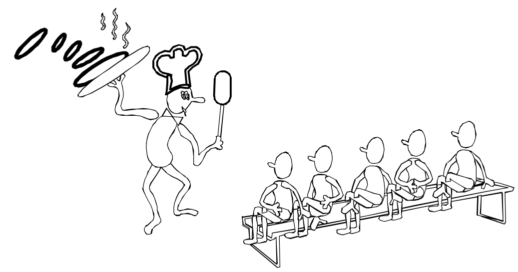
15-251

Cooking for Computer Scientists

Pancakes With A Problem!

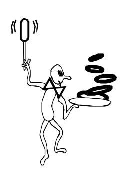
Lecture 3 (January 22, 2008)



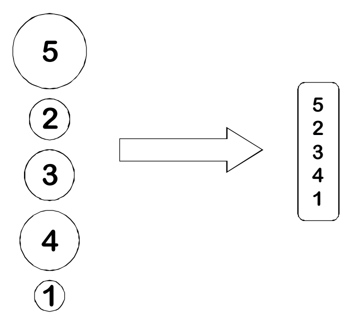
The chefs at our place are sloppy: when they prepare pancakes, they come out all different sizes

When the waiter delivers them to a customer, he rearranges them (so that smallest is on top, and so on, down to the largest at the bottom)

He does this by grabbing several from the top and flipping them over, repeating this (varying the number he flips) as many times as necessary

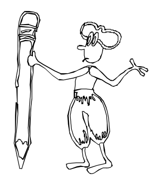


Developing A Notation: Turning pancakes into numbers



How do we sort this stack?
How many flips do we need?

5
2
3
4
1



4 Flips Are Sufficient

5	1	2	4	1
2	4	3	3	2
3	3	4	2	3
4	2	1	1	4
1	5	5	5	5

Best Way to Sort

X = Smallest number of flips required to sort:

5
2
3
4
1

Lower Bound

$? \leq X \leq \boxed{4}$

Upper Bound

Four Flips Are Necessary

5	1	4
2	4	1
3	3	3
4	2	2
1	5	5

If we could do it in three flips:

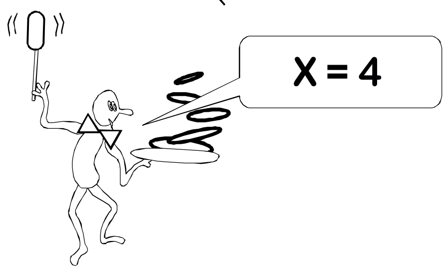
Flip 1 has to put 5 on bottom

Flip 2 must bring 4 to top (if it didn't, we would spend more than 3)

$4 \leq X \leq 4$

Lower Bound

Upper Bound



5^{th} Pancake Number

$P_5 =$ Number of flips required to sort the worst case stack of 5 pancakes

X_1	X_2	X_3		4		X_{119}	X_{120}
1 2 3 4 5	5 4 3 2 1	3	. . .	5 2 3 4 1	. . .	1 1 9	1 2 0

5^{th} Pancake Number

Lower Bound

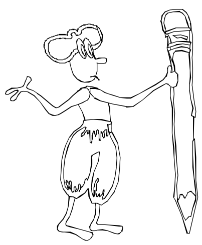
$\boxed{4} \leq P_5 \leq ?$

Upper Bound

$P_n =$ MAX over s $\in$ stacks of n pancakes of MIN # of flips to sort s

$P_n =$ The number of flips required to sort the worst-case stack of n pancakes

What is P_n for small n?



Can you do

n = 0, 1, 2, 3 ?

Initial Values of P_n

n	0	1	2	3
P_n	0	0	1	3

$P_3 = 3$

1
3
2

requires 3 Flips, hence $P_3 \geq 3$

ANY stack of 3 can be done by getting the big one to the bottom ($\leq$ 2 flips), and then using $\leq$ 1 flips to handle the top two

n^{th} Pancake Number

P_n = Number of flips required to sort the worst case stack of n pancakes

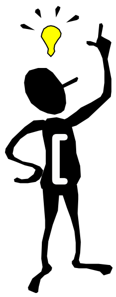
Lower Bound

$? \leq P_n \leq ?$

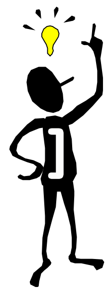
Upper Bound

Bracketing:

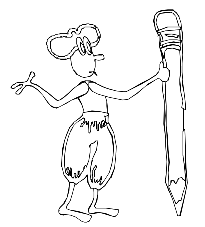
What are the best lower and upper bounds that I can prove?



