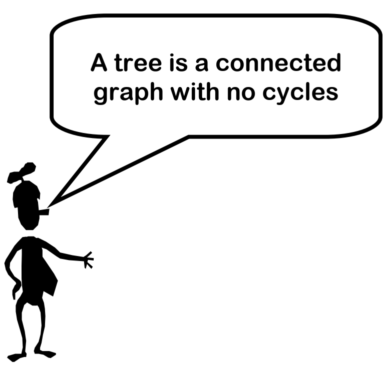
15-251

Great Theoretical Ideas in Computer Science

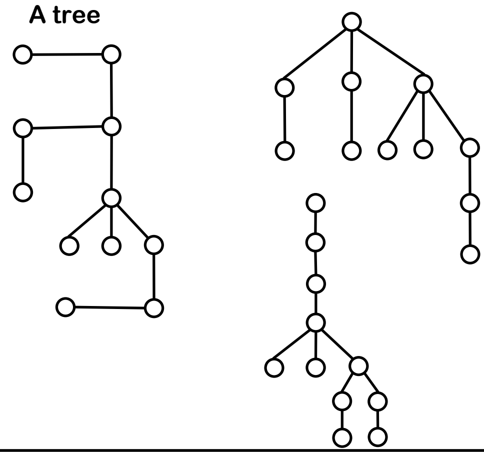
Graphs

Lecture 15 (March 7, 2006)

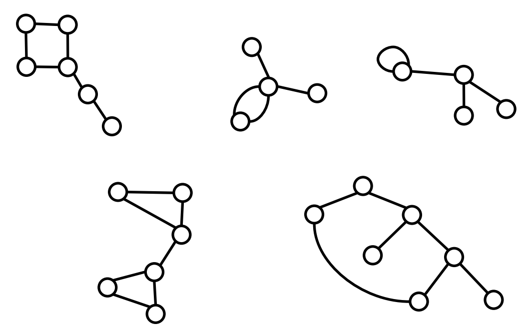
What's a tree?



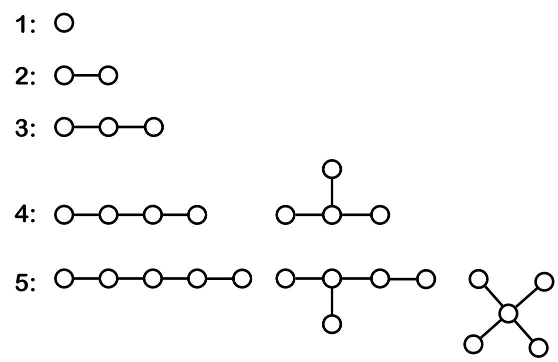
A tree



Not Trees:



How Many n-Node Trees?



Notation

In this lecture:

n will denote the number of nodes in a graph

e will denote the number of edges in a graph

Theorem: Let G be a graph with n nodes and e edges

The following are equivalent:

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$
5. G is acyclic and if any two non-adjacent points are joined by a line, the resulting graph has exactly one cycle

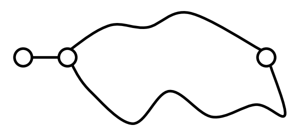
To prove this, it suffices to show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 2$

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:



Then there exists a cycle!

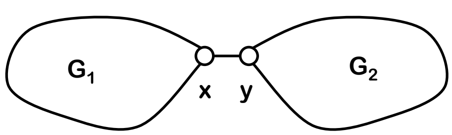
$2 \Rightarrow 3$

2. Every two nodes of G are joined by a unique path
3. G is connected and n = e + 1

Proof: (by induction)

Assume true for every graph with < n nodes

Let G have n nodes and let x and y be adjacent



Let n_1,e_1 be number of nodes and edges in G_1

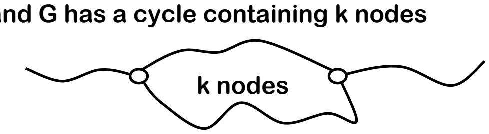
Then $n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1$

