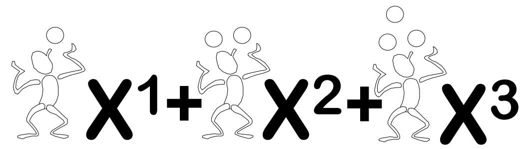
Awesome ~~Some~~ ~~Great~~ 15-251 Theoretical Ideas ~~in~~ for Computer Science with Ben Wolf

Counting III

Lecture 8 (February 7, 2008)



$X^1 + X^2 + X^3$

How many integer solutions to the following equation?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Think of x_k as being the number of gold bars that are allotted to pirate k

$$\binom{34}{4}$$

How many integer solutions to the following equation?

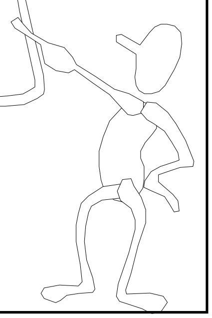
$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

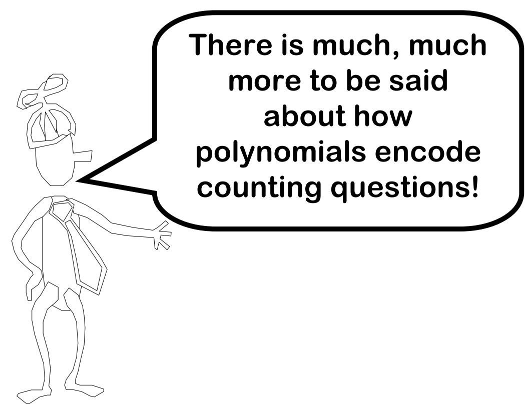
$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^{n-k} Y^k$$





Power Series Representation

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} X^k$$

"Product form" or "Generating form"

For k>n, $\binom{n}{k} = 0$

"Power Series" or "Taylor Series" Expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = 1, $$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

The number of subsets of an n-element set

By varying x, we can discover new identities:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = -1, $$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently, $$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

The number of subsets with even size is the same as the number of subsets with odd size

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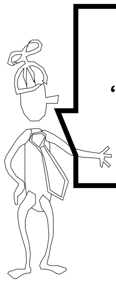
$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a bijection are called "combinatorial" arguments

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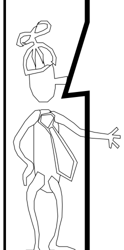
$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones

Let E_n be the set of binary strings of length n with an even number of ones

A combinatorial proof must construct a bijection between O_n and E_n

An Attempt at a Bijection

Let f_n be the function that takes an n-bit string and flips all its bits

f_n is clearly a one-to-one and onto function

for odd n. E.g. in f_7 we have:

0010011 → 1101100
1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

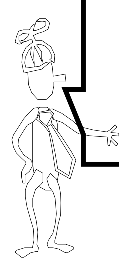
A Correspondence That Works for all n

Let f_n be the function that takes an n-bit string and flips only the first bit. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 = \quad 1$

$(1+X)^1 = \quad 1 + 1X$

$(1+X)^2 = \quad 1 + 2X + 1X^2$

$(1+X)^3 = \quad 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = \quad 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

"Pascal's Triangle"

$\binom{0}{0} = 1$

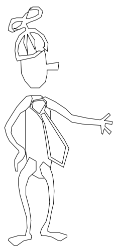
$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$

$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

- Al-Karaji, Baghdad 953-1029
- Chu Shin-Chieh 1303
- Blaise Pascal 1654

Pascal's Triangle



"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	= 1
1 + 1	= 2
1 + 2 + 1	= 4
1 + 3 + 3 + 1	= 8
1 + 4 + 6 + 4 + 1	= 16
1 + 5 + 10 + 10 + 5 + 1	= 32
1 + 6 + 15 + 20 + 15 + 6 + 1	= 64

Odds and Evens

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1 + 15 + 15 + 1 = 6 + 20 + 6

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Summing on kth Avenue

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Fibonacci Numbers

1
1 1 = 2
= 3
1 2 1 = 5
= 8
1 3 3 1 = 13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Sums of Squares

