15-251

Some ~~Great~~ Theoretical Ideas ~~in~~ for Computer Science

Rules of the Game

Each person will have a unique number

For each question, I will first give the class time to work out an answer. Then, I will call three different people at random

They must explain the answer to the TAs (who are all the way in the back). If the TAs are satisfied, the class gets points.

If the class gets 1,700 points, then you win

400 RUNNING TIME	400 AN ALGORITHM
300 ALGO PROPERTIES	300 CONVER-GENTS
200 GCD ALGORITHM	200 EXAMPLES
100 GCD DEFINITION	100 CONTINUED FRACTIONS

1. The Greatest Common Divisor (GCD) of two non-negative integers A and B is defined to be:

The largest positive integer that divides both A and B

2. As an example, what is GCD(12,18) and GCD(5,7)

GCD(12,18) = 6

GCD(5,7)=1

A Naïve method for computing GCD(A,B) is:

Factor A into prime powers.
Factor B into prime powers.

Create GCD by multiplying together each common prime raised to the highest power that goes into both A and B.

Give an algorithm to compute GCD(A,B) that does not require factoring A and B into primes, and does not simply try dividing by most numbers smaller than A and B to find the GCD. Run your algorithm to calculate GCD(67,29)

Euclid(A,B) = Euclid(B, A mod B)
Stop when B=0

Euclid's GCD algorithm can be expressed via the following pseudo-code:

```
Euclid(A,B)
    If B=0 then return A
    else return Euclid(B, A mod B)
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Show that if this algorithm ever stops, then it outputs the GCD of A and B

GCD(A,B) = GCD(B, A mod B)

Proof:

(d | A and d | B) ⇔ (d | (A - kB) and d | B)

The set of common divisors of A, B equals the set of common divisors of B, A - kB

Euclid(A,B) = Euclid(B, A mod B)
Stop when B=0

Show that the running time for this algorithm is bounded above by $2\log_2(\max(A,B))$

Claim: A mod B < ½ A

Proof: If B = ½ A then A mod B = 0

If B < ½ A then any X Mod B < B < ½ A

If B > ½ A then A mod B = A - B < ½ A

Proof of Running Time:

GCD(A,B) calls GCD(B, <½A)

which calls GCD(<½A, B mod <½A)

Every two recursive calls, the input numbers drop by half

DAILY DOUBLE

A simple continued fraction is an expression of the form:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \ldots}}}}$$

Where a,b,c,d,e, … are non-negative integers. We denote this continued fraction by [a,b,c,d,e,…].

What number do the fractions [3,2,1,0,0,0,…] (= [3,2,1]) and [1,1,1,0,0,0,…] (= [1,1,1]) represent? (simplify your answer)

Let r_1 = [1,0,0,0,…]
r_2 = [1,1,0,0,0,…]
r_3 = [1,1,1,0,0,0…]
r_4 = [1,1,1,1,0,0,0…]
:
:

Find a the value of r_n as a ratio of somethig we've seen before (prove your answer)

r_n = Fib(n+1)/F(n)

$$\frac{\text{Fib}(n+1)}{\text{Fib}(n)} = \frac{\text{Fib}(n)+\text{Fib}(n-1)}{\text{Fib}(n)} = 1 + \cfrac{1}{\cfrac{\text{Fib}(n)}{\text{Fib}(n-1)}}$$

Let $\alpha = [a_1, a_2, a_3, ...]$ be a continued fraction

Define: $C_1 = [a_1,0,0,0,0..]$
$C_2 = [a_1,a_2,0,0,0,...]$
$C_3 = [a_1,a_2,a_3,0,0,...]$
:

C_k is called the k-th convergent of α

α is the limit of the sequence $C_1, C_2, C_3,$…

A rational p/q is the best approximator to a real α if no rational number of denominator smaller than q comes closer to α

Given any CF representation of α, each convergent of the CF is a best approximator for α

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Find best approximators for π with denominators 1, 7 and 113

$C_1 = 3$
$C_2 = 22/7$
$C_3 = 333/106$
$C_4 = 355/113$
$C_5 = 103993/33102$
$C_6 = 104348/33215$

1. Write a continued fraction for 67/29

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

2. Write a formula that allows you to calculate the continued fraction of A/B in $2\log_2(\max(A,B))$ steps

$$\frac{A}{B} = \left\lfloor \frac{A}{B} \right\rfloor + \cfrac{1}{\cfrac{B}{A \bmod B}}$$