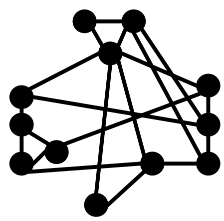
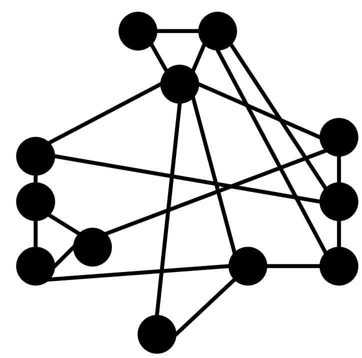
Complexity Theory:

Efficient Reductions Between Computational Problems

Lecture 28 (May 2, 2006)



A Graph Named "Gadget"



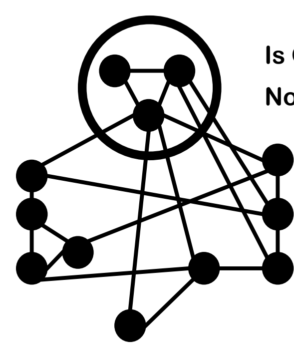
K-Coloring

We define a k-coloring of a graph:

- Each node gets colored with one color
- At most k different colors are used
- If two nodes have an edge between them they must have different colors

A graph is called k-colorable if and only if it has a k-coloring

A 2-CRAYOLA Question!



Is Gadget 2-colorable?

No, it contains a triangle

A 2-CRAYOLA Question!

Given a graph G, how can we decide if it is 2-colorable?

Answer: Enumerate all 2^n possible colorings to look for a valid 2-color

How can we efficiently decide if G is 2-colorable?

Proposition: If G contains an odd cycle, G is not 2-colorable

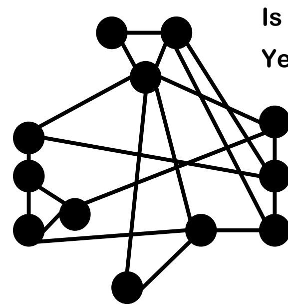
Alternate coloring algorithm:

- To 2-color a connected graph G, pick an arbitrary node v, and color it white
- Color all v's neighbors black
- Color all their uncolored neighbors white, and so on
- If the algorithm terminates without a color conflict, output the 2-coloring
- Else, output an odd cycle

A 2-CRAYOLA Question!

Theorem: G contains an odd cycle if and only if G is not 2-colorable

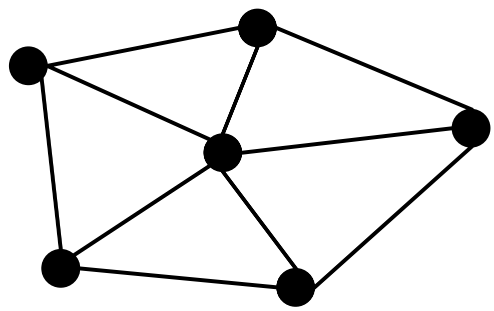
A 3-CRAYOLA Question!



Is Gadget 3-colorable?

Yes!

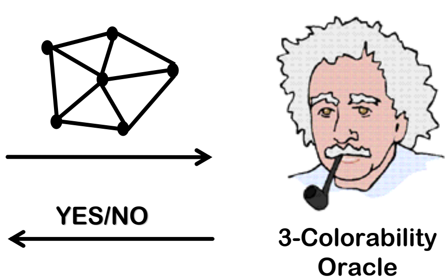
A 3-CRAYOLA Question!



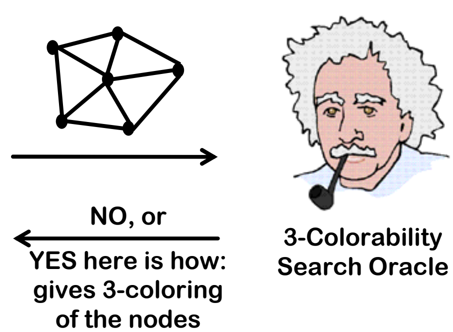
3-Coloring Is Decidable by Brute Force

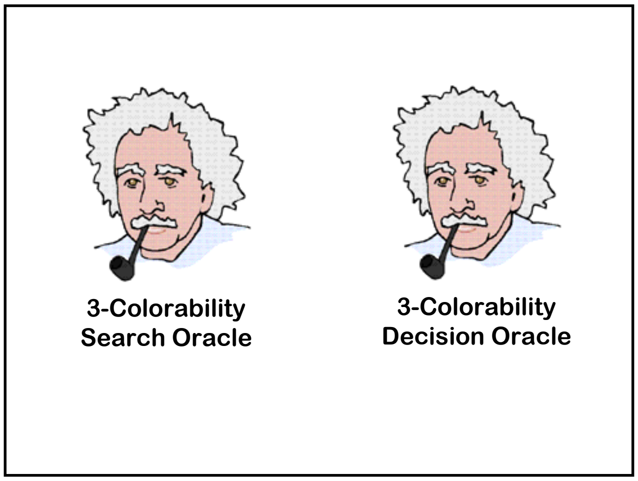
Try out all 3^n colorings until you determine if G has a 3-coloring

