

Great Theoretical Ideas In Computer Science
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Inductive Reasoning



Homework must be Typeset

You may use any typesetting program you wish,

TeX, LaTeX, Mathematica, ...

We Are Here to help!

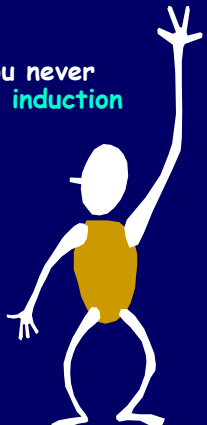
There are many office hours throughout the week

If you have problems with the homework, don't hesitate to ask for help



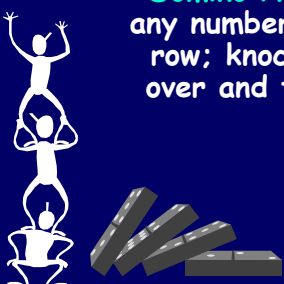
Inductive Reasoning

Raise your hand if you never heard of **mathematical induction**



American Banks

Domino Principle: Line up any number of dominos in a row; knock the first one over and they will all fall



Dominoes Numbered 1 to n

D_k "The k^{th} domino falls"

If we set them up in a row then each one is set up to knock over the next:

For all $1 \leq k < n$, $D_k \Rightarrow D_{k+1}$

$D_1 \Rightarrow D_2 \Rightarrow D_3 \Rightarrow \dots$
All Dominoes Fall



Plain Induction

Suppose we have some property $P(k)$ that may or may not hold for a natural number n .

To demonstrate that $P(k)$ is true for all n is a little problematic.

Inductive Proofs



Base Case: Show that $P(0)$ holds

Induction step: Assume that $P(k)$ holds, show that $P(k+1)$ also holds

In the induction step, the assumption that $P(k)$ holds is called the **Induction Hypothesis**

Proof by Mathematical Induction



In formal notation

$$P(0) \wedge \forall n P(n) \Rightarrow P(n+1)$$

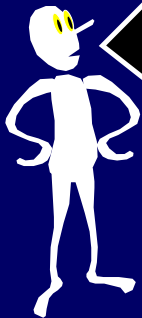
Instead of attacking a problem directly, we only explain how to get a proof for $P(n+1)$ out of a proof for $P(n)$

Theorem

The sum of the first n odd numbers is n^2

Check on small values:

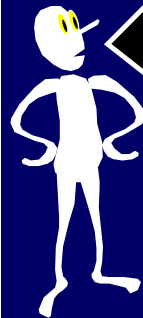
$$\begin{array}{rcl} 1 & = & 1 \\ 1+3 & = & 4 \\ 1+3+5 & = & 9 \\ 1+3+5+7 & = & 16 \end{array}$$

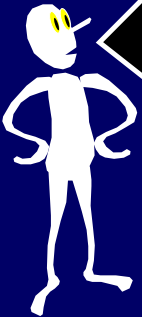


Induction Hypothesis

The sum of the first n odd numbers is n^2

$$1+3+5+\dots+(2n-1) = n^2$$





Induction step:

Assume that $P(n)$ holds, and show that $P(n+1)$ also holds

Assume
 $1+3+5+\dots+(2n-1) = n^2$

Prove
 $1+3+5+\dots+(2n+1) = (n+1)^2$



$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

$$1 + 3 + 5 + \dots + (2n-1) + (2n+1) = n^2 + (2n+1)$$

$$1 + 3 + 5 + \dots + (2n+1) = (n+1)^2$$



Soundness of Induction

How do we know that this really works?




Soundness of Induction

Proof by contradiction.

Assume that for some assertion $P(n)$, we can establish the base case, and the induction step, but nonetheless it is not true that $P(n)$ holds for all n .

So, for some values of n , $P(n)$ is false.

Soundness of Induction

Let n_0 be the **least** such n .

Certainly, n_0 cannot be 0.

Thus, it must be $n_0 = n_1 + 1$, where $n_1 < n_0$.

Soundness of Induction

Now, by our choice of n_0 , this means that $P(n_1)$ holds.

Soundness of Induction

Now, by our choice of n_0 , this means that $P(n_1)$ holds.

because $n_1 < n_0$

Soundness of Induction

But then by **Induction Hypothesis**, $P(n_1+1)$ also holds.

