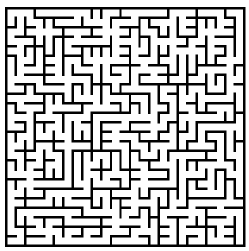
Random Walks

Lecture 24 (April 13, 2006)



Probability Refresher

What does this mean: E[X | Y]?

Is this true:

$\Pr[A] = \Pr[A \mid B]\Pr[B] + \Pr[A \mid \overline{B}]\Pr[\overline{B}]$

Yes!

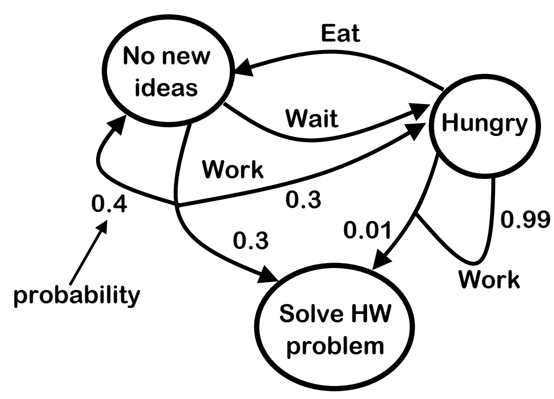
Similarly:

$E[X] = E[X \mid Y]\Pr[Y] + E[X \mid \overline{Y}]\Pr[\overline{Y}]$

Today, we will learn an important lesson:

How to walk drunk home

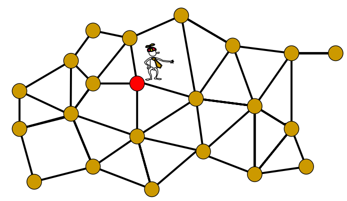
Abstraction of Student Life



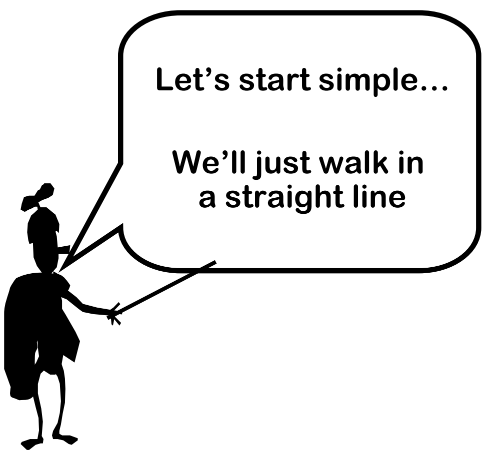
Like finite automata, but instead of a determinisic or non-deterministic action, we have a probabilistic action

Example questions: “What is the probability of reaching goal on string Work,Eat,Work?”

Simpler: Random Walks on Graphs



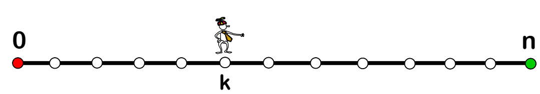
At any node, go to one of the neighbors of the node with equal probability



Random Walk on a Line

You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

You leave when you are broke or have \$n



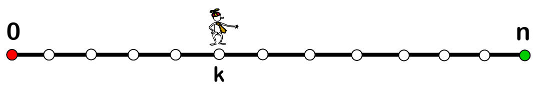
Question 1: what is your expected amount of money at time t?

Let X_t be a R.V. for the amount of \$\$\$ at time t

Random Walk on a Line

You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

You leave when you are broke or have \$n



$X_t = k + \delta_1 + \delta_2 + ... + \delta_t,$

(δ_i is RV for change in your money at time i)

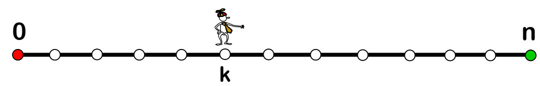
$E[\delta_i] = 0$

So, $E[X_t] = k$

Random Walk on a Line

You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

You leave when you are broke or have \$n



Question 2: what is the probability that you leave with \$n?

Random Walk on a Line

Question 2: what is the probability that you leave with \$n?

