15-251

Some ~~Great~~ Theoretical Ideas ~~in~~ Computer Science for

What does this do?

```
_(__,___,____){___/__<=1?_(__,___+1,___
_):!(___%__)?_(__,___+1,0):___%__==___
/
__&&!____?(printf("%d\t",___/__),_(__,_
__+1,0)):___%__>1&&___%__<___/__?_(
__,1+
___,____+!(___/__%(___%__))):___<__*__
?_(__,___+1,____):0;}main(){_(100,0,0);}
```

1. $0 \in K$.

2. If $x \in K$, then $S(x) \in K$.

3. $\forall x \in K$, $S(x) \neq 0$.

4. If $x, y \in K$ and $S(x) = S(y)$ then $x = y$.

5. If a set of numbers T contains 0 and also $x \in T \Rightarrow S(x) \in T$, then $T = K$.

What is K?

Axioms of Peano Arithmetic

1. Zero is a natural number.

2. If x is a natural number, the successor of x is a natural number.

3. Zero is not the successor of a natural number.

4. Two natural numbers of which the successors are equal are themselves equal.

5. If a set of natural numbers T contains zero and also the successor of every number in T, then every natural number is in T.

Turing's Legacy: The Limits Of Computation

Lecture 25 (April 9, 2009)

This lecture will change the way you think about computer programs…

Many questions which appear easy at first glance are impossible to solve in general

The HELLO assignment

Write a JAVA program to output the words "HELLO WORLD" on the screen and halt.

Space and time are not an issue.

The program is for an ideal computer.

PASS for any working HELLO program, no partial credit.

Grading Script

The grading script G must be able to take any Java program P and grade it.

$$G(P) = \begin{cases} \text{Pass, if P prints only the words “HELLO WORLD” and halts.} \\ \text{Fail, otherwise.} \end{cases}$$

How exactly might such a script work?

What does this do?

```
_(__,___,____){___/__<=1?_(__,___+1,___
_):!(___%__)?_(__,___+1,0):___%__==___
/
__&&!____?(printf("%d\t",___/__),_(__,_
__+1,0)):___%__>1&&___%__<___/__?_(
__,1+
___,____+!(___/__%(___%__))):___<__*__
?_(__,___+1,____):0;}main(){_(100,0,0);}
```

Nasty Program

```
n:=0;
while (n is not a counter-example
     to the Riemann Hypothesis) {
     n++;
}
print "Hello World";
```

The nasty program is a PASS if and only if the Riemann Hypothesis is false.

A TA nightmare: Despite the simplicity of the HELLO assignment, there is no program to correctly grade it!

And we will prove this.

The theory of what can and can't be computed by an ideal computer is called **Computability Theory** or **Recursion Theory**.

From the last lecture:

Are all reals describable? **NO**
Are all reals computable? **NO**

We saw that

computable $\Rightarrow$ describable

but do we also have

describable $\Rightarrow$ computable?

The "grading function" we just described is not computable! (We'll see a proof soon.)

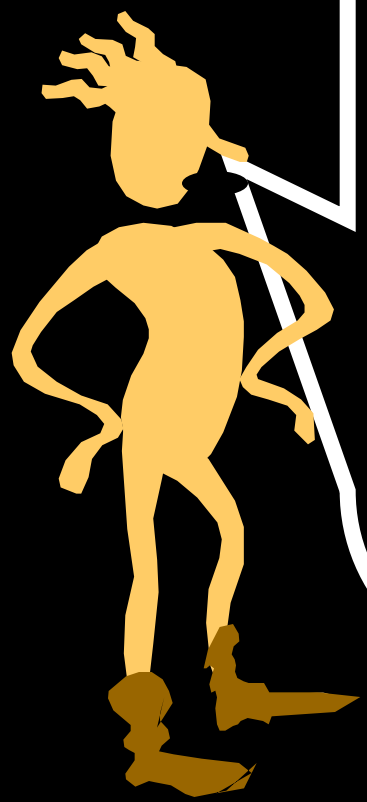
Computable Function

Fix a finite set of symbols, Σ
Fix a precise programming language, e.g., Java

A program is any finite string of characters that is syntactically valid.

A function $f : \Sigma^* \to \Sigma^*$ is **computable** if there is a program P that when executed on an ideal computer, computes f.
That is, for all strings x in Σ^*, f(x) = P(x).

Hence: **countably many** computable functions!



There are only countably many Java programs.

Hence, there are only **countably many** computable functions.

Uncountably Many Functions

The functions f: $\Sigma^* \rightarrow \{0,1\}$ are in 1-1 onto correspondence with the subsets of Σ^* (the powerset of Σ^*).