Soundness of Induction

But then by **Induction Hypothesis**, $P(n_1+1)$ also holds.

But that is the same as $P(n_0)$, and we have a contradiction.

Review that proof

we can pick n_0 to be the **least** n where $P(n)$ fails.

Least Element Principle

Every non-empty subset of the natural numbers must contain a least element.

Some Comments

We have chosen to describe the induction step as moving from n to $n+1$, where $n \geq 0$.

There is the obvious alternative to change the induction step from $n-1$ to n , where $n > 0$.

Some Comments

There is nothing sacred about the base case $n=0$, we could just as well start at $n = 11$.

ATM Machine

Suppose an ATM machine has only **two** dollar and **five** dollar bills. You can type in the amount you want, and it will figure out how to divide things up into the proper number of two's and five's.

Claim: The ATM can generate any output amount $n \geq 4$.

Proof

Base case: $n = 4$. 2 two's, done.

Induction step: suppose the machine can already handle $n \geq 4$ dollars.

How do we proceed for $n+1$ dollars?

Proof

Case 1: The n dollar output contains a five.

Then we can replace the five by 3 two's to get $n+1$ dollars.

Proof

Case 2: The n dollar output contains only two's.

Since $n \geq 4$, there must be at least 2 two's. Remove 2, and replace them by 1 five. Done.

Theorem

Every natural number > 1 can be factored into primes

Base case:

2 is prime $\Rightarrow P(2)$ is true

Inductive hypothesis:

$P(n)$ can be factored into primes



This shows a technical point about mathematical induction

Theorem?

Every natural number > 1 can be factored into primes

A different approach:

Assume $2, 3, \dots, n-1$ all can be factored into primes

Then show that n can be factored into primes



Strong Induction

Establish Base Case: $P(0)$

Establish Domino Effect:

Assume $\forall j < n, P(j)$

use that to derive $P(n)$

Theorem

Every natural number $n > 1$ can be factored into primes

Base case:

2 is prime $\Rightarrow P(2)$ is true

Inductive hypothesis:

$P(j)$, $j < n$ can be factored into primes

Case 1: n is prime

Case 2: n is composite, $n = p \cdot q$

Faulty Induction

Claim. $6^n = 0$ for all $n \geq 0$.

Base step: Clearly $6^0 = 0$.

Induction step: Assume that $6^k = 0$ for all $0 \leq k \leq n$.

We need to show that $6^{(n+1)}$ is 0.

Write $n+1 = a+b$, where $a, b > 0$.

$6^{(n+1)} = 6^{(a+b)} = 6^a + 6^b = 0 + 0 = 0$



And there are more ways to do inductive proofs



Yet another way of packaging inductive reasoning is to define "invariants"

Invariant (n):

1. Not varying; constant.
2. *Mathematics*. Unaffected by a designated operation, as a transformation of coordinates.

Invariant (n):

3. *Programming*. A rule, such as the ordering of an ordered list, that applies throughout the life of a data structure or procedure. Each change to the data structure maintains the correctness of the invariant

Odd/Even Handshaking Theorem

At any party at any point in time define a person's **parity** as ODD/EVEN according to the number of hands they have shaken

Statement: The number of people of odd parity must be even

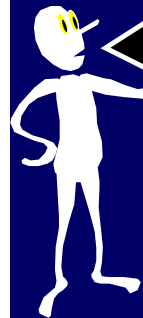
Statement: The number of people of odd parity must be even

Initial case: Zero hands have been shaken at the start of a party, so zero people have odd parity

Invariant Argument:

If 2 people of the same parity shake, they both change and hence the odd parity count changes by 2 - and remains even

If 2 people of different parities shake, then they both swap parities and the odd parity count is unchanged



Inductive reasoning is the high level idea

"Standard" Induction
"Strong" Induction
"Least Element Principal"
"Invariants"
all just different packaging



Induction is also how we can define and construct our world

So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages



Inductive Definition

A linked list is either empty list or a node followed by a linked list

A binary tree is either empty tree or a node containing left and right binary trees.

$F(1) = 1$ *recursive function*
 $F(n) = F(n/2) + 1$

Pancakes With A Problem!

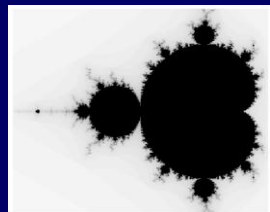
Bring-to-top Method



$P(n) = 2 + P(n-1)$
 $P(2) = 1$

Fractals

Fractals are geometric objects that are self-similar, i.e. composed of infinitely many pieces, all of which look the same.