A 3-CRAYOLA Oracle

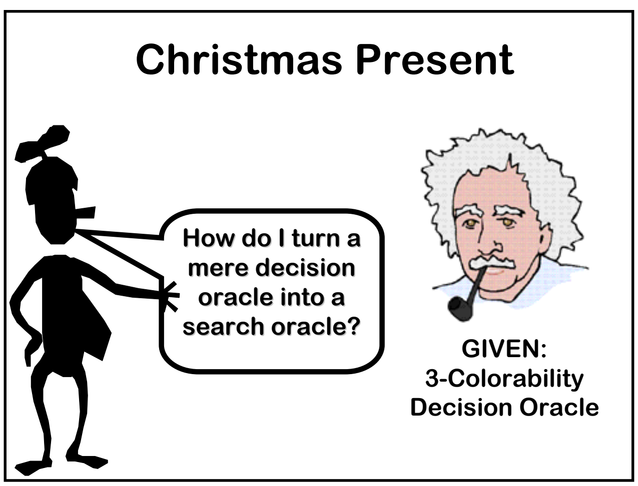


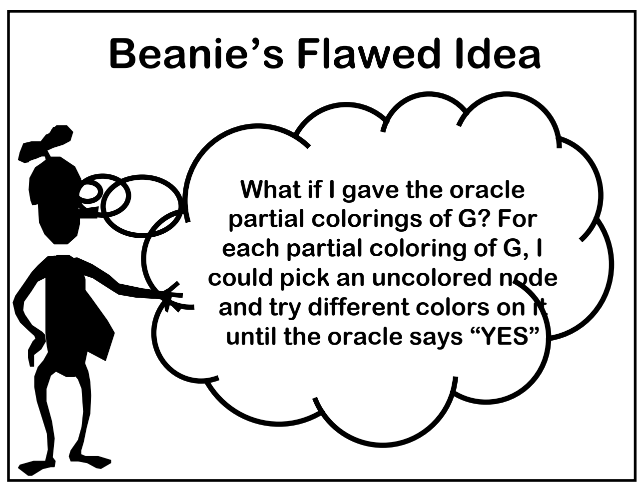
Better 3-CRAYOLA Oracle



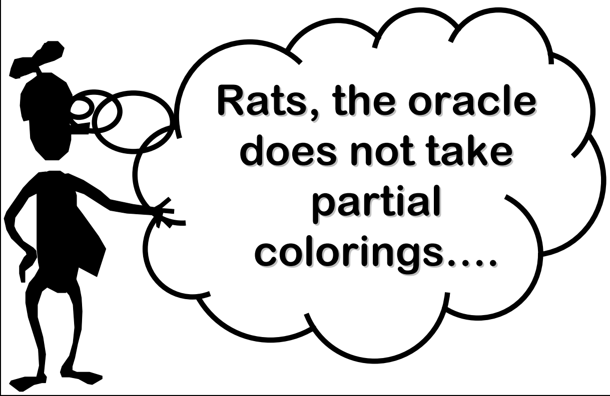

3-Colorability
Search Oracle
3-Colorability
Decision Oracle


Christmas Present
BUT I WANTED a SEARCH oracle for Christmas
I am really bummed out
GIVEN:
3-Colorability
Decision Oracle

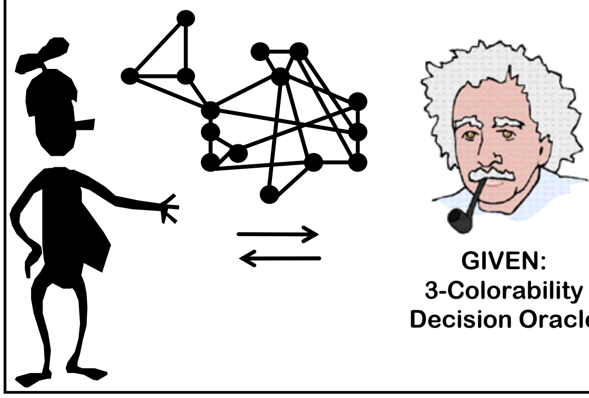

Christmas Present
How do I turn a mere decision oracle into a search oracle?
GIVEN:
3-Colorability
Decision Oracle

