Great Theoretical Ideas In Computer Science		
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Lecture 3	Jan 19, 2010	Carnegie Mellon University

Unary and Binary

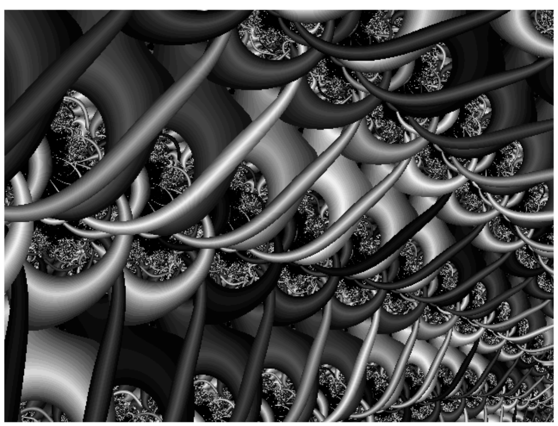
Oh No!

Homework #1 is due today at 11:59pm
Give yourself sufficient time to make PDF

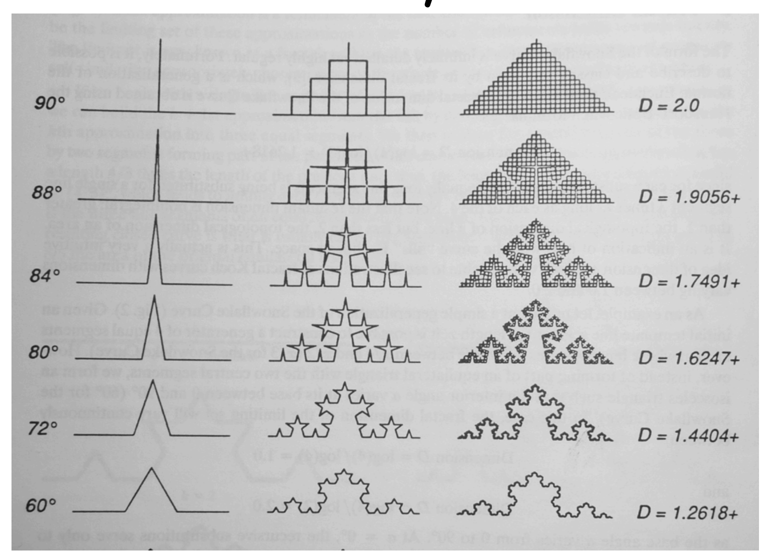
Quiz #1 is next Thursday during lecture

More on Fractals

Fractals are geometric objects that are self-similar, i.e. composed of infinitely many pieces, all of which look the same.



The Koch Family of Curves



Fractal Dimension

We can break a line segment into N self-similar pieces, and each of which can be magnified by a factor of N to yield the original segment.

We can break a square into N^2 self-similar pieces, and each of which can be magnified by a factor of N.

Fractal Dimension

The dimension is the exponent of the number of self-similar pieces with magnification factor into which the figure may be broken.

$$\boxed{\#\text{ of self-similar pieces} = mf^{\dim}}$$

$$\boxed{\dim = \frac{\ln(\#\text{ of self-similar pieces})}{\ln(\text{magnification factor})}}$$

Hausdorff dimension

Fractal Dimension of the Plane

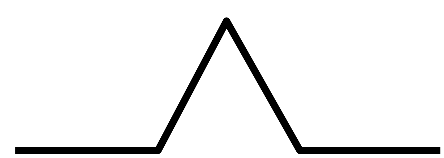
We can break a square into N^2 self-similar pieces, and each of which can be magnified by a factor of N.

$$\dim = \frac{\ln(N^2)}{\ln(N)} = 2$$

Fractal Dimension of the Koch Curve

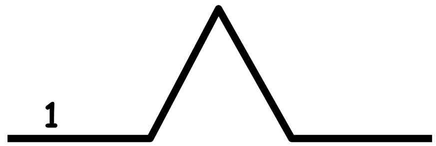
We begin with a straight line of length 1

Remove the middle third of the line, and replace it with two lines that each have the same length



Repeat infinitely

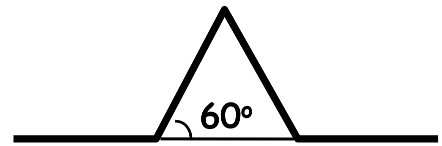
Fractal Dimension of the Koch Curve



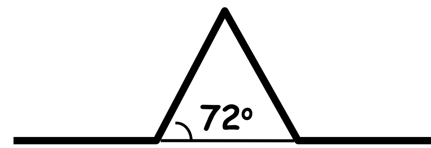
$$\dim = \frac{\ln(\#\text{ of self-similar pieces})}{\ln(\text{magnification factor})}$$

$$\dim = \frac{\ln(4)}{\ln(3)} = 1.26$$

The Koch Family of Curves

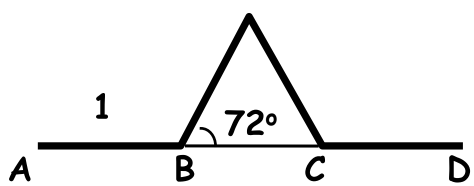


What if we increase that angle but keep all sides of the equal length?



Compute its dimension!

