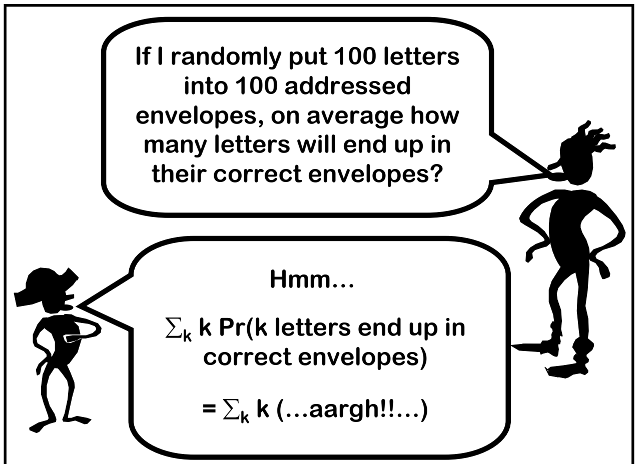
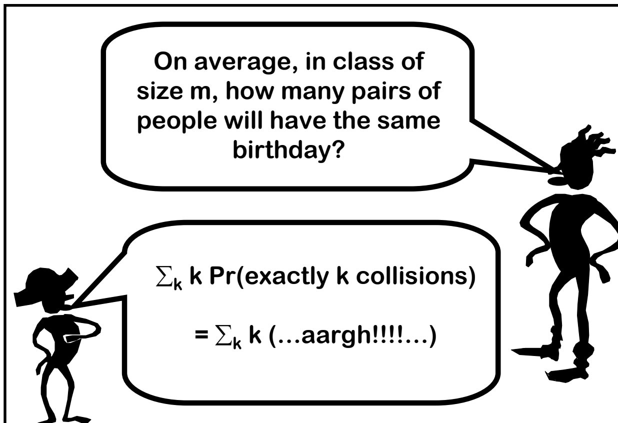
Lecture 22 (April 6, 2006)

Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem really really messy…





The new tool is called "Linearity of Expectation"

Random Variable

To use this new tool, we will also need to understand the concept of a Random Variable

Today's lecture: not too much material, but need to understand it well

Random Variable

Let S be sample space in a probability distribution

A Random Variable is a real-valued function on S

Examples:

X = value of white die in a two-dice roll

X(3,4) = 3, X(1,6) = 1

Y = sum of values of the two dice

Y(3,4) = 7, Y(1,6) = 7

W = (value of white die)$^{\text{value of black die}}$

W(3,4) = 3^4, Y(1,6) = 1^6

Tossing a Fair Coin n Times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S

$\Rightarrow D(x) = (½)^n$ for all $x \in S$

Random Variables (say n = 10)

X = # of heads

X(HHHTTHTHTT) = 5

Y = (1 if #heads = #tails, 0 otherwise)

Y(HHHTTHTHTT) = 1, Y(THHHHTTTTT) = 0

Notational Conventions

Use letters like A, B, E for events

Use letters like X, Y, f, g for R.V.'s

R.V. = random variable

Two Views of Random Variables

Think of a R.V. as

Input to the function is random

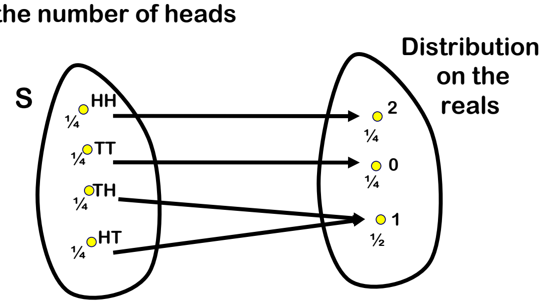
A function from S to the reals $\mathbb{R}$