Subset S of Σ^* $\Leftrightarrow$ Function f_S

x in S	$\Leftrightarrow$	$f_S(x) = 1$
x not in S	$\Leftrightarrow$	$f_S(x) = 0$

Hence, the set of all f:$\Sigma^* \rightarrow \{0,1\}$ has the same size as the power set of Σ^*, which is uncountable.

Countably many computable functions.

Uncountably many functions from Σ^* to {0,1}.

Thus, most functions from Σ^* to {0,1} are not computable.

Can we explicitly describe an uncomputable function?

Notation And Conventions

Fix a single programming language (Java)

When we write program P we are talking about the text of the source code for P

P(x) means the output that arises from running program P on input x, assuming that P eventually halts.

P(x) = $\perp$ means P did not halt on x

The meaning of P(P)

It follows from our conventions that P(P) means the output obtained when we run P on the text of its own source code

The Halting Problem

Is there a program HALT such that:

HALT(P) = yes, if P(P) halts
HALT(P) = no, if P(P) does not halt

THEOREM: There is no program to solve the halting problem (Alan Turing 1937)

Suppose a program HALT existed that solved the halting problem.

HALT(P) = yes, if P(P) halts
HALT(P) = no, if P(P) does not halt

We will call HALT as a subroutine in a new program called CONFUSE.

CONFUSE

```
CONFUSE(P)
{  if (HALT(P))
        then loop forever;          //i.e., we dont halt
   else exit;                       //i.e., we halt
   // text of HALT goes here
}
```

Does CONFUSE(CONFUSE) halt?

CONFUSE

```
CONFUSE(P)
{  if (HALT(P))
        then loop forever;          //i.e., we dont halt
   else exit;                       //i.e., we halt
   // text of HALT goes here  }
```

Suppose CONFUSE(CONFUSE) halts:
then HALT(CONFUSE) = TRUE
$\Rightarrow$ CONFUSE will loop forever on input CONFUSE

Suppose CONFUSE(CONFUSE) does not halt
then HALT(CONFUSE) = FALSE
$\Rightarrow$ CONF

CONTRADICTION

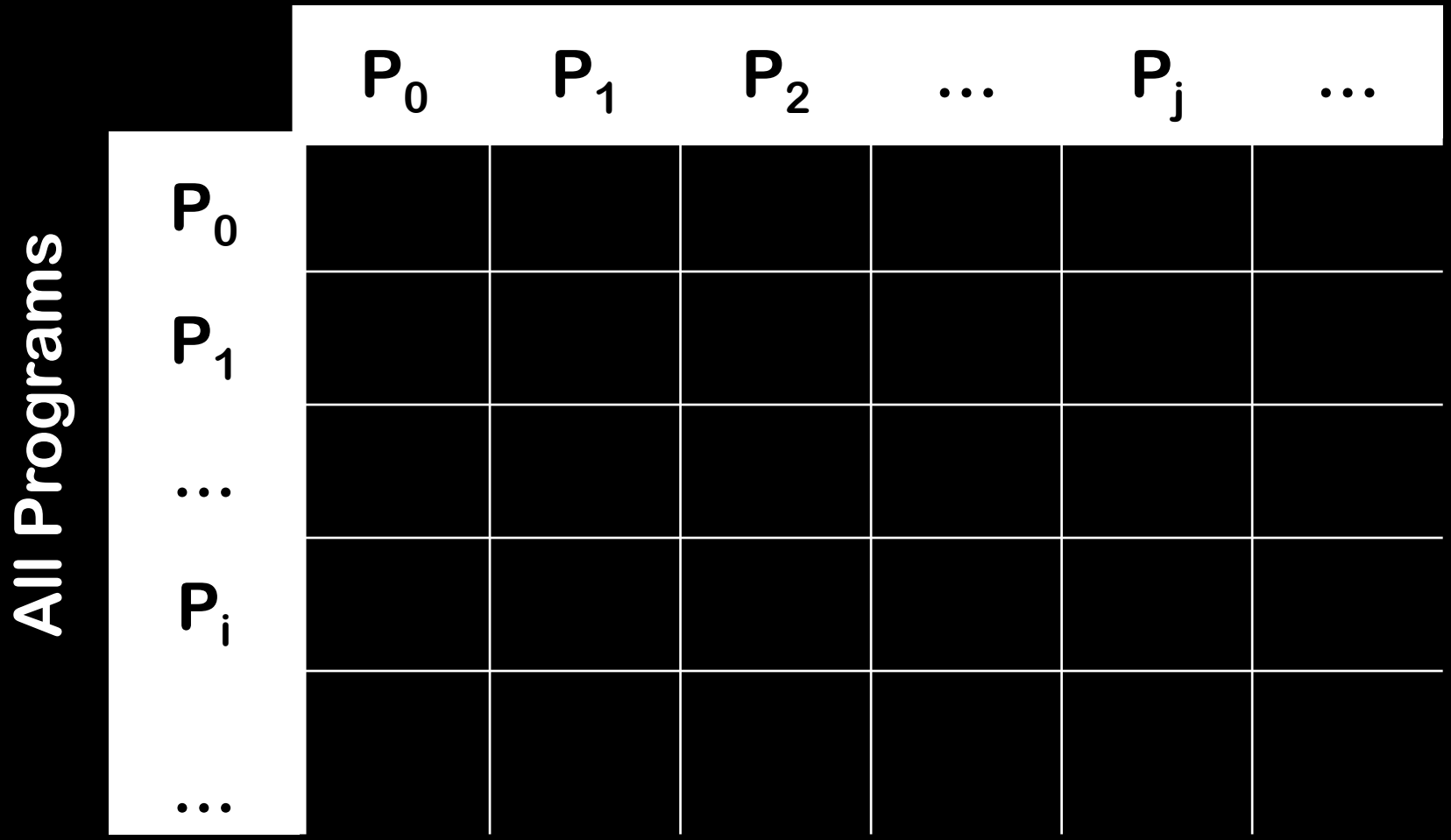
Alan Turing (1912-1954)