The Koch Game

Alphabet: { F, +, - }

Start word: F

Productions Rules: $\text{Sub}(F) = F+F--F+F$
 $\text{Sub}(+) = +$
 $\text{Sub}(-) = -$

$\text{NEXT}(w_1 w_2 \dots w_n) = \text{Sub}(w_1) \text{Sub}(w_2) \dots \text{Sub}(w_n)$

Time 0: F
 Time 1: F+F--F+F
 Time 2: F+F--F+F+F--F+F--F+F--F+F--F+F--F+

The Koch Game



F+F--F+F

Visual representation:
 F draw forward one unit
 + turn 60 degree left
 - turn 60 degrees right

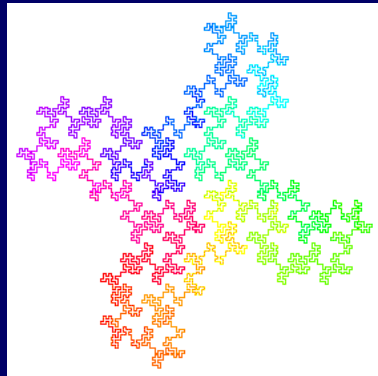
The Koch Game



F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F

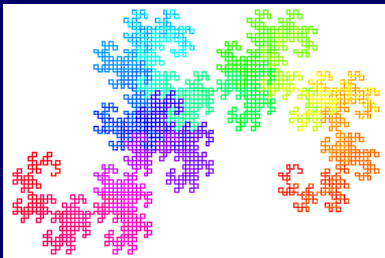
Visual representation: _

F draw forward one unit
+ turn 60 degree left
- turn 60 degrees right



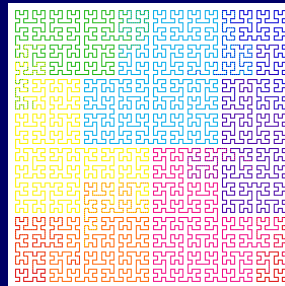
Dragon Game

Sub(X) = X+YF+ FX-Y Sub(Y) = -

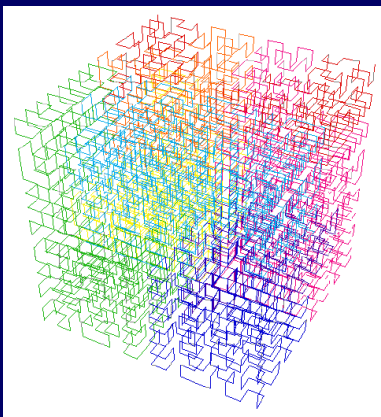


Hilbert Game

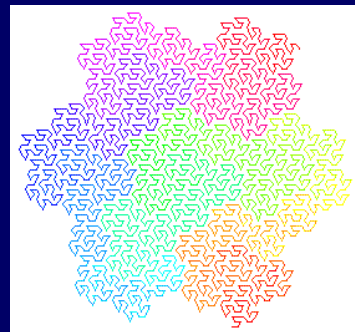
Sub(L) = +RF-LFL-FR+
Sub(R) = -LF+RFR+FL-



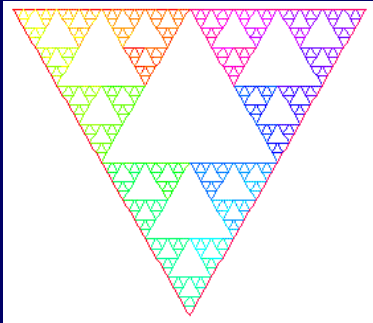
Note: Make 90 degree turns instead of 60 degrees



Peano-Gossamer Curve



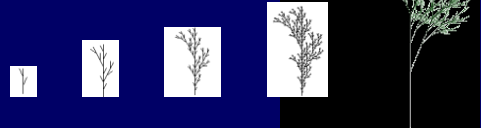
Sierpinski Triangle



Lindenmayer (1968)

$$\text{Sub}(F) = F[-F]F[+F][F]$$

Interpret the stuff inside brackets as a branch



Study Bee

Inductive Proof
Standard Form
Strong Form
Least Element Principal
Invariant Form

Inductive Definition
Recurrence Relations
Fractals