Or think of the induced distribution on $\mathbb{R}$

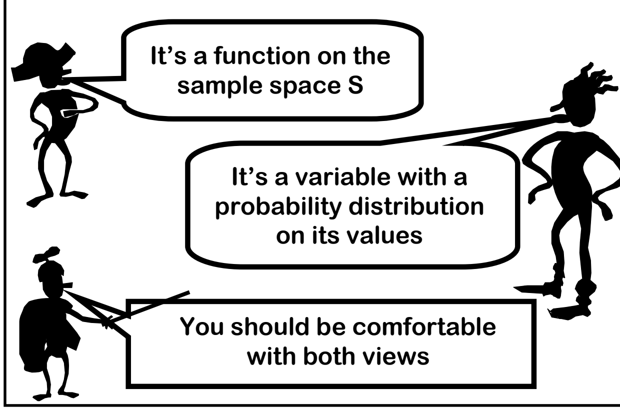
Randomness is "pushed" to the values of the function

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts the number of heads



It's a Floor Wax And a Dessert Topping



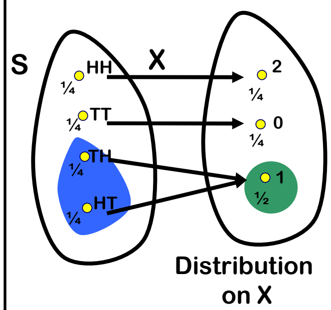
From Random Variables to Events

For any random variable X and value a, we can define the event A that X = a

$Pr(A) = Pr(X=a) = Pr(\{x \in S \mid X(x)=a\})$

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts # of heads



$Pr(X = a) = Pr(\{x \in S \mid X(x) = a\})$

$Pr(X = 1)$

$= Pr(\{x \in S \mid X(x) = 1\})$

$= Pr(\{TH, HT\}) = ½$

From Random Variables to Events

For any random variable X and value a, we can define the event A that X = a

$Pr(A) = Pr(X=a) = Pr(\{x \in S \mid X(x)=a\})$

X has a distribution on its values

X is a function on the sample space S

From Events to Random Variables

For any event A, can define the indicator random variable for A:

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition: Expectation

The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x)\, X(x) = \sum_{k} k \Pr[X = k]$$

X is a function on the sample space S

X has a distribution on its values

A Quick Calculation…

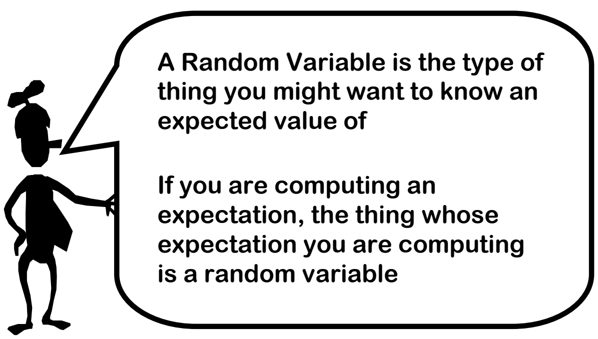
What if I flip a coin 3 times? What is the expected number of heads?

$E[X] = (1/8)\times 0 + (3/8)\times 1 + (3/8)\times 2 + (1/8)\times 3 = 1.5$

But $\Pr[X = 1.5] = 0$

Moral: don't always expect the expected.
$\Pr[X = E[X]]$ may be 0 !

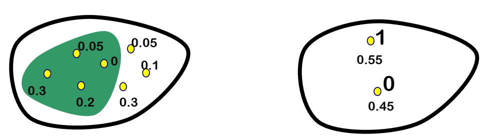
Type Checking



Indicator R.V.s: $E[X_A] = \Pr(A)$

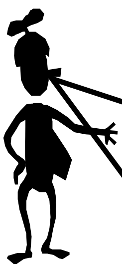
For any event A, can define the indicator random variable for A:

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$



$E[X_A] = 1 \times \Pr(X_A = 1) = \Pr(A)$

Adding Random Variables

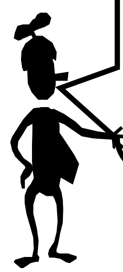


If X and Y are random variables (on the same set S), then Z = X + Y is also a random variable

$Z(x) = X(x) + Y(x)$

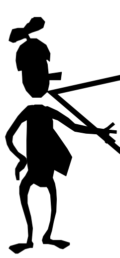
E.g., rolling two dice. X = 1st die, Y = 2nd die, Z = sum of two dice

Adding Random Variables



Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let Z = X+Y. Z measures the combined arm lengths

Independence



Two random variables X and Y are independent if for every a,b, the events X=a and Y=b are independent

How about the case of X=1st die, Y=2nd die? X = left arm, Y=right arm?