Theorem: [1937]

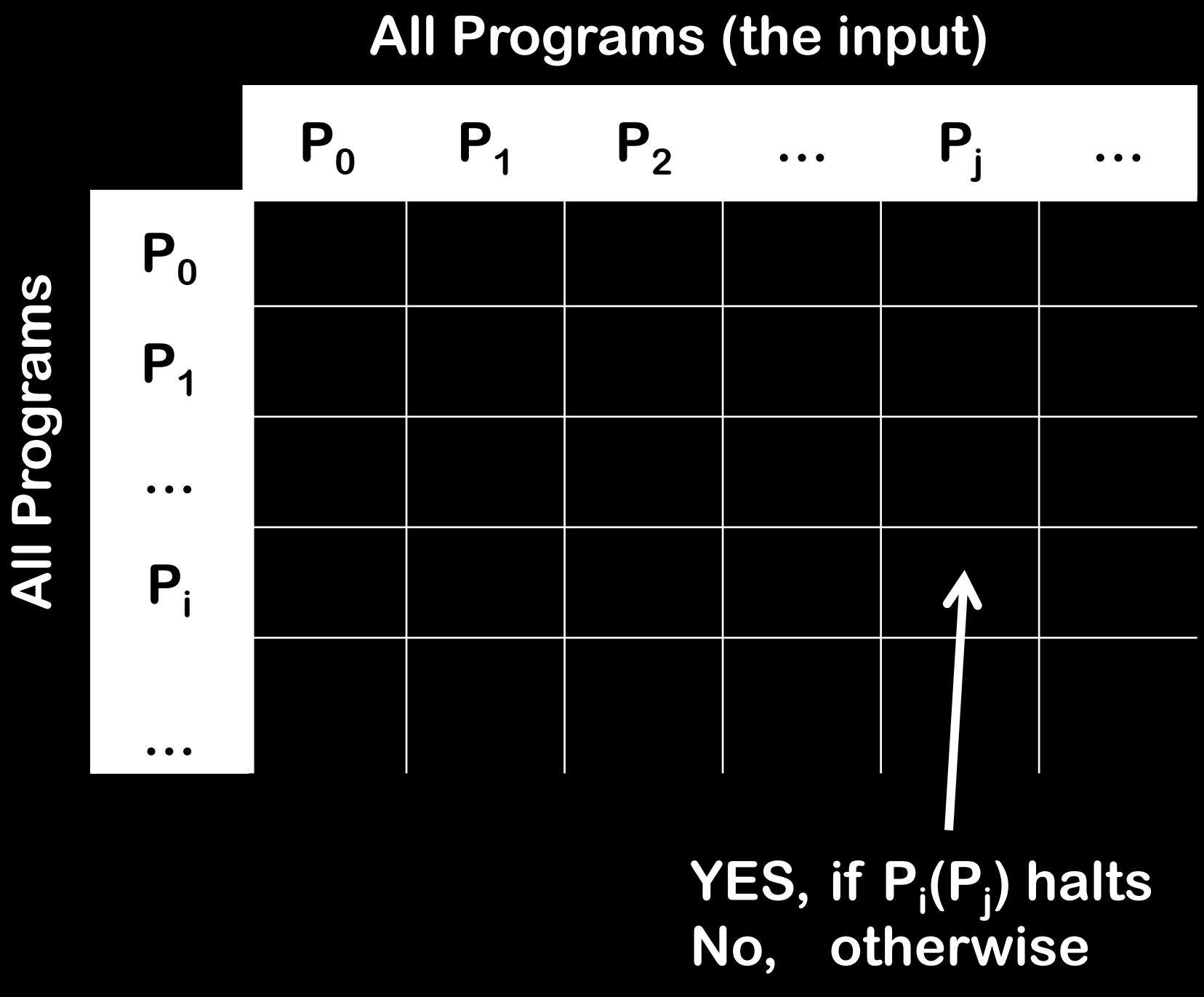
There is no program to solve the halting problem