The Koch Family of Curves

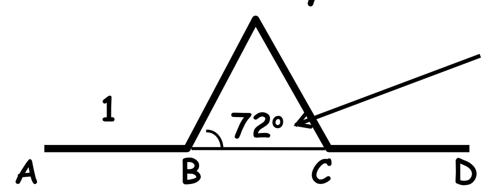


|AD| = 2 + |BC|

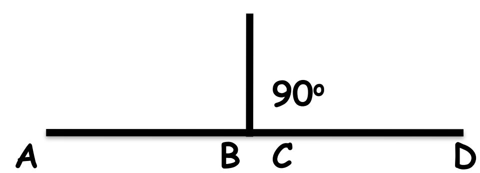
|BC| = 2 cos(72°)

$$\dim = \frac{\ln(4)}{\ln(2 + 2\cos(72^\circ))} = 1.44$$

The Koch Family of Curves



Increase this angle



$$\dim = \frac{\ln(4)}{\ln(2 + 2\cos(90^\circ))} = 2$$

Plane-filling curve

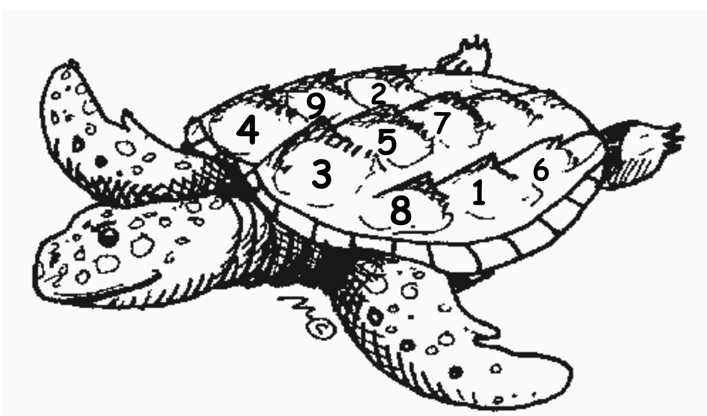
Unary and Binary

How to play the 9 stone game?

9 stones, numbered 1-9
Two players alternate moves.
Each move a player gets to take a new stone

Any subset of 3 stones adding to 15, wins.

Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu in ancient China



Magic Square: Any 3 in a vertical, horizontal, or diagonal line add up to 15.

4	9	2
3	5	7
8	1	6

Conversely,
any 3 that add to 15 must be on a line.

4	9	2
3	5	7
8	1	6

TIC-TAC-TOE on a Magic Square
Represents The Nine Stone Game

Alternate taking squares 1-9.
Get 3 in a row to win.

4	9	2
3	5	7
8	1	6

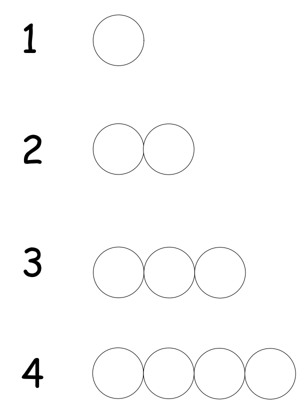
Basic Idea of This Lecture

Don't stick with the representation in which you encounter problems!

Always seek the more useful one!

This idea requires a lot of practice

Prehistoric Unary



Consider the problem of finding a formula for the sum of the first n numbers

You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$

A different approach…

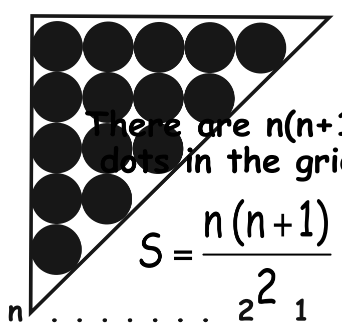
$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \dots & + & 2 & + & 1 & = & S \\
\hline
n+1 & + & n+1 & + & n+1 & + & \dots & + & n+1 & + & n+1 & = & 2S
\end{array}
$$

$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

$$\boxed{1 + 2 + 3 + \dots + n-1 + n = S}$$

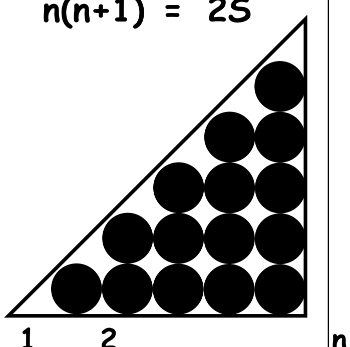
$$\boxed{n + n-1 + n-2 + \dots + 2 + 1 = S}$$



There are n(n+1) dots in the grid!

$$S = \frac{n(n+1)}{2}$$

$$n(n+1) = 2S$$



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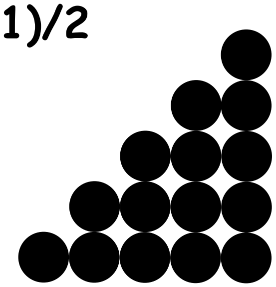
n^{th} Triangular Number