Linearity of Expectation

If Z = X+Y, then

$$E[Z] = E[X] + E[Y]$$

Even if X and Y are not independent

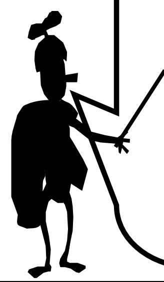
$$E[Z] = \sum_{x \in S} \Pr[x]\, Z(x)$$

$$= \sum_{x \in S} \Pr[x]\, (X(x) + Y(x))$$

$$= \sum_{x \in S} \Pr[x]\, X(x) + \sum_{x \in S} \Pr[x]\, Y(x))$$

$$= E[X] + E[Y]$$

Linearity of Expectation



E.g., 2 fair flips:

X = 1st coin, Y = 2nd coin

Z = X+Y = total # heads

What is E[X]? E[Y]? E[Z]?

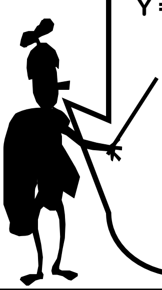
1,1,2 HH

1,0,1 HT

0,1,1 TH

0,0,0 TT

Linearity of Expectation



E.g., 2 fair flips:

X = at least one coin is heads

Y = both coins are heads, Z = X+Y

Are X and Y independent?

What is E[X]? E[Y]? E[Z]?