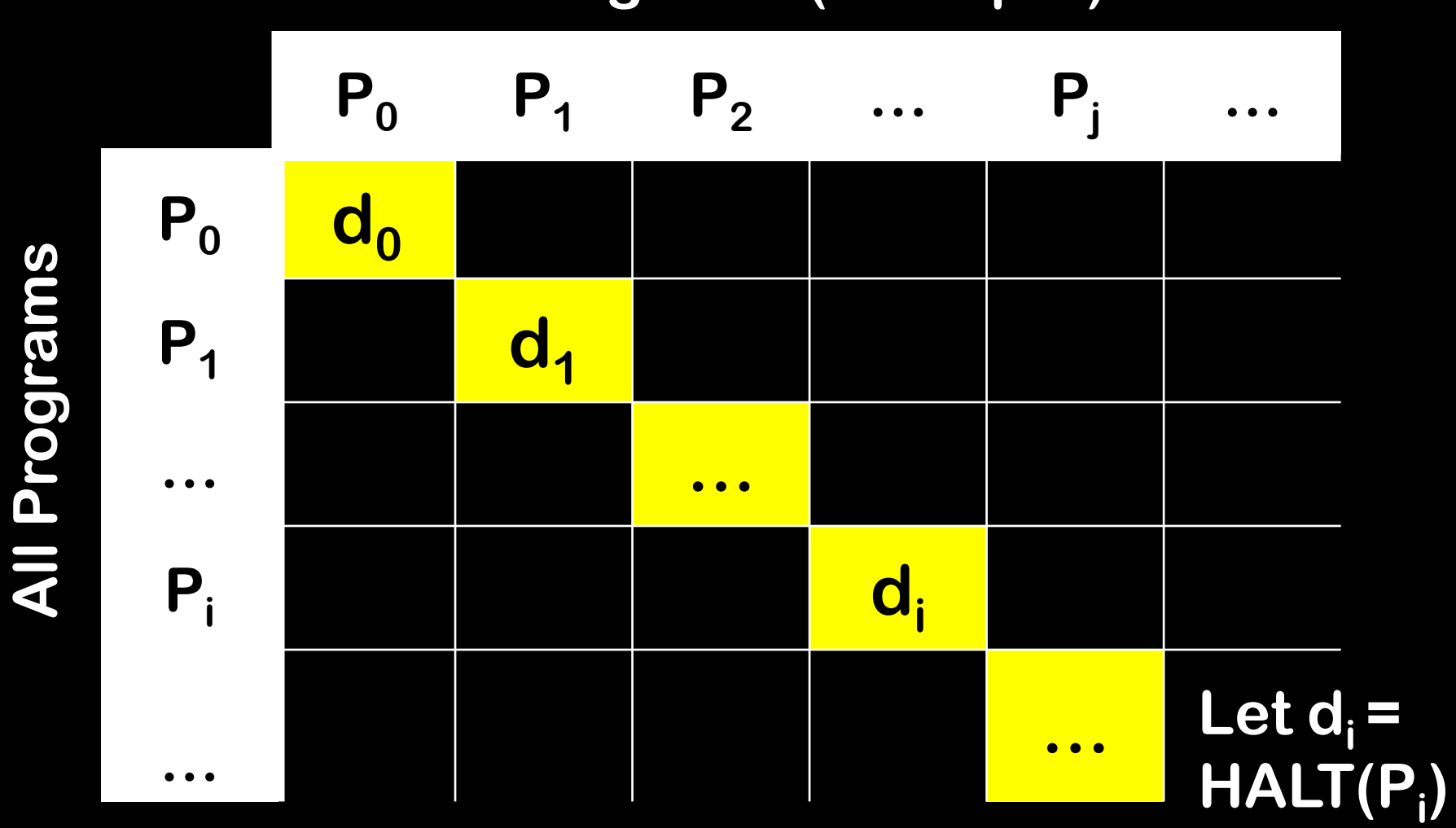
Turing's argument is essentially the reincarnation of Cantor's **Diagonalization argument** that we saw in the previous lecture.