$$\Delta_n = 1 + 2 + 3 + \dots + n-1 + n$$

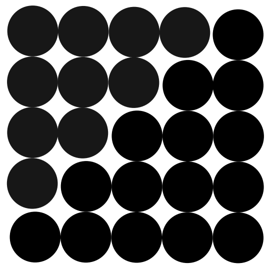
$$= n(n+1)/2$$

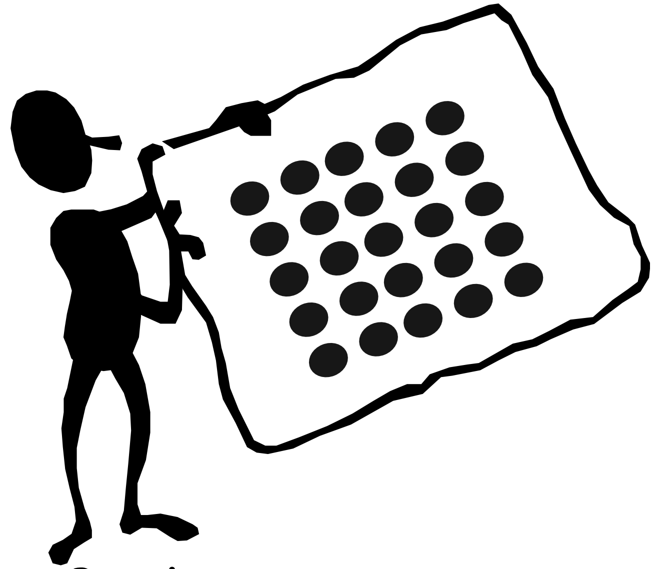


n^{th} Square Number

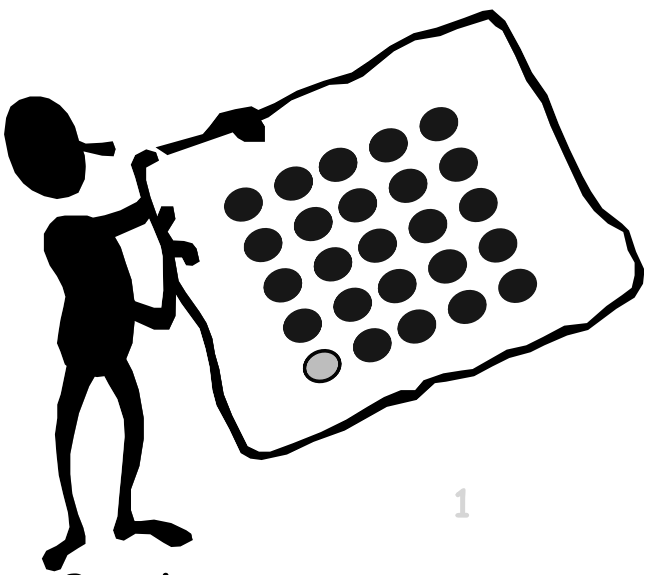
$$\sim_n = n^2$$

$$= \Delta_n + \Delta_{n-1}$$



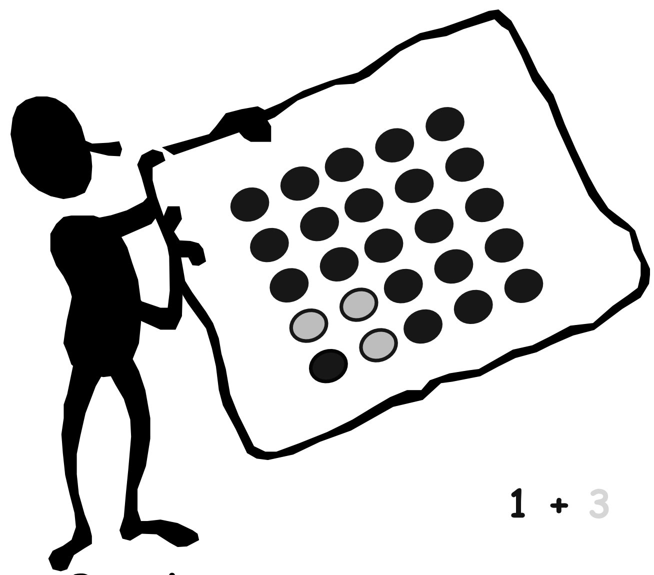


Breaking a square up in a new way



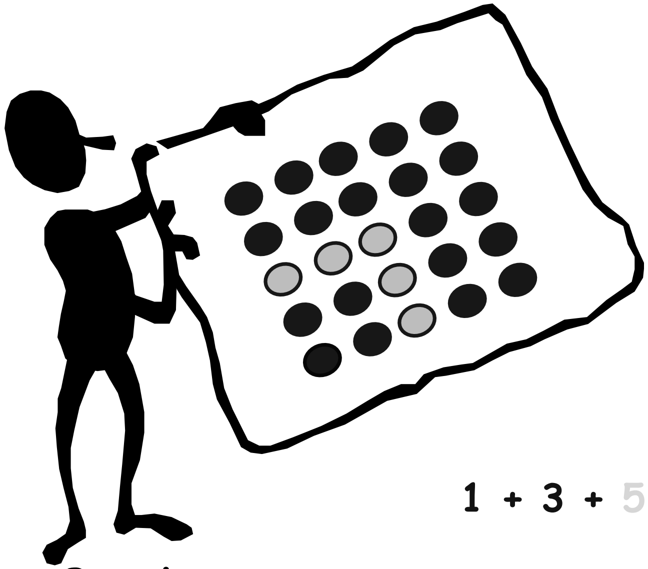
1

Breaking a square up in a new way



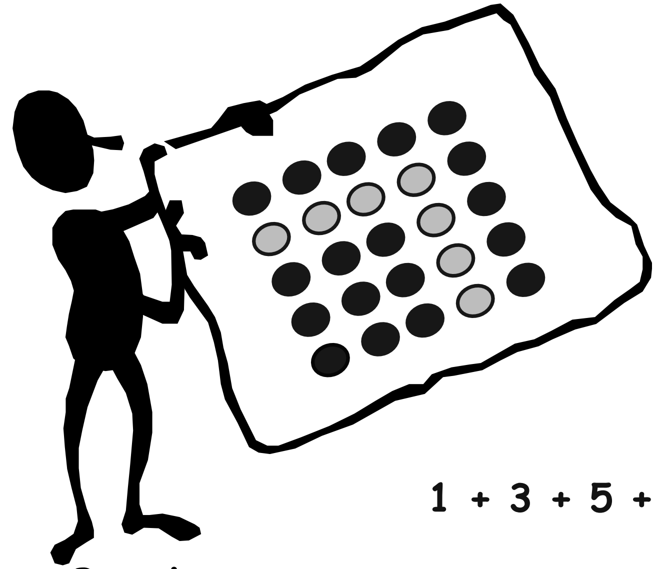
1 + 3

Breaking a square up in a new way



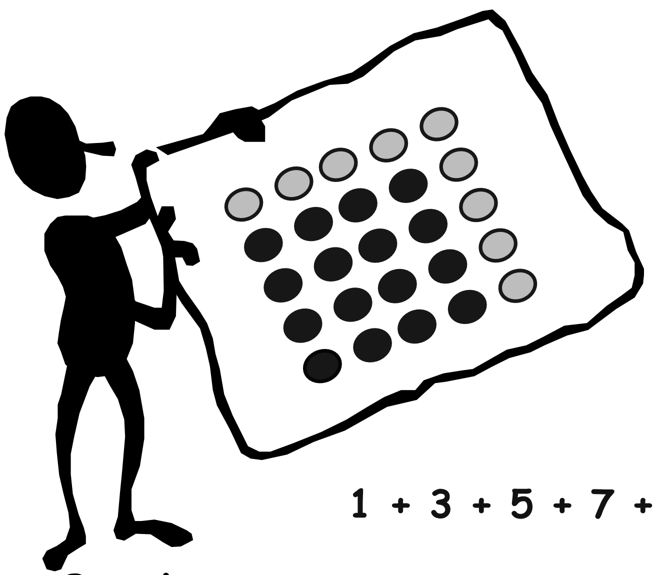
$1 + 3 + 5$

Breaking a square up in a new way



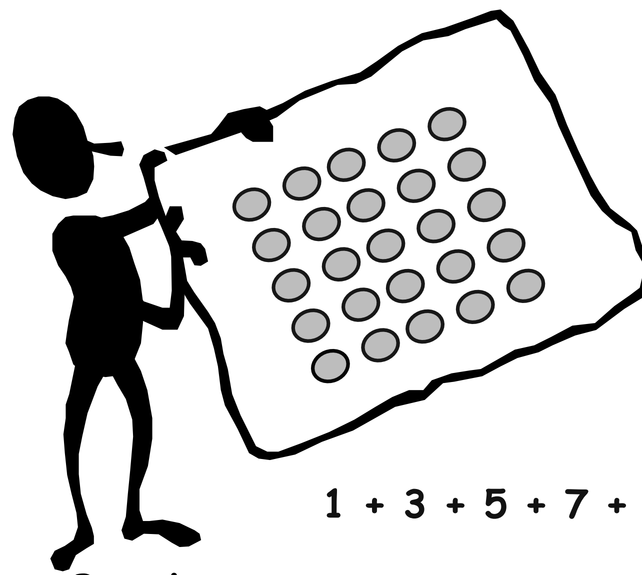
$1 + 3 + 5 + 7$

Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9 = ?$

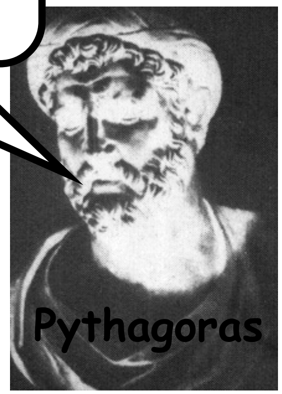
Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9 = 5^2$

Breaking a square up in a new way

The sum of the first n odd numbers is n^2

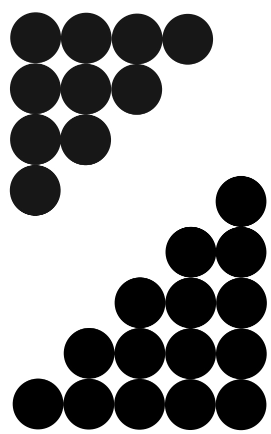


Here is an alternative dot proof of the same sum….

n^{th} Square Number

$\tilde{}_n = \Delta_n + \Delta_{n-1}$

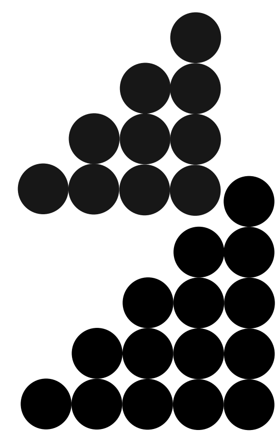
$= n^2$



n^{th} Square Number

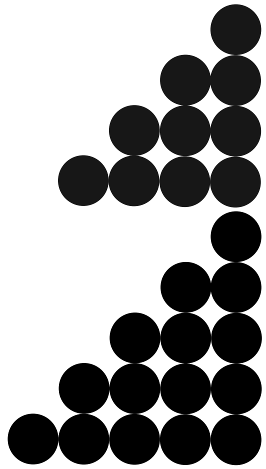
$\tilde{}_n = \Delta_n + \Delta_{n-1}$

$= n^2$



n^{th} Square Number

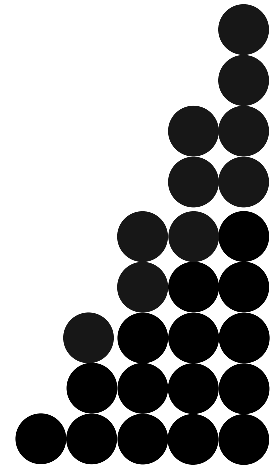
$\tilde{\ }_n = \Delta_n + \Delta_{n-1}$



n^{th} Square Number

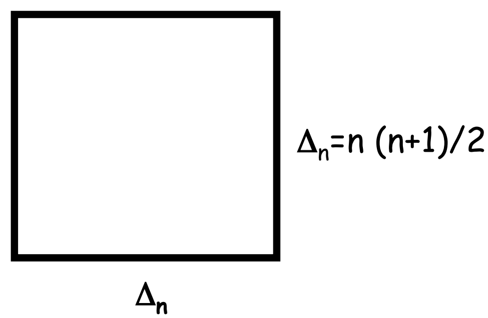
$\tilde{\ }_n = \Delta_n + \Delta_{n-1}$

= Sum of first n odd numbers

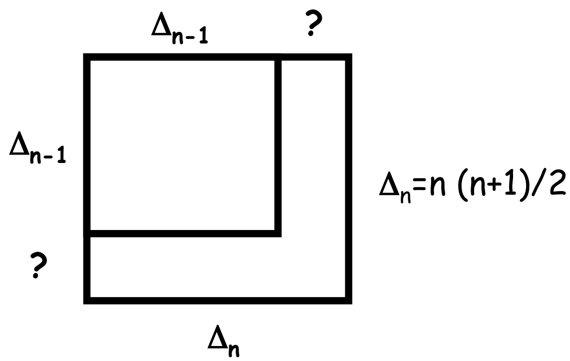


We find a formula for the sum of the first n cubes.