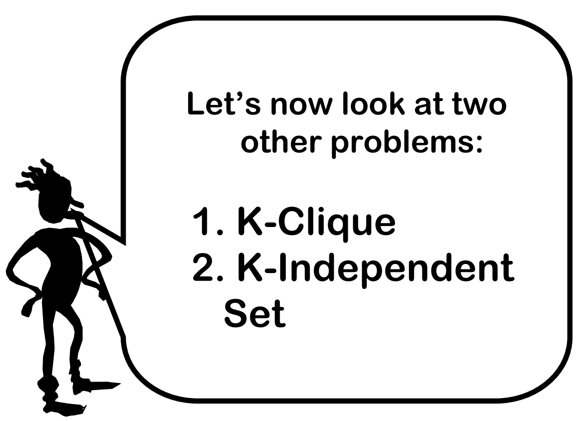

Beanie’s Flawed Idea
What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says “YES”

Beanie's Flawed Idea



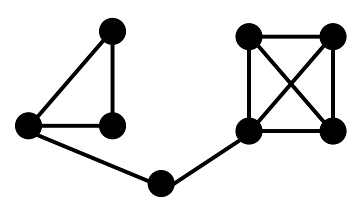
Beanie's Fix





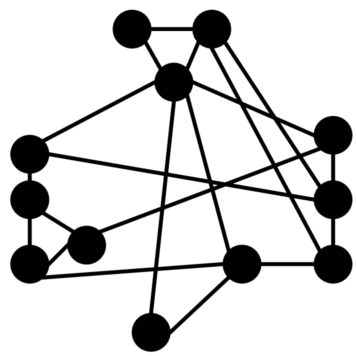
K-Cliques

A K-clique is a set of K nodes with all K(K-1)/2 possible edges between them



This graph contains a 4-clique

A Graph Named “Gadget”

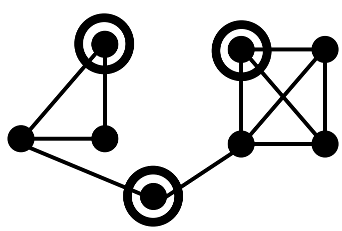


Given: (G, k)
Question: Does G contain a k-clique?

BRUTE FORCE: Try out all n choose k possible locations for the k clique

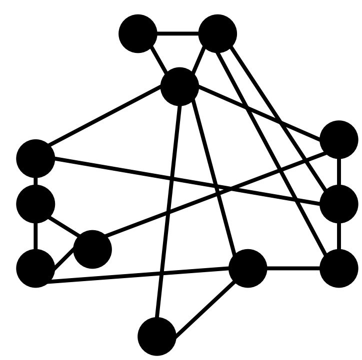
Independent Set

An independent set is a set of nodes with no edges between them



This graph contains an independent set of size 3

A Graph Named “Gadget”



Given: (G, k)
Question: Does G contain an independent set of size k?

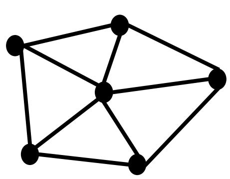
BRUTE FORCE: Try out all n choose k possible locations for the k independent set

Clique / Independent Set

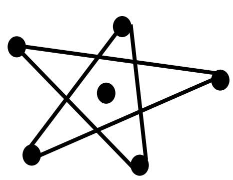
Two problems that are cosmetically different, but substantially the same

Complement of G

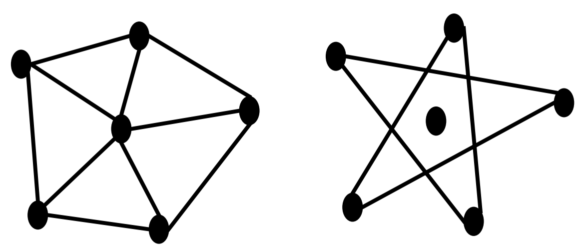
Given a graph G, let G*, the complement of G, be the graph obtained by the rule that two nodes in G* are connected if and only if the corresponding nodes of G are not connected