All Programs (the input)



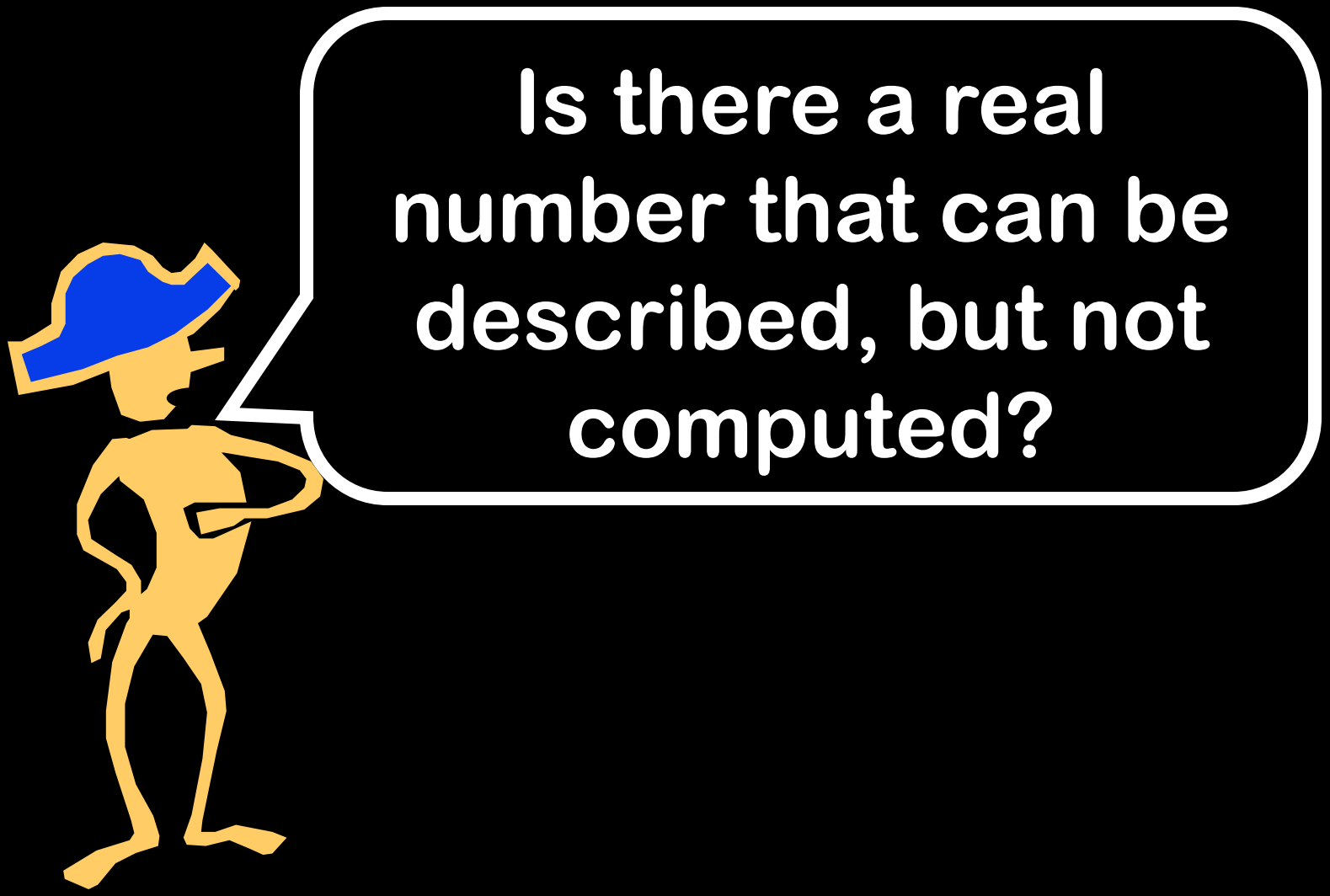
Programs (computable functions) are countable, so we can put them in a (countably long) list


All Programs (the input)
P0
P1
P2
…
Pj
…
All Programs
P0
P1
…
Pi
…
YES, if Pi(Pj) halts
No, otherwise



CONFUSE(P_i) halts iff d_i = no
(The CONFUSE function is the negation of the diagonal.)