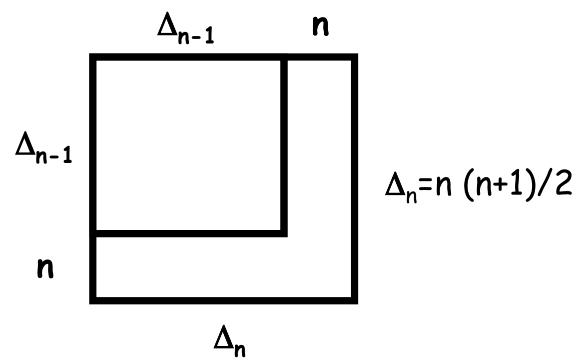
Area of square = $(\Delta_n)^2$



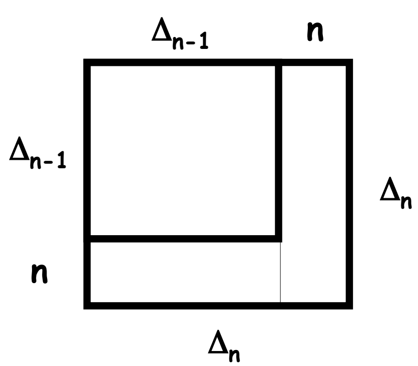
Area of square = $(\Delta_n)^2$



Area of square = $(\Delta_n)^2$

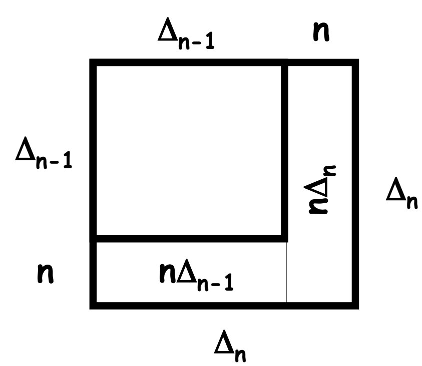


Area of square = $(\Delta_n)^2$

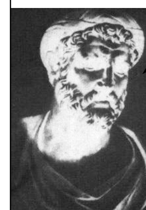


Area of square = $(\Delta_n)^2$

$$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$$
$$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$$
$$= (\Delta_{n-1})^2 + n(n^2)$$
$$= (\Delta_{n-1})^2 + n^3$$



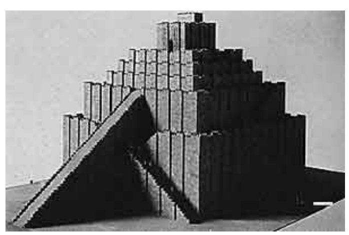
$$(\Delta_n)^2 = n^3 + (\Delta_{n-1})^2$$
$$= n^3 + (n-1)^3 + (\Delta_{n-2})^2$$
$$= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2$$
$$= n^3 + (n-1)^3 + (n-2)^3 + \dots + 1^3$$



$$(\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3$$
$$= [\ n(n+1)/2\ ]^2$$

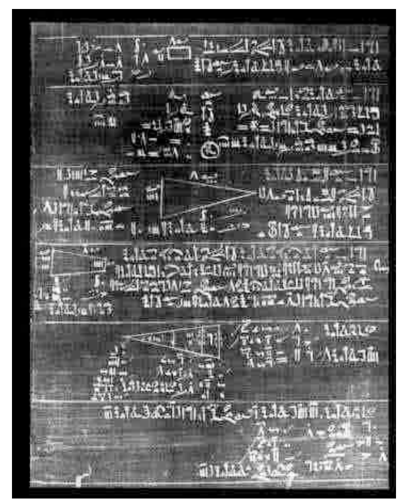
Can you find a formula for the sum of the first n squares?

Babylonians needed this sum to compute the number of blocks in their pyramids



Rhind Papyrus

Scribe Ahmes was Martin Gardner of his day!



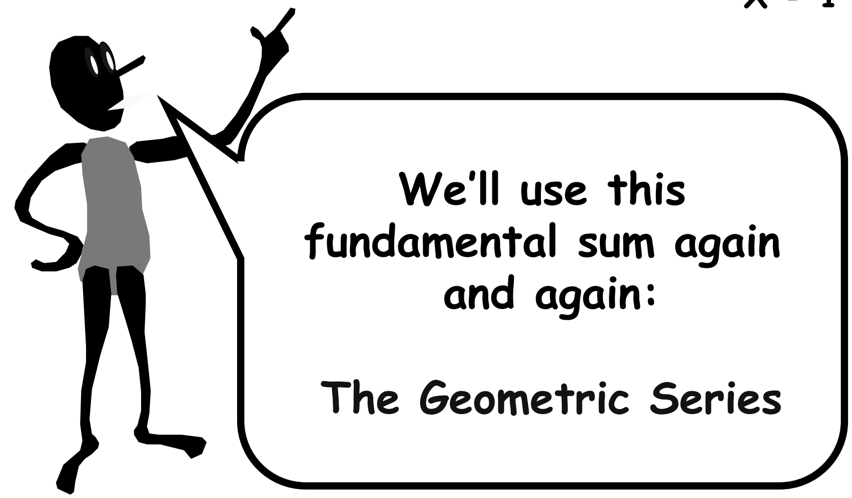
Rhind Papyrus

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

What is a closed form of the sum of powers of integers?