1

1 1

$\boxed{1^2 \quad 2^2 \quad 1^2}$

$\boxed{1^2 \quad 3^2 \mid 3^2 \quad 1^2}$

1 4 6 4 1

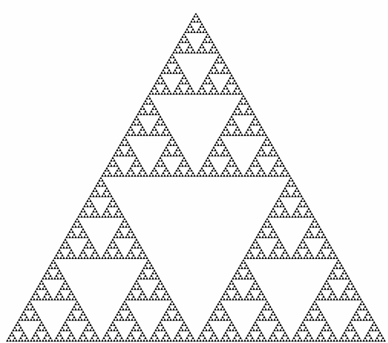
1 5 10 10 5 1

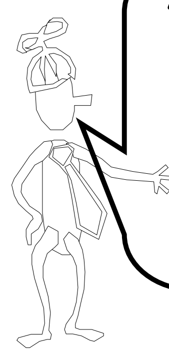
1 6 15 20 15 6 1

Al-Karaji Squares

1	
1 1	= 1
1 2 +2·1	= 4
1 3 +2·3 1	= 9
1 4 +2·6 4 1	= 16
1 5 +2·10 10 5 1	= 25
1 6 +2·15 20 15 6 1	= 36

Pascal Mod 2

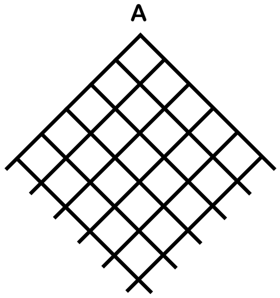




All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients

How many shortest routes from A to B?

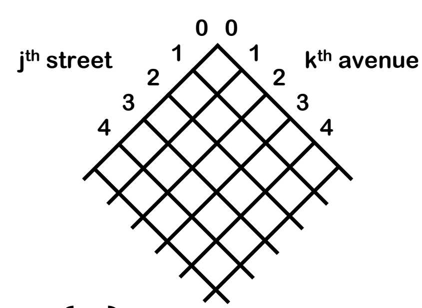
A



B

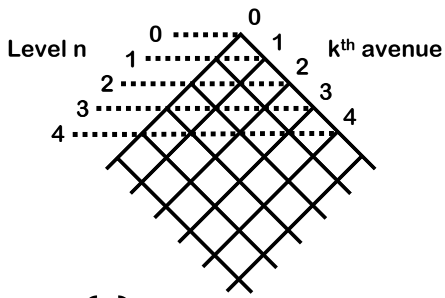
$\binom{10}{5}$

Manhattan



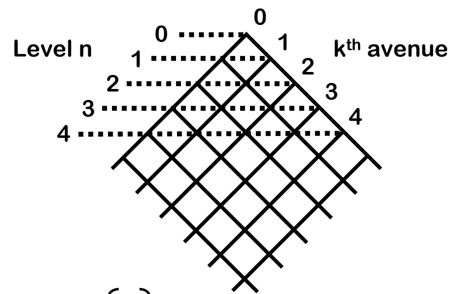
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

Manhattan

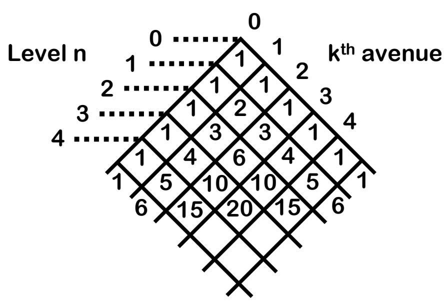


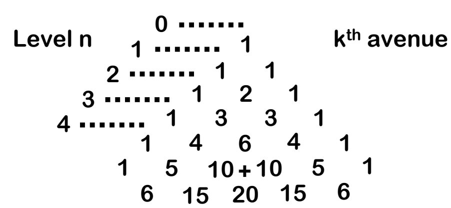
There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

Manhattan

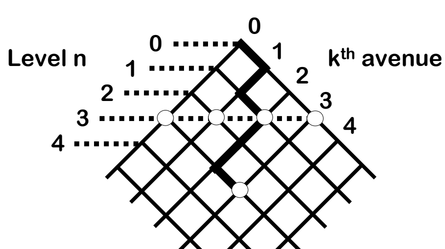


There are $\binom{n}{k}$ shortest routes from (0,0) to level n and kth avenue





$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $\vec{V}$ can be thought of as:

$< * , * , * , * , * , * , \ldots >$

Vector Programs

Let k stand for a scalar constant
<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,…>
<1> = <1,0,0,0,…>

$\vec{V} + \vec{T}$ means to add the vectors position-wise

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Vector Programs

RIGHT($\vec{V}$) means to shift every number in $\vec{V}$ one position to the right and to place a 0 in position 0

RIGHT(<1,2,3, …>) = <0,1,2,3,…>

Vector Programs

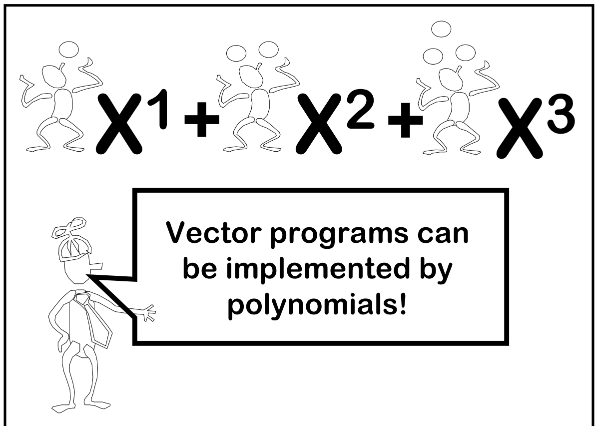
Example:	Store:
$\vec{V}$:= <6>;	$\vec{V}$ = <6,0,0,0,…>
$\vec{V}$:= RIGHT($\vec{V}$) + <42>;	$\vec{V}$ = <42,6,0,0,…>
$\vec{V}$:= RIGHT($\vec{V}$) + <2>;	$\vec{V}$ = <2,42,6,0,…>
$\vec{V}$:= RIGHT($\vec{V}$) + <13>;	$\vec{V}$ = <13,2,42,6,…>

$\vec{V}$ = < 13, 2, 42, 6, 0, 0, 0, … >

Vector Programs

Example:	Store:
$\vec{V}$:= <1>;	$\vec{V}$ = <1,0,0,0,…>
	$\vec{V}$ = <1,1,0,0,…>
Loop n times	$\vec{V}$ = <1,2,1,0,…>
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$);	$\vec{V}$ = <1,3,3,1,…>

$\vec{V}$ = n^{th} row of Pascal's triangle



Programs → Polynomials

The vector $\vec{V} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the polynomial:

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

Formal Power Series

The vector $\vec{V} = < a_0, a_1, a_2, \ldots >$ will be represented by the formal power series:

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

$$\vec{V} = < a_0, a_1, a_2, \ldots >$$

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

<0> is represented by	0
<k> is represented by	k
$\vec{V} + \vec{T}$ is represented by	$(P_V + P_T)$
RIGHT($\vec{V}$) is represented by	$(P_V\, X)$

Vector Programs

Example:

$\vec{V} := <1>;$	$P_V := 1;$
Loop n times $\vec{V} := \vec{V} + \text{RIGHT}(\vec{V});$	$P_V := P_V + P_V\, X;$

$\vec{V}$ = n^{th} row of Pascal's triangle

Vector Programs

Example:

$\vec{V} := <1>;$	$P_V := 1;$
Loop n times $\vec{V} := \vec{V} + \text{RIGHT}(\vec{V});$	$P_V := P_V(1+X);$

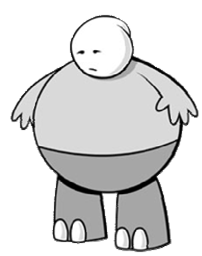
$\vec{V}$ = n^{th} row of Pascal's triangle

Vector Programs

Example:

$$\left.\begin{array}{l} \vec{V} := <1>; \\ \\ \text{Loop n times} \\ \quad \vec{V} := \vec{V} + \text{RIGHT}(\vec{V}); \end{array}\right\} P_V = (1+X)^n$$

$\vec{V}$ = n^{th} row of Pascal's triangle



Here's What You Need to Know…

- Polynomials count
- Binomial formula
- Combinatorial proofs of binomial identities
- Vector programs