1,1,2 HH

1,0,1 HT

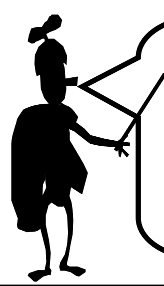
1,0,1 TH

0,0,0 TT

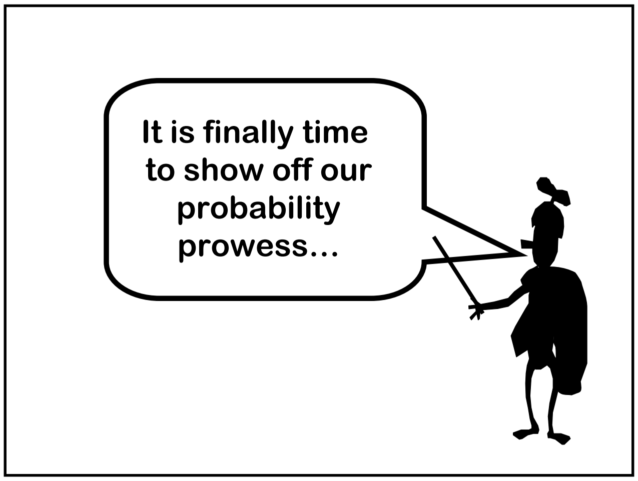
By Induction

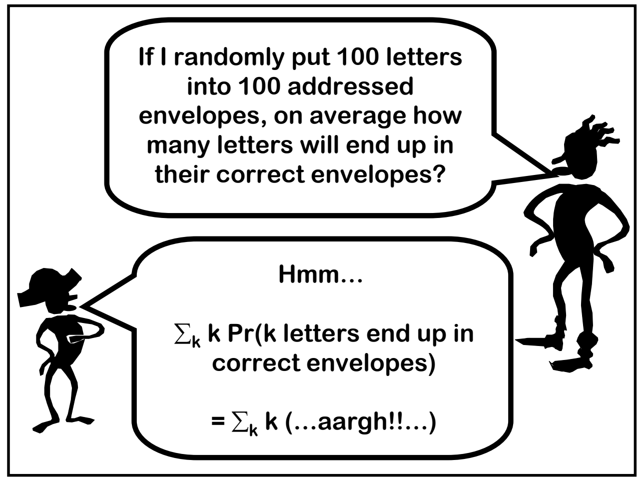
$$E[X_1 + X_2 + \ldots + X_n] =$$

$$E[X_1] + E[X_2] + \ldots. + E[X_n]$$



The expectation of the sum = The sum of the expectations





Use Linearity of Expectation

Let A_i be the event the i^{th} letter ends up in its correct envelope

Let X_i be the indicator R.V. for A_i

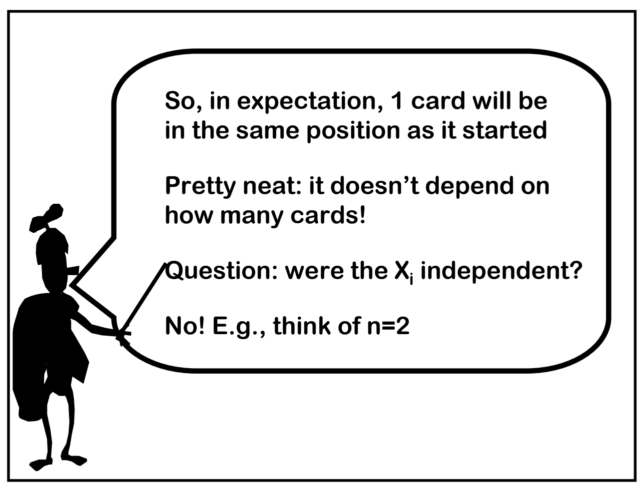
$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$

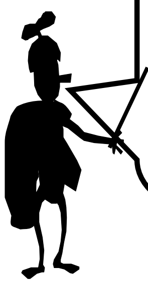
We are asking for E[Z]

$E[X_i] = Pr(A_i) = 1/100$

So E[Z] = 1



Use Linearity of Expectation



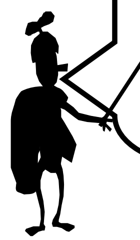
General approach:

View thing you care about as expected value of some RV

Write this RV as sum of simpler RVs (typically indicator RVs)

Solve for their expectations and add them up!

Example



We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$$\sum_k k \Pr(X = k) = \sum_k k \binom{n}{k} p^k(1-p)^{n-k}$$

But now we know a better way!

Linearity of Expectation!

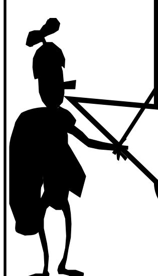
Let X = number of heads when n independent coins of bias p are flipped

Break X into n simpler RVs:

$$X_i = \begin{cases} 1 & \text{if the } j^{th} \text{ coin is tails} \\ 0 & \text{if the } j^{th} \text{ coin is heads} \end{cases}$$

$E[X] = E[\Sigma_i X_i] = np$

What About Products?



If Z = XY, then
E[Z] = E[X] × E[Y]?

No!

X=indicator for “1st flip is heads”
Y=indicator for “1st flip is tails”

