}$$

2) Choose the set of 6 positions to put
the 1's. The 0's are forced.

$$\binom{8}{6}$$

How many 8 bit sequences
have two 0's and six 1's?

Tempting, but incorrect:

8 ways to place first 0 times

7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and
then position j for the second 0 gives
the same sequence as choosing position
 j for the first 0 and position i for the
second.

How Many Hands Have at Least 3 As?

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

= 4 ways of picking 3 out of 4 aces

$$\begin{pmatrix} 49 \\ 2 \end{pmatrix}$$

= 1176 ways of picking 2 cards out of the remaining $52-3=49$ cards

$$4 \times 1176 = 4704$$

How Many Hands Have at Least 3 As?

Second counting...

How many hands have exactly 3 aces?

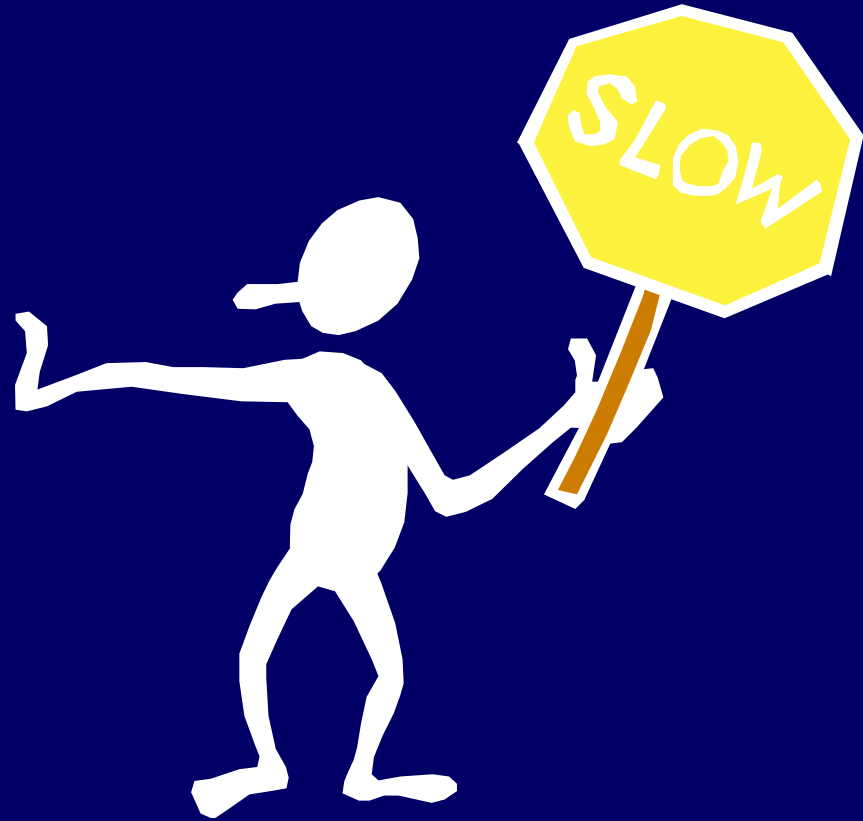
$$\binom{4}{3} \binom{48}{2} = 4 \times 1128 = 4512$$

How many hands have exactly 4 aces? 48

$$\text{Thus, } 4512 + 48 = 4560 \neq 4704$$

4704 $\neq$ 4560

One of the two
counting
arguments is not
correct.