G

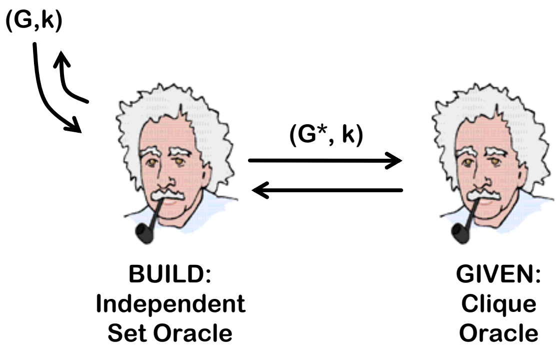


G*

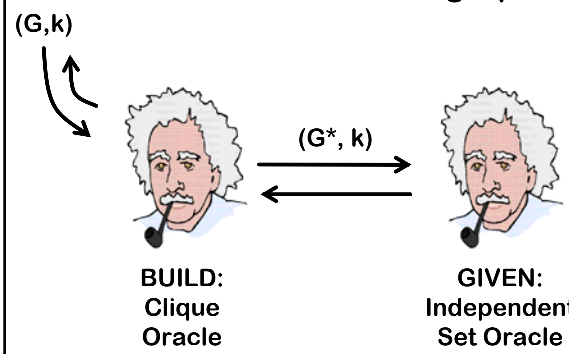


G has a k-clique ⇔ G* has an independent set of size k

Let G be an n-node graph

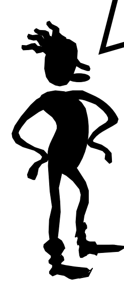


Let G be an n-node graph



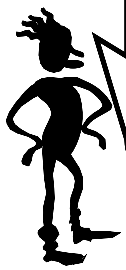
Clique / Independent Set

Two problems that are cosmetically different, but substantially the same



Thus, we can quickly reduce a clique problem to an independent set problem and vice versa

There is a fast method for one if and only if there is a fast method for the other

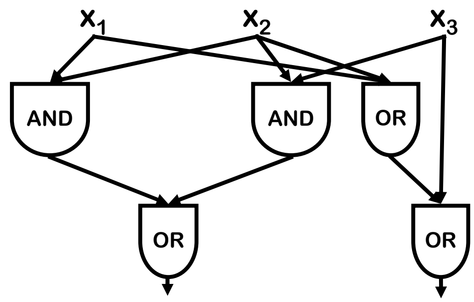


Let's now look at two other problems:

1. Circuit Satisfiability
2. Graph 3-Colorability

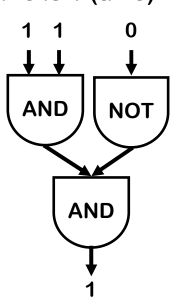
Combinatorial Circuits

AND, OR, NOT, 0, 1 gates wired together with no feedback allowed



Circuit-Satisfiability

Given a circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?



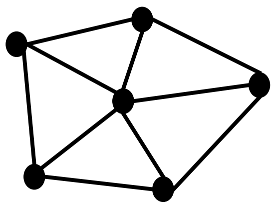
Yes, this circuit is satisfiable: 110

Circuit-Satisfiability

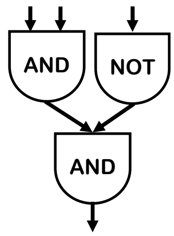
Given: A circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?

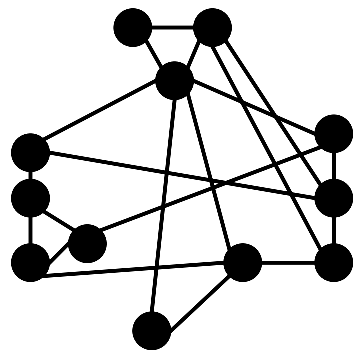
BRUTE FORCE: Try out all 2^n assignments