$E[X_t] = k$

$E[X_t] = E[X_t | X_t = 0] \times Pr(X_t = 0)$
$\quad + E[X_t | X_t = n] \times Pr(X_t = n)$
$\quad + E[X_t | \text{neither}] \times Pr(\text{neither})$

$E[X_t] = 0 + n \times Pr(X_t = n)$
$\quad + (\text{something}_t) \times Pr(\text{neither})$

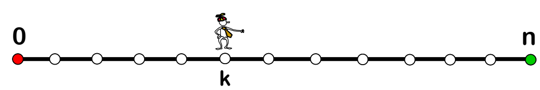
As $t \to \infty$, $Pr(\text{neither}) \to 0$, also $\text{something}_t < n$

Hence $Pr(X_t = n) \to k/n$

Another Way To Look At It

You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

You leave when you are broke or have \$n

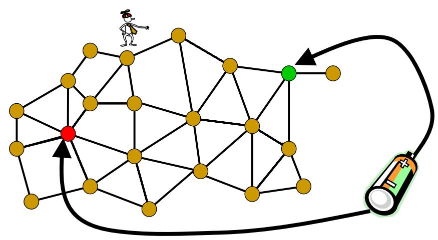


Question 2: what is the probability that you leave with \$n?

= probability that I hit green before I hit red

Random Walks and Electrical Networks

What is chance I reach green before red?



Same as voltage if edges are resistors and we put 1-volt battery between green and red

Random Walks and Electrical Networks

p_x = Pr(reach green first starting from x)

p_{green}= 1, p_{red} = 0

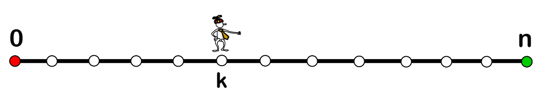
And for the rest p_x = $\text{Average}_{y\in \text{Nbr}(x)}(p_y)$

Same as equations for voltage if edges all have same resistance!

Another Way To Look At It

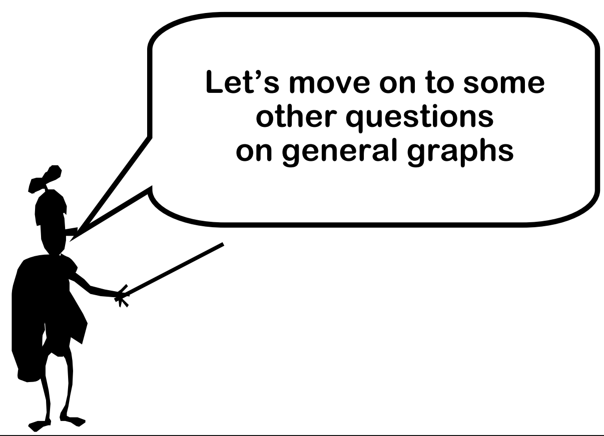
You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

You leave when you are broke or have \$n

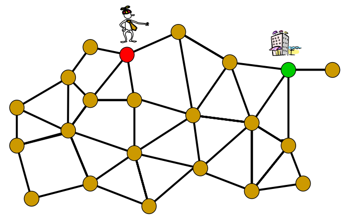


Question 2: what is the probability that you leave with \$n?

voltage(k) = k/n
= Pr[hitting n before 0 starting at k] !!!



Getting Back Home

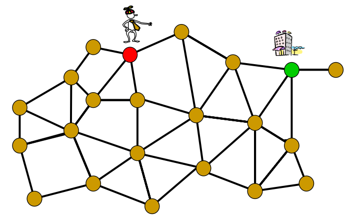


Lost in a city, you want to get back to your hotel
How should you do this?

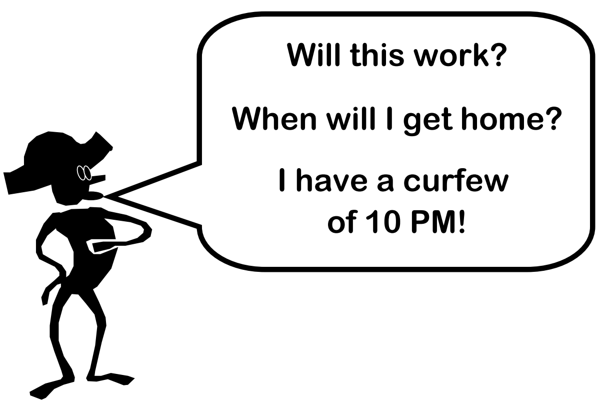
Depth First Search!

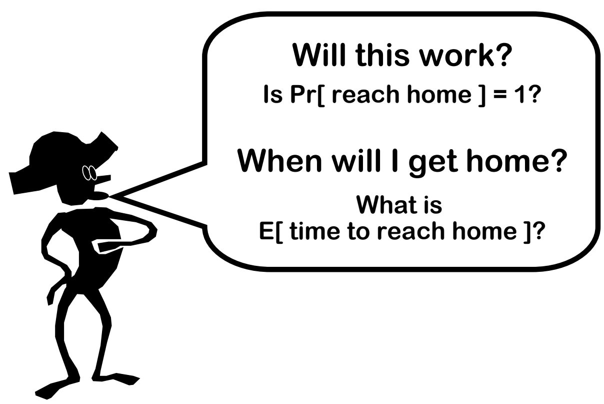
Requires a good memory and a piece of chalk

Getting Back Home



How about walking randomly?