Review the first counting argument

1. Choose 3 of 4 aces

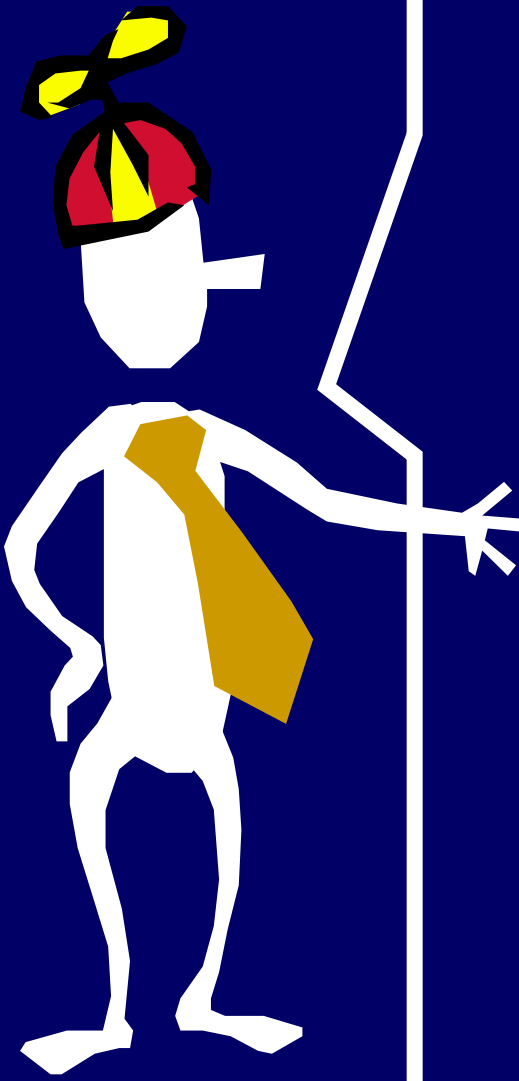
2. Choose 2 of the remaining 49 cards

$A_{\clubsuit} A_{\diamondsuit} A_{\heartsuit}$	$A_{\spadesuit} K_{\diamondsuit}$
$A_{\clubsuit} A_{\diamondsuit} A_{\spadesuit}$	$A_{\heartsuit} K_{\diamondsuit}$
$A_{\clubsuit} A_{\spadesuit} A_{\heartsuit}$	$A_{\diamondsuit} K_{\diamondsuit}$
$A_{\spadesuit} A_{\diamondsuit} A_{\heartsuit}$	$A_{\clubsuit} K_{\diamondsuit}$

Four different sequences of choices produce the same hand

Is the other
argument correct?
How do I avoid
fallacious reasoning?

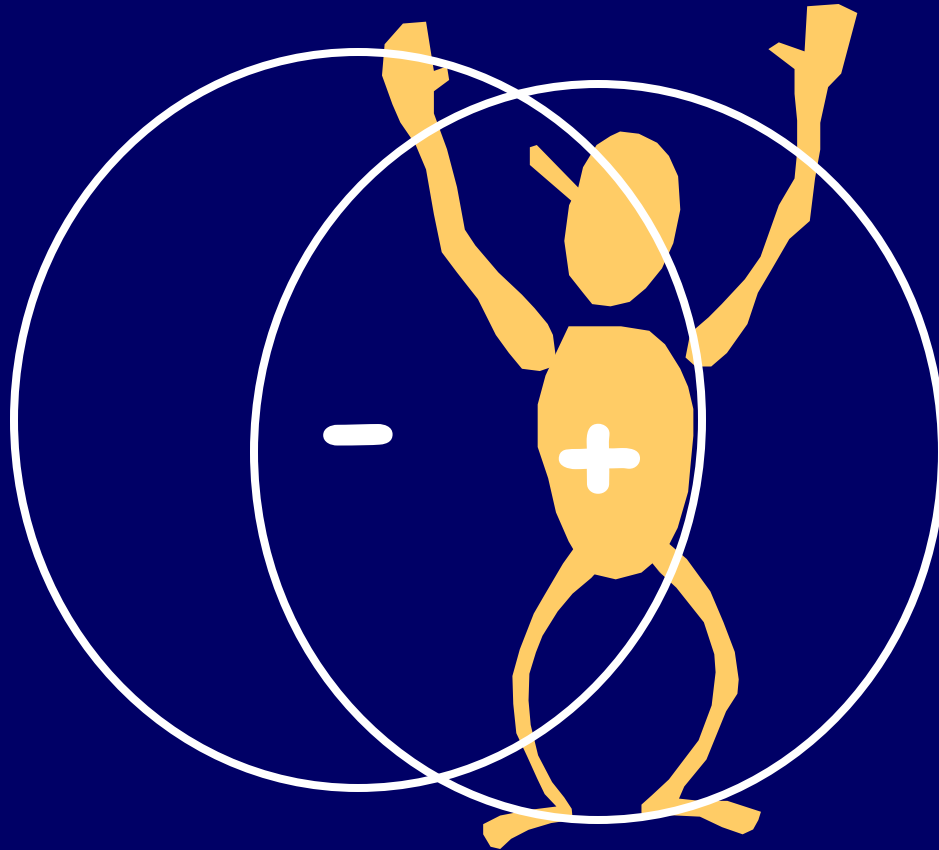




The three big mistakes people make in counting are:

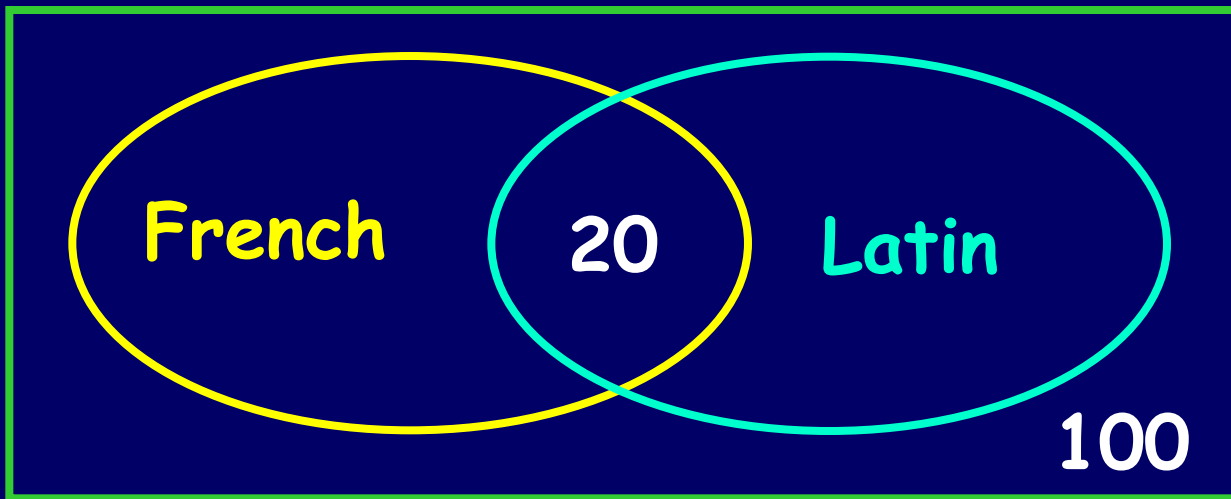
1. Creating objects not in S
2. Missing out some objects from the set S
3. Creating the same object two different ways

To Exclude Or Not To Exclude?





A school has 100 students.
50 take French, 40 take
Latin, and 20 take both.
How many students take
neither language?



F $\equiv$ French students

L $\equiv$ Latin students

$$|F|=50 \quad |L|=40$$

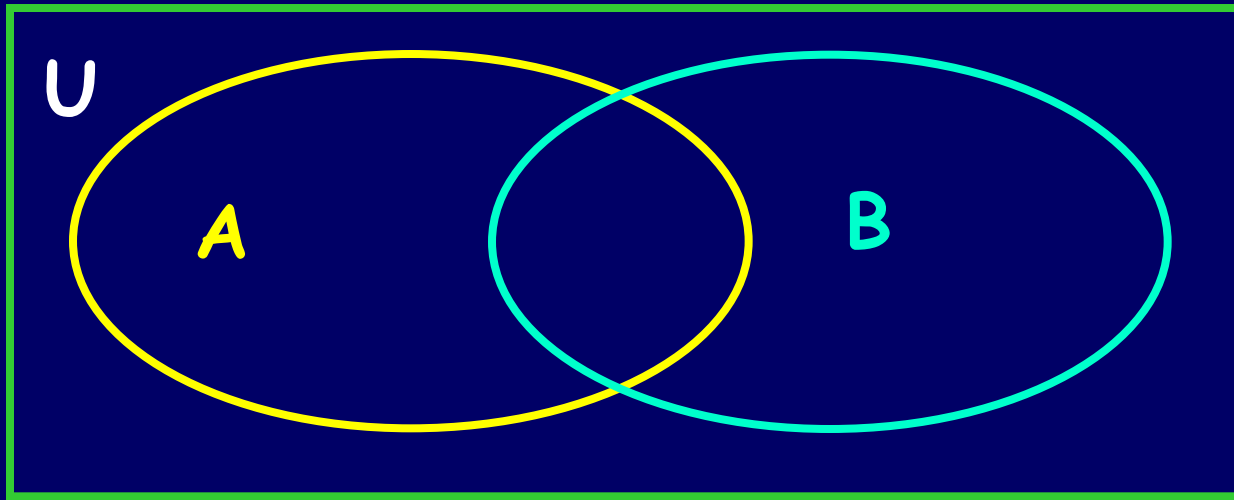
French AND Latin students: $|F \cap L|=20$

French OR Latin students:

$$\begin{aligned} |F \cup L| &= |F| + |L| - |F \cap L| \\ &= 50 + 40 - 20 = 70 \end{aligned}$$

Neither language:

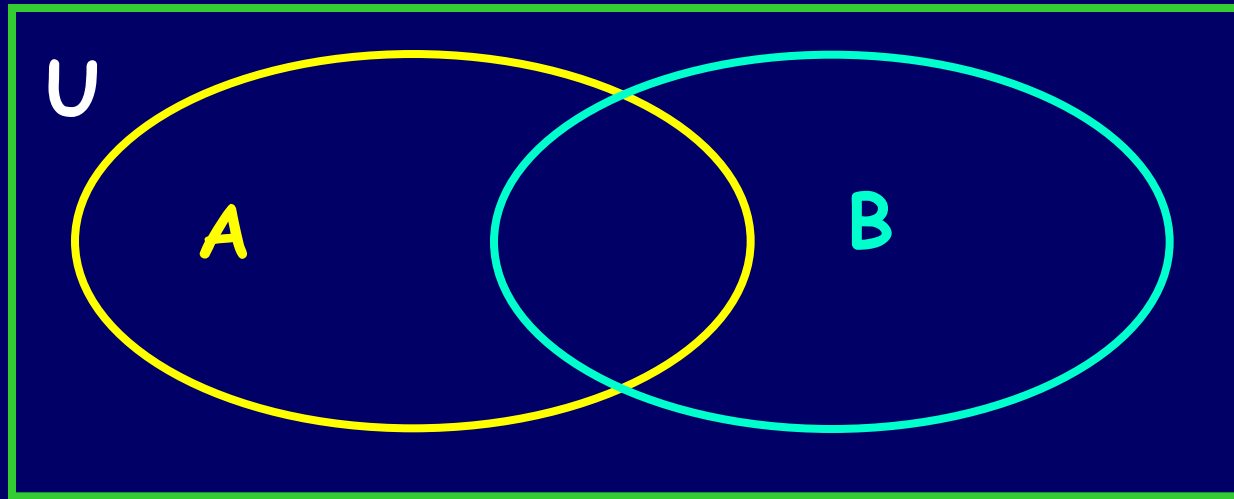
$$100 - 70 = 30$$



Lesson: The Principle of Inclusion and Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The number of elements in at least one of sets A or B



$U \equiv$ universe of elements

$$\bar{A} = U - A$$

$$\bar{B} = U - B$$

Lesson: The Principle of Inclusion and Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|$$

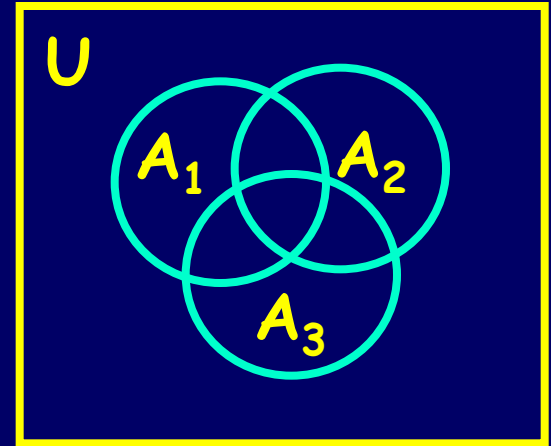
How many positive integers less than 70 are relatively prime to 70?

$$U = [1..70], 70 = 2 \times 5 \times 7$$

$A_1 \equiv$ integers in U divisible by 2

$A_2 \equiv$ integers in U divisible by 5

$A_3 \equiv$ integers in U divisible by 7



$$|A_1| = 35 \quad |A_2| = 14 \quad |A_3| = 10$$

$$|A_1 \cap A_2| = 7$$

$$|A_1 \cap A_3| = 5$$

$$|A_2 \cap A_3| = 2$$

$$|A_1 \cap A_2 \cap A_3| = 1$$

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - \\ &|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + \\ &|A_1 \cap A_2 \cap A_3| \end{aligned}$$

How many positive integers less than 70 are relatively prime to 70?

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3| = 35 + 14 + 10 - 7 - 5 - 2 + 1 = 46$$

Thus, the number of relatively prime to 70 is $70 - 46 = 24$.

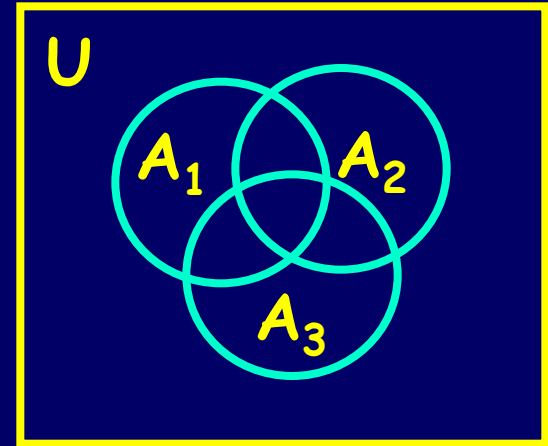
The Principle of Inclusion and Exclusion

Let S_k be the sum of the sizes of
All k -tuple intersections of the A_i 's.

$$S_1 = |A_1| + |A_2| + |A_3|$$

$$S_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$S_3 = |A_1 \cap A_2 \cap A_3|$$



$$|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3$$



- Correspondence Principle
- Choice Tree
- Product Rule
- Binomial coefficients
- The Principle of Inclusion and Exclusion

Study Bee