$\leq$ **f(x)** $\leq$

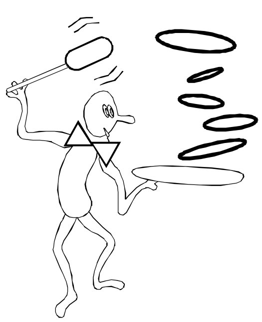


$? \leq P_n \leq ?$



Try to find upper and lower bounds on P_n, for $n > 3$

Bring-to-top Method



Bring biggest to top
Place it on bottom

Bring next largest to top
Place second from bottom

And so on…

Upper Bound On P_n:

Bring-to-top Method For n Pancakes

If n=1, no work required — we are done!

Otherwise, flip pancake n to top and then flip it to position n

Now use: $\boxed{\text{Bring To Top Method For n-1 Pancakes}}$

Total Cost: at most 2(n-1) = 2n –2 flips

Better Upper Bound On P_n:

Bring-to-top Method For n Pancakes

If n=2, at most one flip and we are done!

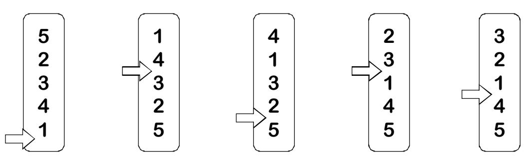
Otherwise, flip pancake n to top and then flip it to position n

Now use: Bring To Top Method For n-1 Pancakes

Total Cost: at most 2(n-2) + 1 = 2n – 3 flips

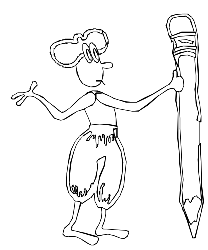
$$? \leq P_n \leq 2n\text{-}3$$

Bring-to-top not always optimal for a particular stack



Bring-to-top takes 5 flips, but we can do in 4 flips

$$? \leq P_n \leq 2n\text{-}3$$



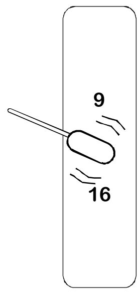
What other bounds can you prove on P_n?

Breaking Apart Argument

Suppose a stack S has a pair of adjacent pancakes that will not be adjacent in the sorted stack

Any sequence of flips that sorts stack S must have one flip that inserts the spatula between that pair and breaks them apart

Furthermore, this is true of the "pair" formed by the bottom pancake of S and the plate



S $\quad n \leq P_n$

2
4
6
8
.
.
n
1
3
5
.
.
n-1

Suppose n is even

S contains n pairs that will need to be broken apart during any sequence that sorts it

Detail: This construction only works when n>2

2
1

S $\quad n \leq P_n$

1
3
5
7
.
.
n
2
4
6
.
.
n-1

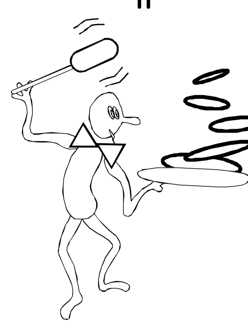
Suppose n is odd

S contains n pairs that will need to be broken apart during any sequence that sorts it

Detail: This construction only works when n>3

1
3
2

$n \leq P_n \leq 2n - 3$ for n > 3



Bring-to-top is within a factor of 2 of optimal!

From ANY stack to sorted stack in $\leq P_n$

From sorted stack to ANY stack in $\leq P_n$?

Reverse the sequences we use to sort

Hence, from ANY stack to ANY stack in $\leq 2P_n$

Can you find a faster way than $2P_n$ flips to go from ANY to ANY?