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



Proof

$$(X-1)(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$= X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n$$

$$- \quad 1 - X^1 - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1}$$

$$= X^n - 1$$

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

(when $x \neq 1$)

Geometric Series for x=2

$$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$$

Geometric Series for $x=\frac{1}{2}$

$$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}=2-\frac{1}{2^n}$$

Two Case Studies

Bases and Representation

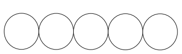
BASE X Representation

$S = a_{n-1}\ a_{n-2}\ \dots\ a_1\ a_0$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ .\ .\ .\ + a_0 X^0$$

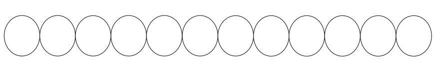
Base 2 [Binary Notation]

101 represents: $1\ (2)^2 + 0\ (2^1) + 1\ (2^0)$

= 

Base 7

015 represents: $0\ (7)^2 + 1\ (7^1) + 5\ (7^0)$

= 

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE X Representation

$S = (a_{n-1}\ a_{n-2} \dots a_1\ a_0)_X$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ .\ .\ .\ + a_0 X^0$$

Largest number representable in base-X with n "digits"

$$= (X-1\ X-1\ X-1\ X-1\ X-1 \dots X-1)_X$$

$$= (X-1)(X^{n-1} + X^{n-2} + \ .\ .\ .\ + X^0)$$

$$= (X^n - 1)$$

Fundamental Theorem For Binary

Each of the numbers from 0 to 2^n-1 is uniquely represented by an n-bit number in binary

k uses $\lceil \log_2(k+1) \rceil = \lfloor \log_2 k \rfloor + 1$ digits

Fundamental Theorem For Base X

Each of the numbers from 0 to X^n-1 is uniquely represented by an n-"digit" number in base X

k uses $\lfloor \log_X k \rfloor + 1$ digits in base X

$X > 0$

Proof of the Uniqueness

Let the integer Z has two representations

$Z = a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ .\ .\ .\ + a_1 X + a_0$

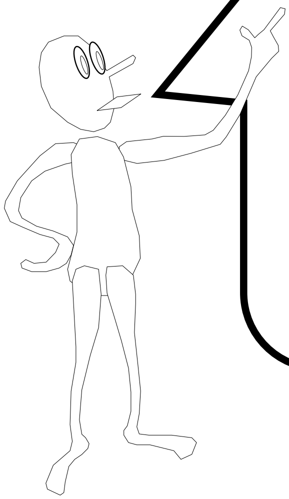
and

$Z = b_{m-1} X^{m-1} + b_{m-2} X^{m-2} + \ .\ .\ . + b_1 X + b_0$

This implies that $a_0 = b_0 \pmod X$

and thus $a_0 = b_0$, since all coefficients are $< X$

We continue in similar manner for a_1 and b_1:
$a_1 X = b_1 X \pmod{X^2}$, so $a_1 = b_1$.
And so on. This olso proofs that $m = n$.



n has length n in unary, but has length $\lfloor \log_2 n \rfloor + 1$ in binary

Unary is exponentially longer than binary

Other Representations: Egyptian Base 3

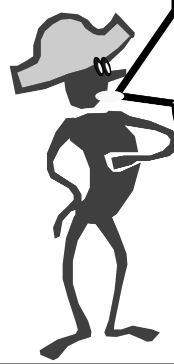
Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

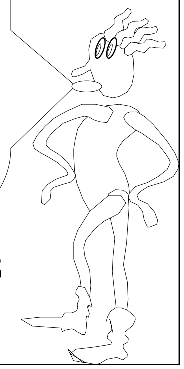
Example: $(1\ {-1}\ {-1})_{EB3} = 9 - 3 - 1 = 5$

We can prove a unique representation theorem

We can prove a unique representation theorem



How could this be Egyptian? Historically, negative numbers first appear in the writings of the Hindu mathematician Brahmagupta (628 AD)



One weight for each power of 3
Left = "negative". Right = "positive"

Study Bee

- Unary and Binary
- Triangular Numbers
- Dot proofs
- $(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$
- Base-X representations
- unique binary representations
- proof for no-leading zero binary
- k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ digits in base 2
- Largest length n number in base X

Two Case Studies

Bases and Representation

Solving Recurrences using a good representation

Example

T(1) = 1
T(n) = 4T(n/2) + n

Notice that T(n) is inductively defined only for positive powers of 2, and undefined on other values

T(1) = 1 T(2) = 6 T(4) = 28 T(8) = 120

Give a closed-form formula for T(n)

Technique 1

Guess Answer, Verify by Induction

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

Base Case: $G(1) = 1$ and $T(1) = 1$

Induction Hypothesis: $T(x) = G(x)$ for $x < n$

Hence: $T(n/2) = G(n/2) = 2(n/2)^2 - n/2$

$$
\begin{aligned}
T(n) &= 4\,T(n/2) + n \\
&= 4\,G(n/2) + n \\
&= 4\,[2(n/2)^2 - n/2] + n \\
&= 2n^2 - 2n + n \\
&= 2n^2 - n = G(n)
\end{aligned}
$$

Guess:
$G(n) = 2n^2 - n$

Technique 2

Guess Form, Calculate Coefficients

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

Guess: $T(n) = an^2 + bn + c$
for some a,b,c

Calculate: $T(1) = 1$, so $a + b + c = 1$

$$T(n) = 4\,T(n/2) + n$$

$$
\begin{aligned}
an^2 + bn + c &= 4\,[a(n/2)^2 + b(n/2) + c] + n \\
&= an^2 + 2bn + 4c + n
\end{aligned}
$$