Hence CONFUSE cannot be on this list.


Is there a real number that can be described, but not computed?

Consider the real number R whose binary expansion has a 1 in the j^{th} position iff the j^{th} program halts on input itself.

Proof that R cannot be computed

Suppose it is, and program FRED computes it.
then consider the following program:

```
MYSTERY(program text P)
  for j = 0 to forever do {
    if (P == Pj)
        then use FRED to compute jth bit of R
    return YES if (bit == 1), NO if (bit == 0)
  }
```

MYSTERY solves the halting problem!

The Halting Set K

Definition:

K is the set of all programs P such that P(P) halts.

K = { Java P | P(P) halts }

Computability Theory: Vocabulary Lesson

We call a set $S \subseteq \Sigma^*$ **decidable** or **recursive** if there is a program P such that:

$P(x) = \text{yes}$, if $x \in S$

$P(x) = \text{no}$, if $x \notin S$

We already know: **the halting set K is undecidable**

Decidable and Computable

Subset S of Σ^* $\Leftrightarrow$ Function f_S

x in S	$\Leftrightarrow$	$f_S(x) = 1$
x not in S	$\Leftrightarrow$	$f_S(x) = 0$

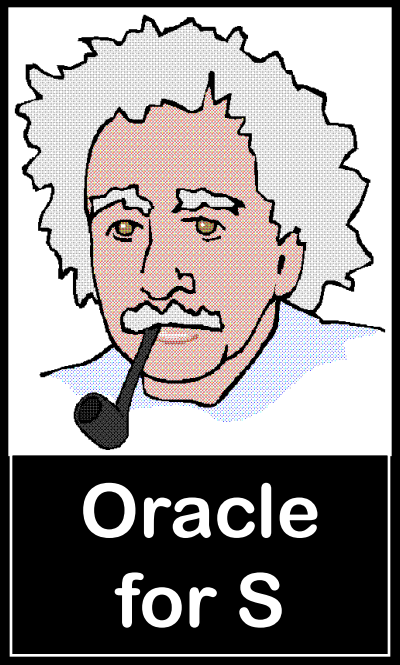
Set S is decidable $\Leftrightarrow$ function f_S is computable

Sets are “decidable” (or undecidable), whereas functions are “computable” (or not)

Oracles and Reductions

Oracle For Set S

Is $x \in S$?



YES/NO

Oracle for S

Example Oracle
S = Odd Naturals

4?