E[XY]=0

But It's True If RVs Are Independent

Proof: $E[X] = \sum_a a \times Pr(X=a)$
$E[Y] = \sum_b b \times Pr(Y=b)$

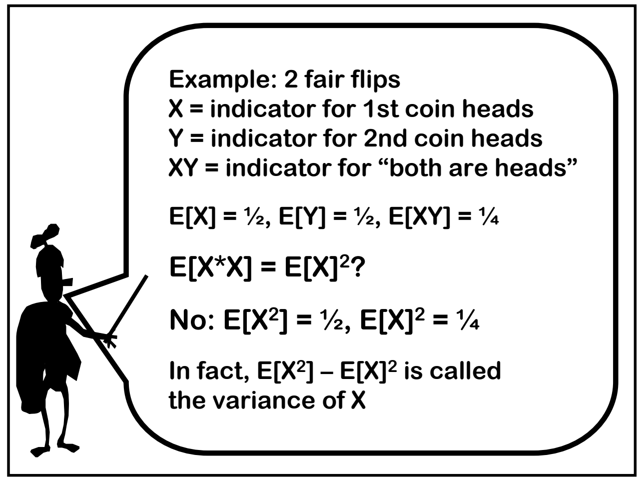
$E[XY] = \sum_c c \times Pr(XY = c)$

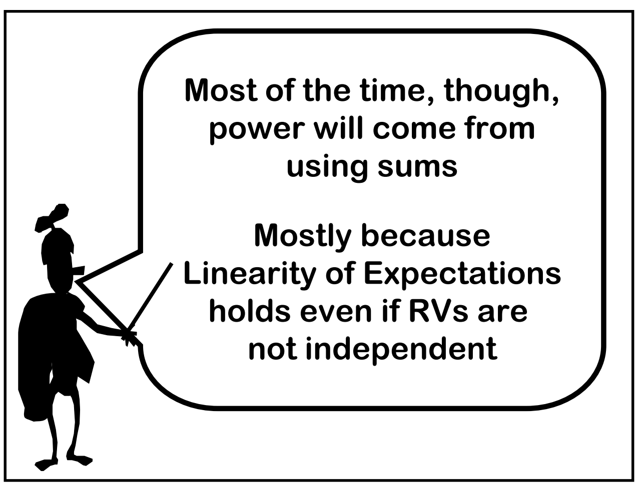
$= \sum_c \sum_{a,b:ab=c} c \times Pr(X=a \cap Y=b)$

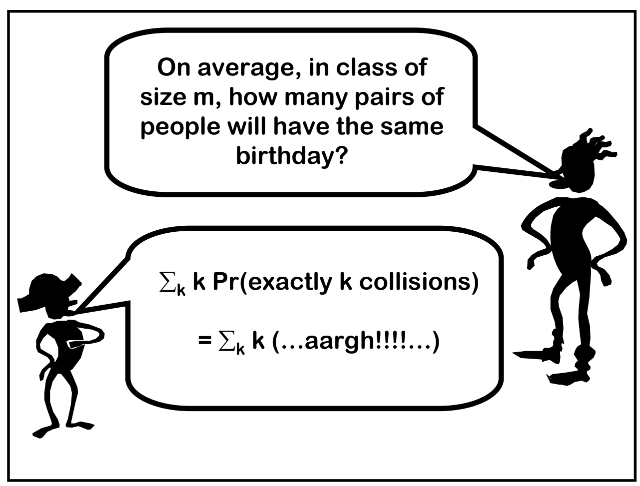
$= \sum_{a,b} ab \times Pr(X=a \cap Y=b)$

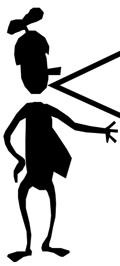
$= \sum_{a,b} ab \times Pr(X=a)\, Pr(Y=b)$

$= E[X]\, E[Y]$







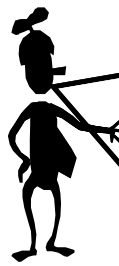


Use linearity of expectation

Suppose we have m people each with a uniformly chosen birthday from 1 to 366

X = number of pairs of people with the same birthday

E[X] = ?



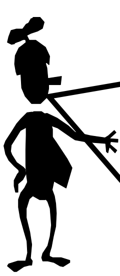
X = number of pairs of people with the same birthday

E[X] = ?

Use m(m-1)/2 indicator variables, one for each pair of people

X_{jk} = 1 if person j and person k have the same birthday; else 0

$E[X_{jk}] = (1/366)\ 1 + (1 – 1/366)\ 0$
$= 1/366$



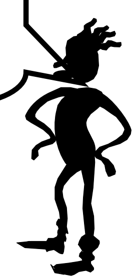
X = number of pairs of people with the same birthday

X_{jk} = 1 if person j and person k have the same birthday; else 0

$E[X_{jk}] = (1/366)\ 1 + (1 – 1/366)\ 0$
$= 1/366$

$$\begin{aligned} E[X] &= E[\ \Sigma_{j \le k \le m} X_{jk}\ ] \\ &= \Sigma_{j \le k \le m} E[\ X_{jk}\ ] \\ &= m(m-1)/2 \times 1/366 \end{aligned}$$

Step Right Up…



You pick a number n ∈ [1..6]. You roll 3 dice. If any match n, you win \$1. Else you pay me \$1. Want to play?

Hmm… let’s see

Analysis

A_i = event that i-th die matches

X_i = indicator RV for A_i

Expected number of dice that match:
$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = ½$

But this is not the same as
Pr(at least one die matches)

Analysis

Pr(at least one die matches)
= 1 – Pr(none match)
= 1 – $(5/6)^3$ = 0.416

Study Bee

Random Variables

Definition

Two Views of R.V.s

Expectation

Definition

Linearity

How to solve problems using it