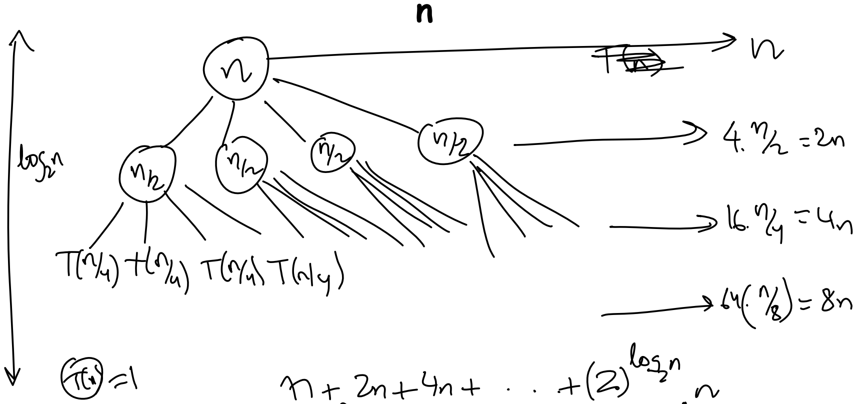
$$(b+1)n + 3c = 0$$

Therefore: $b = -1 \qquad c = 0 \qquad a = 2$

Technique 3

The Recursion Tree Approach

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

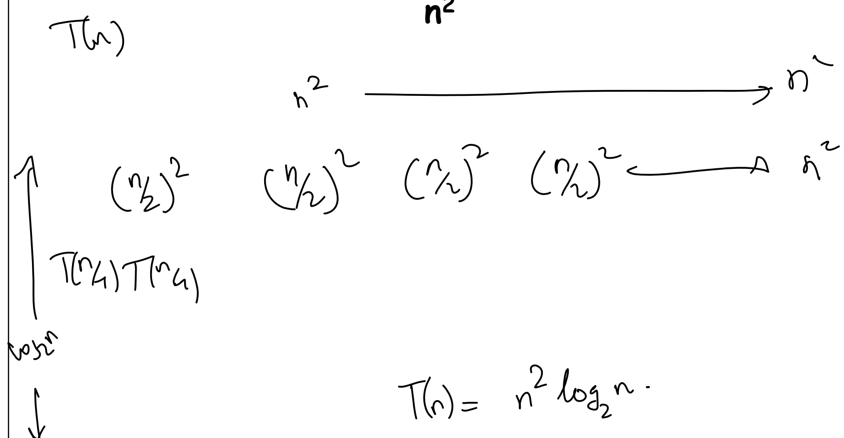


$n + 2n + 4n + \ldots + (2)^{\log_2 n} n$

$= n(1 + 2 + 4 + 8 \ldots + 2^{\log_2 n}) =$

A slight variation

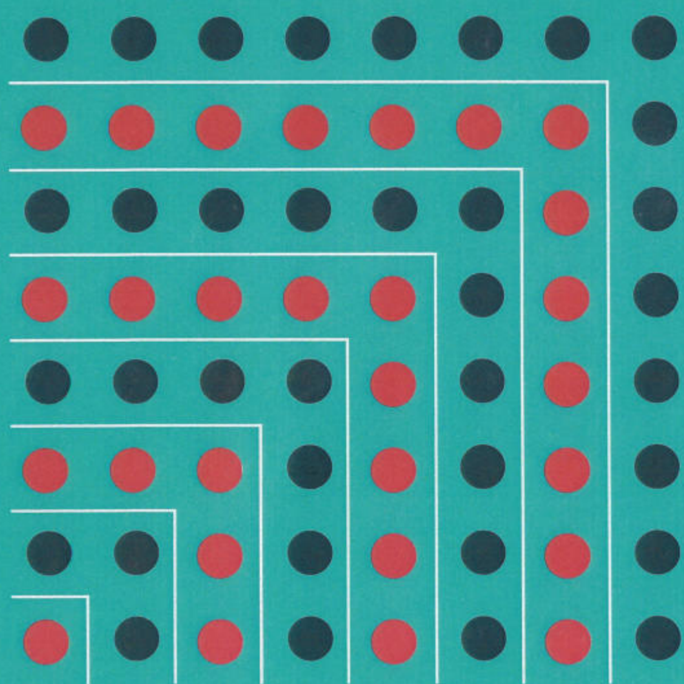
$T(1) = 1,\ T(n) = 4\,T(n/2) + n^2$



$T(n) = n^2 \log_2 n.$

PROOFS WITHOUT WORDS

EXERCISES IN VISUAL THINKING



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

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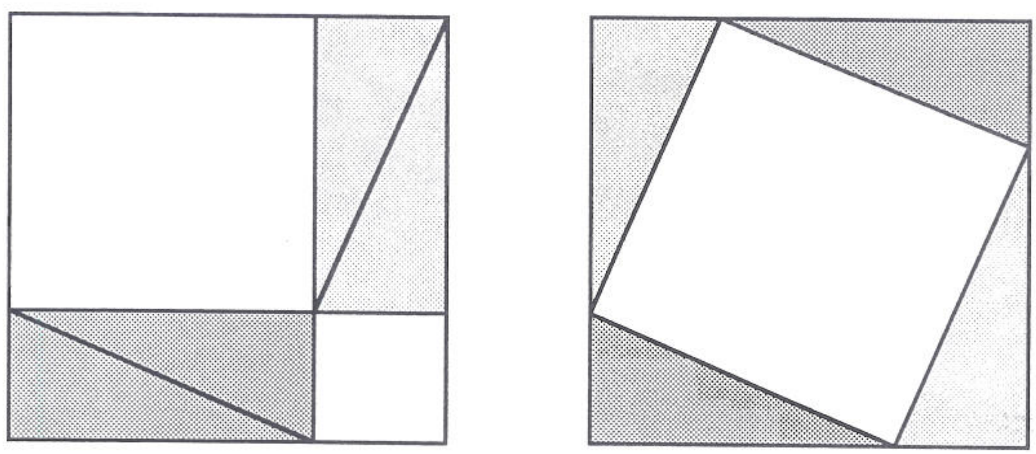
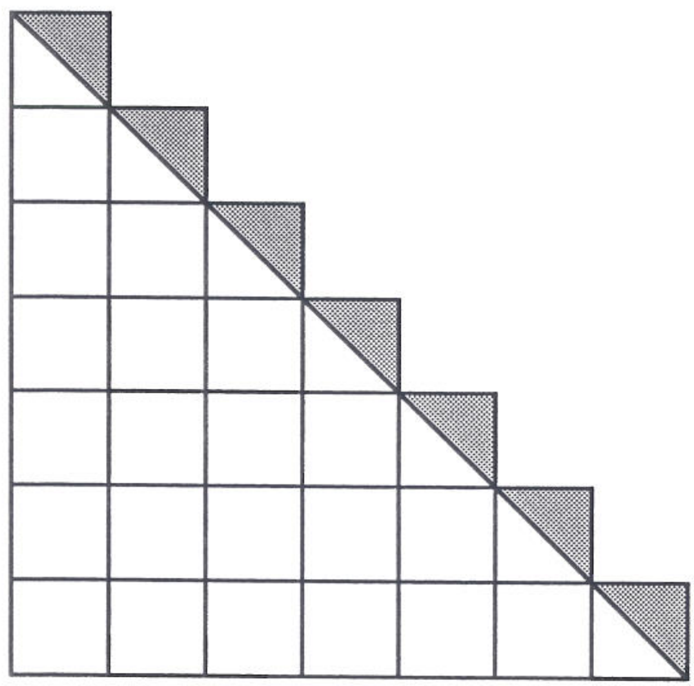
The Pythagorean Theorem I