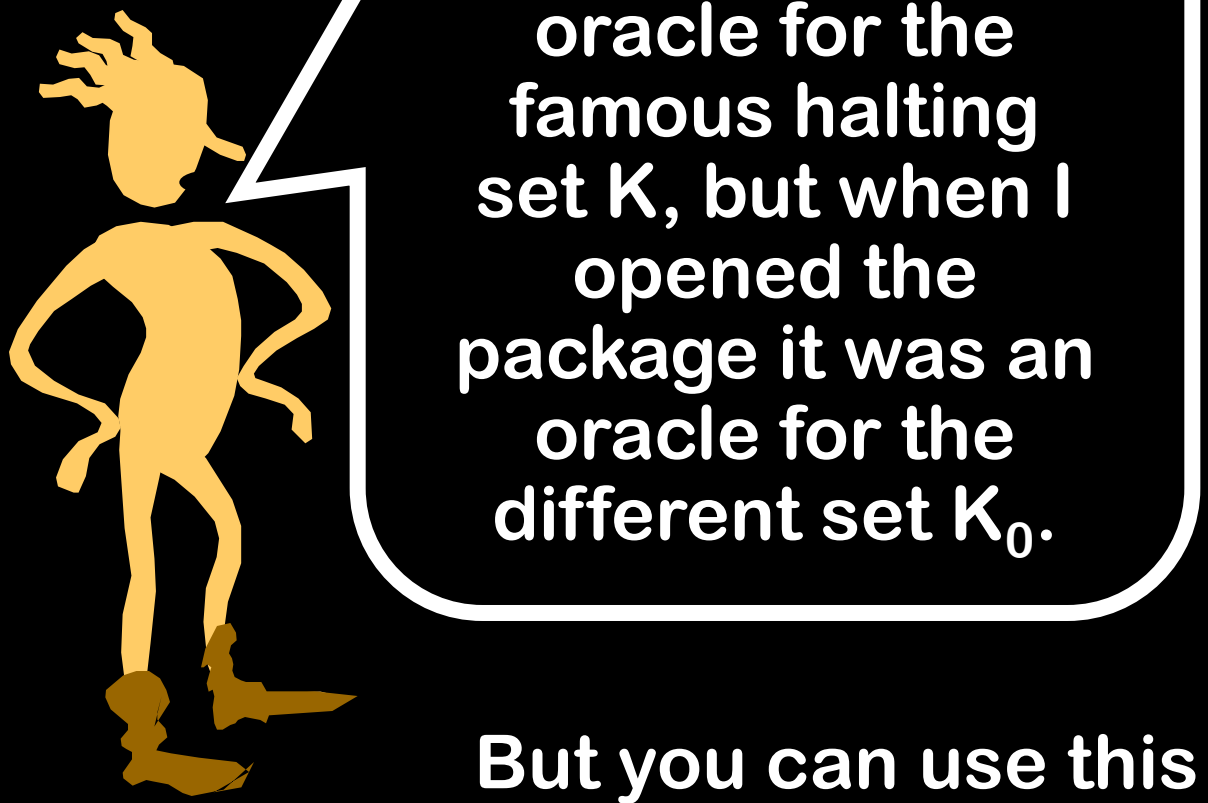
No

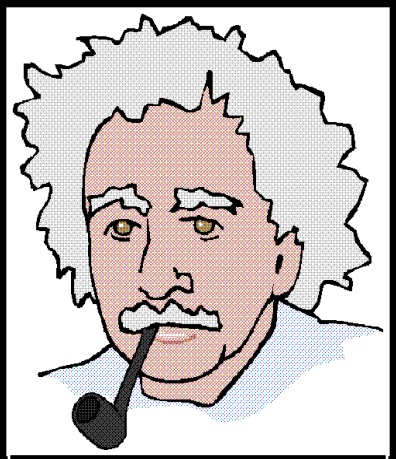
81?

Yes

Oracle for S

K_0= the set of programs that take no input and halt



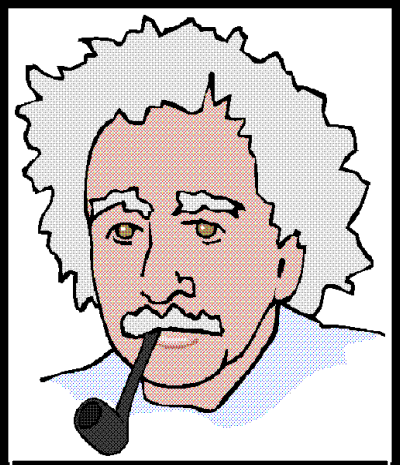


GIVEN: Oracle for K_0

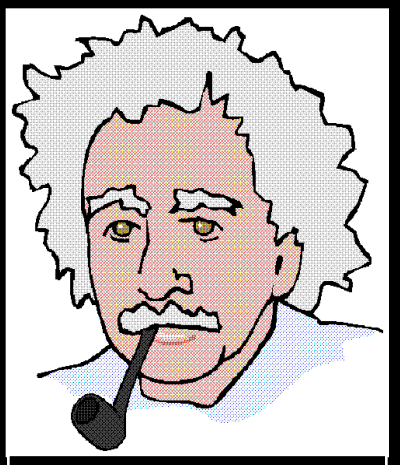
But you can use this oracle for K_0 to build an oracle for K.

K_0= the set of programs that take no input and halt

P = [input I; Q]
Does P(P) halt?



Does [I:=P;Q] halt?



BUILD:
Oracle
for K

GIVEN:
Oracle
for K_0

We've **reduced** the problem of deciding membership in K to the problem of deciding membership in K_0.

Hence, deciding membership for K_0 must be **at least as hard** as deciding membership for K.

Thus if K_0 were decidable then K would be as well.

We already know K is not decidable, hence K_0 is not decidable.

HELLO = the set of programs that print hello and halt

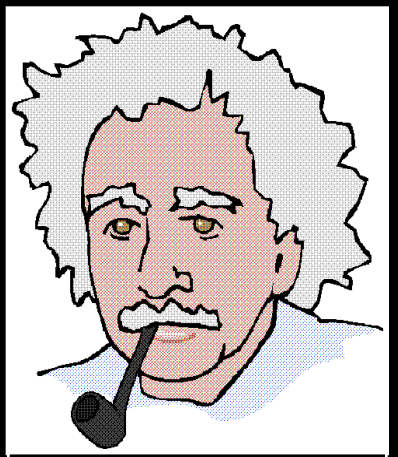
Does P halt?

Let P' be P with all print statements removed.

(assume there are no side effects)

Is [P'; print HELLO] a hello program?