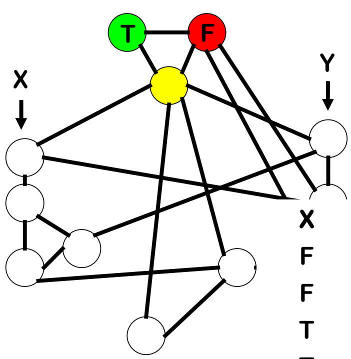
3-Colorability



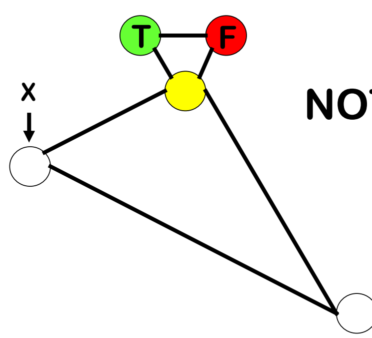
Circuit Satisfiability



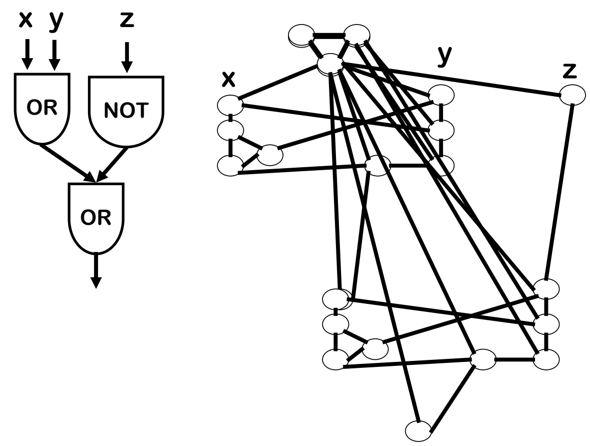




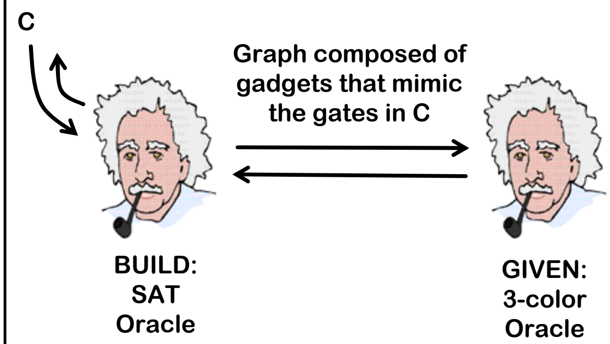
X	Y	OR
F	F	F
F	T	T
T	F	T
T	T	T



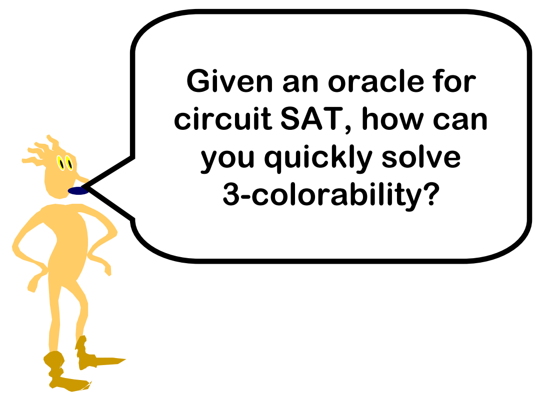
NOT gate!

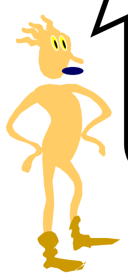


Let C be an n-input circuit



You can quickly transform a method to decide 3-coloring into a method to decide circuit satifiability!





Can you make a circuit that takes a description of a graph and a node coloring, and checks if it is a valid 3-coloring?

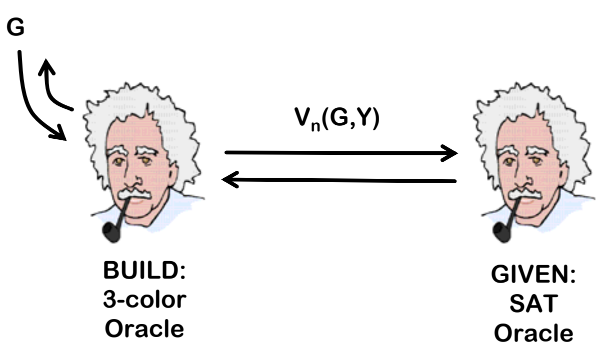
$V_n(X,Y)$

Let V_n be a circuit that takes an n-node graph X and an assignment of colors to nodes Y, and **verifies** that Y is a valid 3 coloring of X. I.e., $V_n(X,Y) = 1$ iff Y is a 3 coloring of X

X is expressed as an n choose 2 bit sequence. Y is expressed as a 2n bit sequence

Given n, we can construct V_n in time $O(n^2)$

Let G be an n-node graph



Circuit-SAT / 3-Colorability

Two problems that are cosmetically different, but substantially the same

Circuit-SAT / 3-Colorability

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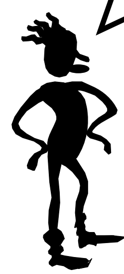
Clique / Independent Set



Given an oracle for circuit SAT, how can you quickly solve k-clique?

Circuit-SAT / 3-Colorability

↓ ↑

Clique / Independent Set

Four problems that are cosmetically different, but substantially the same

FACT: No one knows a way to solve any of the 4 problems that is fast on all instances

Summary

Many problems that appear different on the surface can be efficiently reduced to each other, revealing a deeper similarity