Relax, Bonzo!

Yes,
Pr[will reach home] = 1

Furthermore:

If the graph has
n nodes and m edges, then

E[time to visit all nodes]
$\le$ 2m × (n-1)

E[time to reach home] is at most this

Cover Times

Let us define a couple of useful things:

Cover time (from u)
C_u = E [time to visit all vertices | start at u]

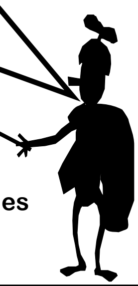
Cover time of the graph
$C(G) = \max_u \{ C_u \}$

(worst case expected time to see all vertices)

Cover Time Theorem

If the graph G has
n nodes and m edges, then
the cover time of G is

$C(G) \le 2m (n - 1)$



Any graph on n vertices has < $n^2/2$ edges

Hence $C(G) < n^3$ for all graphs G

We Will Eventually Get Home

Look at the first n steps

There is a non-zero chance p_1 that we get home

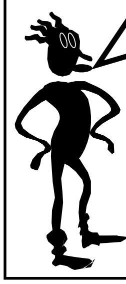
Also, $p_1 \geq (1/n)^n$

Suppose we fail

Then, wherever we are, there is a chance $p_2 \geq (1/n)^n$ that we hit home in the next n steps from there

Probability of failing to reach home by time kn

$$= (1 - p_1)(1 - p_2) \ldots (1 - p_k) \to 0 \text{ as } k \to \infty$$



Actually, we get home pretty fast…

Chance that we don't hit home by $2k \times 2m(n-1)$ steps is $(½)^k$

A Simple Calculation

True of False:

If the average income of people is \$100 then more than 50% of the people can be earning more than \$200 each

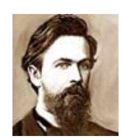
False! else the average would be higher!!!

Markov's Inequality

If X is a non-negative r.v. with mean E[X], then

$$\Pr[\, X > 2\,E[X]\,] \leq ½$$

$$\Pr[\, X > k\,E[X]\,] \leq 1/k$$



Andrei A. Markov

Markov's Inequality

Non-neg random variable X has expectation $A = E[X]$

$A = E[X] = E[X \mid X > 2A]\ \Pr[X > 2A]$
$\quad + E[X \mid X \le 2A]\ \Pr[X \le 2A]$

$\ge E[X \mid X > 2A]\ \Pr[X > 2A]$ (since X is non-neg)

Also, $E[X \mid X > 2A] > 2A$

$\Rightarrow A \ge 2A \times \Pr[X > 2A]$
$\Rightarrow ½ \ge \Pr[X > 2A]$

$\Pr[\ X > k \times \text{expectation}\ ] \le 1/k$

An Averaging Argument

Suppose I start at u

E[time to hit all vertices | start at u] $\le$ C(G)

Hence, by Markov's Inequality:
Pr[time to hit all vertices > 2C(G) | start at u] $\le ½$

So Let's Walk Some Mo!

Pr [time to hit all vertices > 2C(G) | start at u] $\le ½$

Suppose at time 2C(G), I'm at some node with more nodes still to visit

Pr [haven't hit all vertices in 2C(G) <u>more</u> time | start at v] $\le ½$

Chance that you failed both times $\le ¼ = (½)^2$

The Power of Independence

It's like flipping a coin with tails probability $q \le ½$

The probability that you get k tails is $q^k \le (½)^k$
(because the trials are independent!)

Hence,
Pr[havent hit everyone in time k × 2C(G)] $\le (½)^k$

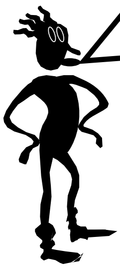
Exponential in k

Hence, if we know that

Expected Cover Time
$C(G) < 2m(n-1)$

then

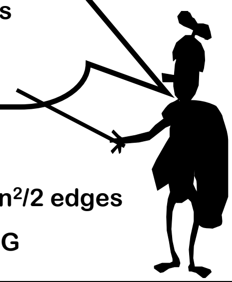
Pr[home by time $4k\ m(n-1)$]
$\geq 1 - (½)^k$