BUILD:
Oracle for K_0



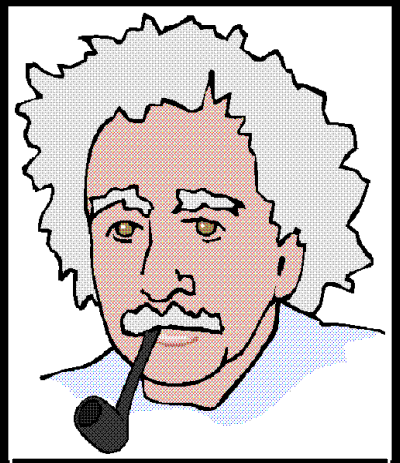
GIVEN:
HELLO Oracle

Hence, the set HELLO is not decidable.

EQUAL = All <P,Q> such that P and Q have identical output behavior on all inputs

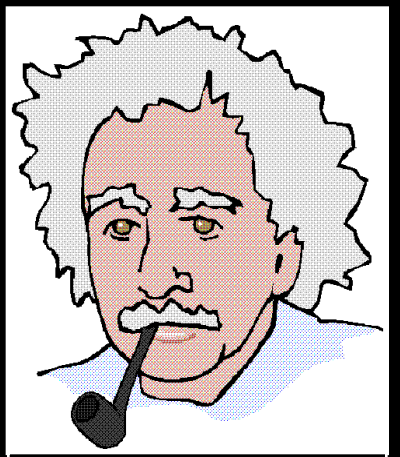
Is P in set HELLO?

Let HI = [print HELLO]



BUILD:
HELLO
Oracle

Are P and HI equal?

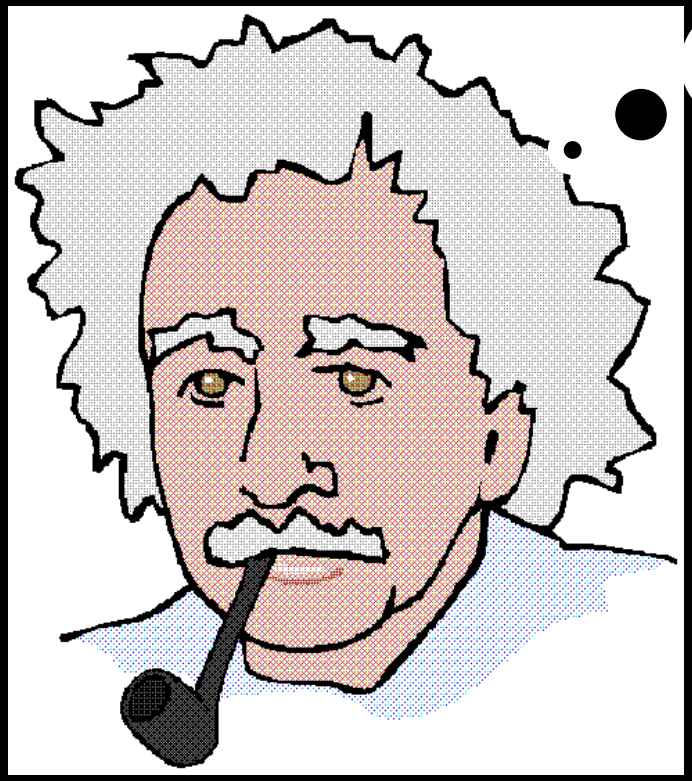


GIVEN:

EQUAL
Oracle

Halting with input, Halting without input, HELLO, and EQUAL are all **undecidable**.

PHILOSOPHICAL INTERLUDE



CHURCH-TURING THESIS

Any well-defined procedure that can be grasped and performed by the human mind and pencil/paper, can be performed on a conventional digital computer with no bound on memory.

The Church-Turing Thesis is NOT a theorem. It is a statement of belief concerning the universe we live in.

Your opinion will be influenced by your religious, scientific, and philosophical beliefs…

…mileage may vary

Empirical Intuition

No one has ever given a counter-example to the Church-Turing thesis. I.e., no one has given a concrete example of something humans compute in a consistent and well defined way, but that can't be programmed on a computer. The thesis is true.

Mechanical Intuition

The brain is a machine. The components of the machine obey fixed physical laws. In principle, an entire brain can be simulated step by step on a digital computer. Thus, any thoughts of such a brain can be computed by a simulating computer. The thesis is true.

Quantum Intuition

The brain is a machine, but not a classical one. It is inherently quantum mechanical in nature and does not reduce to simple particles in motion. Thus, there are inherent barriers to being simulated on a digital computer. The thesis is false. However, the thesis is true if we allow quantum computers.


"That's all Folks!"
ACME ACRES
U.S.A.
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USA
33
2001