Sums of Integers II

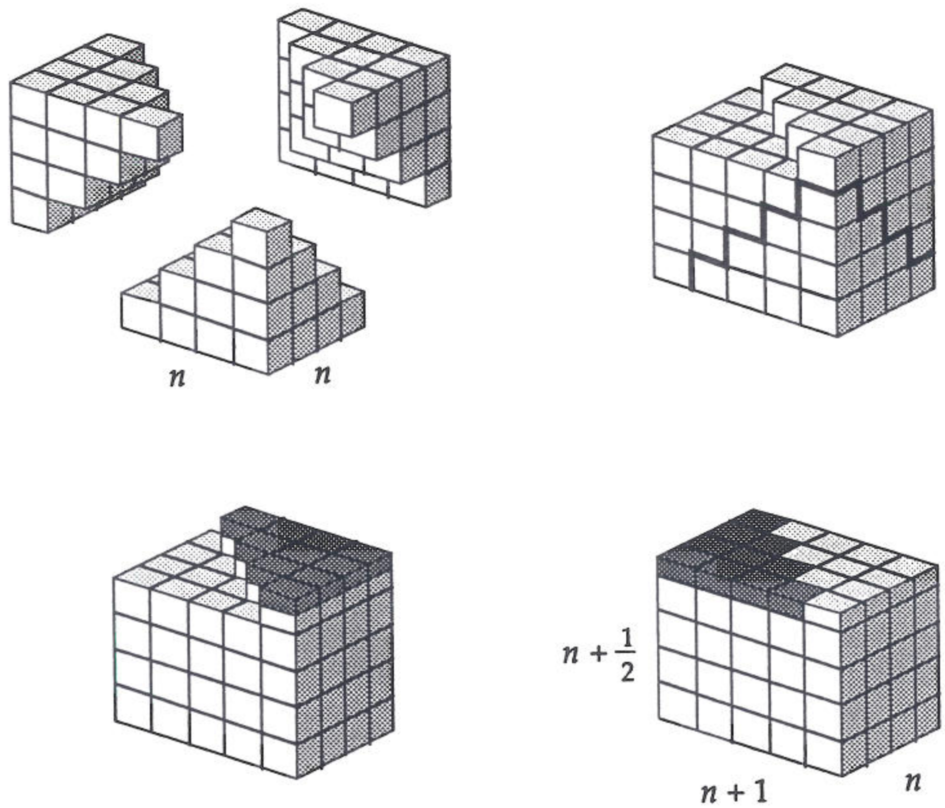


$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

—Ian Richards

Sums of Squares I

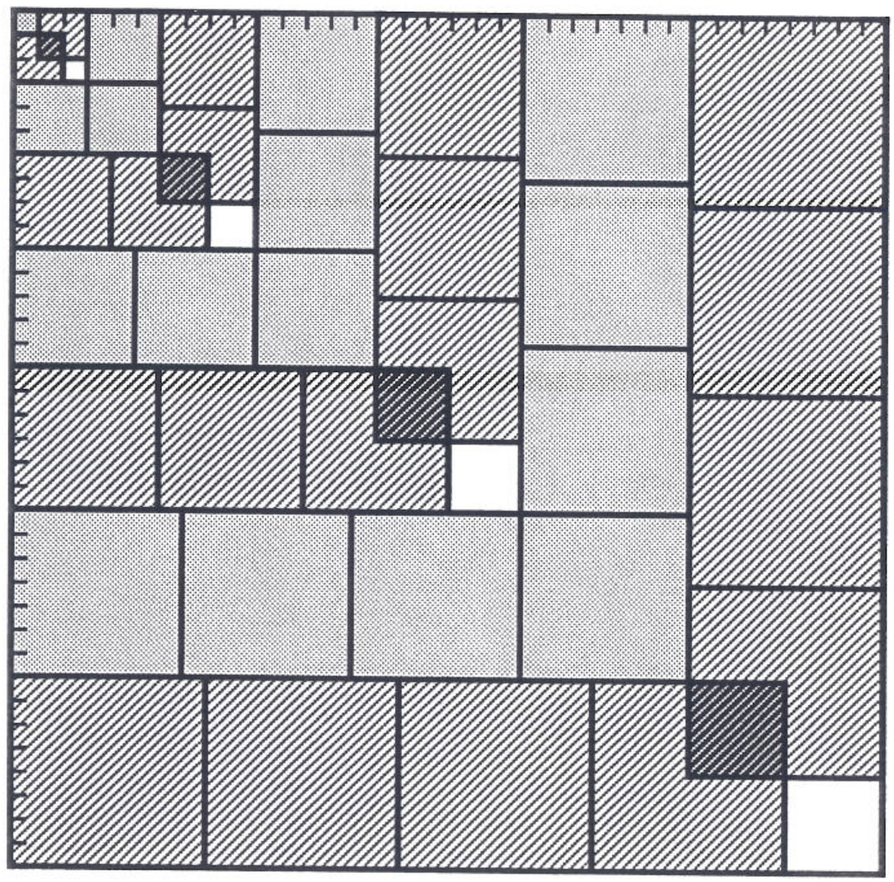
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\tfrac{1}{2})$$



—Man-Keung Siu

Sums of Cubes I

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Solomon W. Golomb