Cover Time Theorem

If the graph G has
n nodes and m edges, then
the cover time of G is

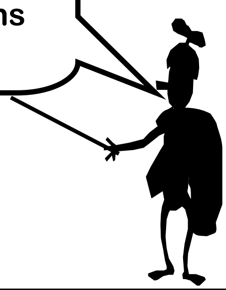
$C(G) \leq 2m\ (n – 1)$



Any graph on n vertices has < $n^2/2$ edges

Hence $C(G) < n^3$ for all graphs G

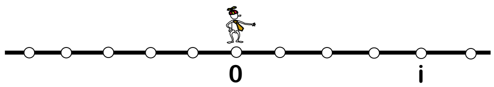
Random walks on infinite graphs



Drunk man will find his
way home, but drunk bird
may get lost forever

- Shizuo Kakutani

Random Walk On a Line



Flip an unbiased coin and go left/right

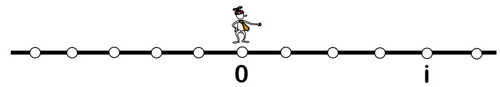
Let X_t be the position at time t

$\Pr[X_t = i] = \Pr[\#\text{heads} - \#\text{tails} = i]$

$= \Pr[\#\text{heads} - (t - \#\text{heads}) = i]$

$= \binom{t}{(t-i)/2} / 2^t$

Random Walk On a Line



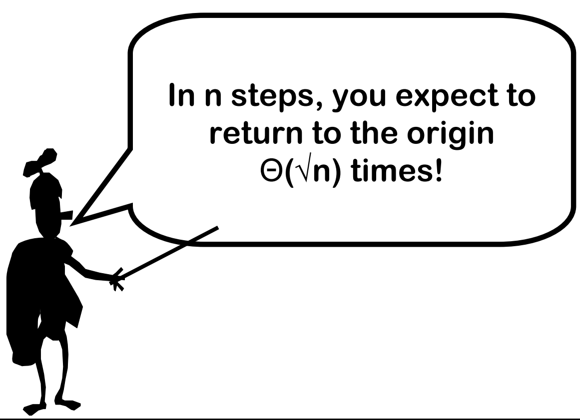
$\Pr[X_{2t} = 0] = \binom{2t}{t} / 2^{2t} \le \Theta(1/\sqrt{t})$ Sterling's approx

Y_{2t} = indicator for $(X_{2t} = 0) \Rightarrow E[Y_{2t}] = \Theta(1/\sqrt{t})$

Z_{2n} = number of visits to origin in 2n steps

$E[Z_{2n}] = E[\sum_{t = 1 \ldots n} Y_{2t}]$

$= \Theta(1/\sqrt{1} + 1/\sqrt{2} + \ldots + 1/\sqrt{n}) = \Theta(\sqrt{n})$



Simple Claim

If we repeatedly flip coin with bias p
E[# of flips till heads] = 1/p

Theorem: If Pr[not return to origin] = p, then
E[number of times at origin] = 1/p

Proof: H = never return to origin. T = we do.

Hence returning to origin is like getting a tails

E[# of returns] = E[# tails before a head]

= 1/p – 1

(But we started at the origin too!)

We Will Return…

Theorem: If Pr[not return to origin] = p, then E[number of times at origin] = 1/p

Theorem: Pr[we return to origin] = 1

Proof: Suppose not

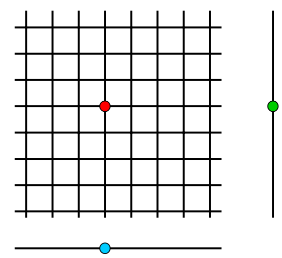
Hence p = Pr[never return] > 0

E [#times at origin] = 1/p = constant

But we showed that E[Z_n] = $\Theta(\sqrt{n}) \to \infty$

How About a 2-d Grid?

Let us simplify our 2-d random walk:
move in both the x-direction and y-direction…



In The 2-d Walk

Returning to the origin in the grid
$\Leftrightarrow$ both "line" random walks return to their origins

Pr[visit origin at time t] $= \Theta(1/\sqrt{t}) \times \Theta(1/\sqrt{t})$
$= \Theta(1/t)$

E[# of visits to origin by time n]
$= \Theta(1/1 + 1/2 + 1/3 + \ldots + 1/n) = \Theta(\log n)$

We Will Return (Again)

Theorem: If Pr[not return to origin] = p, then E[number of times at origin] = 1/p

Theorem: Pr[we return to origin] = 1

Proof: Suppose not

Hence p = Pr[never return] > 0

E [#times at origin] = 1/p = constant

But we showed that E[Z_n] = $\Theta(\log n) \to \infty$

But In 3D

Pr[visit origin at time t] = $\Theta(1/\sqrt{t})^3 = \Theta(1/t^{3/2})$

$\lim_{n \to \infty}$ E[# of visits by time n] < K (constant)

Hence Pr[never return to origin] > 1/K

A Cycle Game

Suppose we walk on the cycle at random till we see all the nodes

Is x more likely than y to be the last node we see?