ANY Stack S to ANY stack T in $\leq P_n$

S: 4,3,5,1,2 — T: 5,2,4,3,1

1,2,3,4,5 — 3,5,1,2,4 ← "new T"

Rename the pancakes in S to be 1,2,3,..,n

Rewrite T using the new naming scheme that you used for S

The sequence of flips that brings the sorted stack to the "new T" will bring S to T

The Known Pancake Numbers

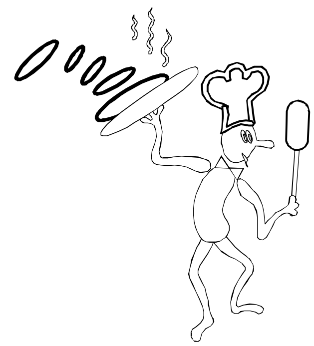
n	P_n
1	0
2	1
3	3
4	4
5	5
6	**7**
7	**8**
8	**9**
9	**10**
10	**11**
11	**13**
12	**14**
13	**15**

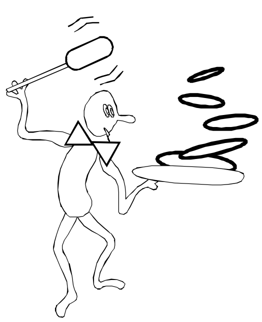
P_{14} is Unknown

$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 13 \cdot 14 = 14!$ orderings of 14 pancakes

$14! = 87{,}178{,}291{,}200$

Is This Really Computer Science?



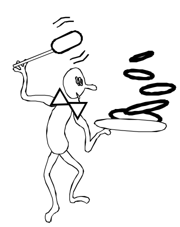


Sorting By Prefix Reversal

Posed in *Amer. Math. Monthly* 82 (1) (1975),
"Harry Dweighter" a.k.a. Jacob Goodman

$(17/16)n \leq P_n \leq (5n+5)/3$

William Gates and Christos Papadimitriou.
Bounds For Sorting By Prefix Reversal.
Discrete Mathematics, vol 27, pp 47-57, 1979.



$(15/14)n \le P_n \le (5n+5)/3$

H. Heydari and H. I. Sudborough. On the Diameter of the Pancake Network. *Journal of Algorithms,* vol 25, pp 67-94, 1997.

How many different stacks of n pancakes are there?

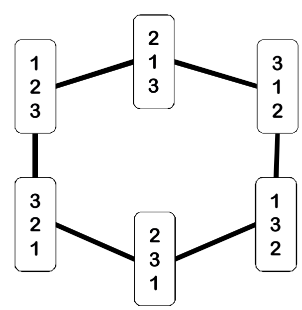
$n! = 1 \times 2 \times 3 \times \ldots \times n$

Pancake Network: Definition For n! Nodes

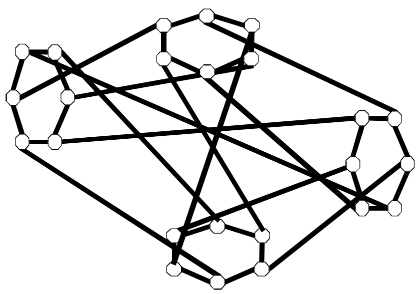
For each node, assign it the name of one of the n! stacks of n pancakes

Put a wire between two nodes if they are one flip apart

Network For n = 3

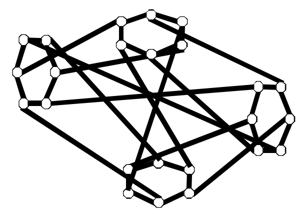


Network For n=4



Pancake Network: Message Routing Delay

What is the maximum distance between two nodes in the pancake network?



P_n

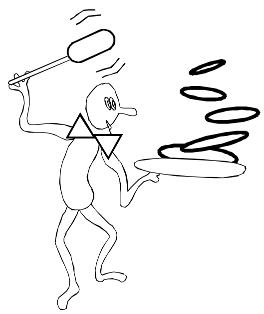
Pancake Network: Reliability

If up to n-2 nodes get hit by lightning, the network remains connected, even though each node is connected to only n-1 others

The Pancake Network is optimally reliable for its number of edges and nodes

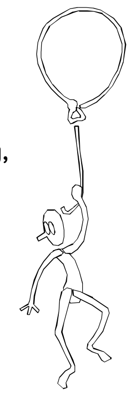
Mutation Distance

One “Simple” Problem



A host of problems and applications at the frontiers of science

High Level Point



Computer Science is not merely about computers and programming, it is about mathematically modeling our world, and about finding better and better ways to solve problems

Today’s lecture is a microcosm of this exercise

Here’s What You Need to Know…

Definitions of:
- n^{th} pancake number
- lower bound
- upper bound

Proof of:
- ANY to ANY in $\leq P_n$

Important Technique:
- Bracketing