$3 \Rightarrow 4$

3. G is connected and n = e + 1
4. G is acyclic and n = e + 1

Proof: (by contradiction)

Assume G is connected with n = e + 1, and G has a cycle containing k nodes



Note that the cycle has k nodes and k edges

Start adding nodes and edges until you cover the whole graph

Number of edges in the graph will be at least n

$4 \Rightarrow 5$ 4. G is acyclic and $n = e + 1$

5. G is acyclic and if any two non-adjacent points are joined by a line, the resulting graph has exactly one cycle

Proof:

If there are k connected components, each is a tree

For each tree: $n_i = e_i + 1$

Therefore, $n = e + k$

So $k = 1$

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

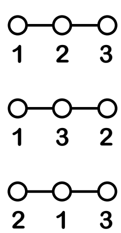
Proof:

Assume all but one of the points in the tree have degree at least 2

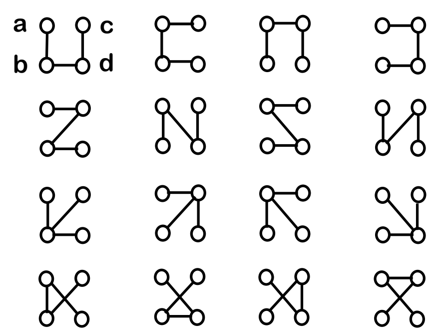
In any graph, sum of the degrees = 2e

Then the total number of edges in the tree is at least $(2n-1)/2 = n - 1/2 > n - 1$

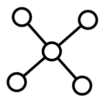
How many labeled trees are there with three nodes?



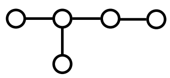
How many labeled trees are there with four nodes?



How many labeled trees are there with five nodes?

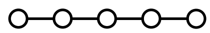


5 labelings



$5 \binom{4}{3} 3$

labelings



$5 \binom{4}{2} 2$

labelings

125 labeled trees

How many labeled trees are there with n nodes?

3 labeled trees with 3 nodes

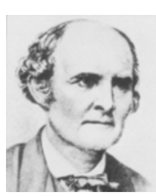
16 labeled trees with 4 nodes

125 labeled trees with 5 nodes

n^{n-2} labeled trees with n nodes

Cayley's Formula

The number of labeled trees on n nodes is n^{n-2}



The proof will use the correspondence principle

Each labeled tree on n nodes

corresponds to

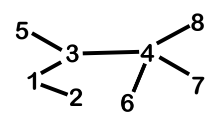
A sequence in $\{1,2,\ldots,n\}^{n-2}$ (that is, n-2 numbers, each in the range [1..n])

How to make a sequence from a tree?

Loop through i from 1 to n-2

- Let L be the degree-1 node with the lowest label
- Define the i^{th} element of the sequence as the label of the node adjacent to L
- Delete the node L from the tree

Example:



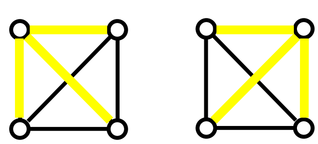
1 3 3 4 4 4

How to reconstruct the unique tree from a sequence S:

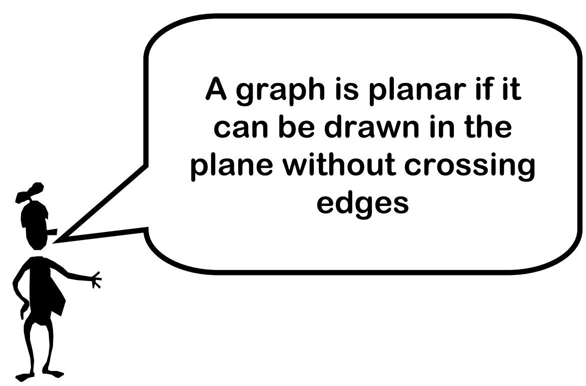
- Let I = {1, 2, 3, …, n}
- Loop until S is empty
  - Let i = smallest # in I but not in S
  - Let s = first label in sequence S
    - Add edge {i, s} to the tree
    - Delete i from I
    - Delete s from S

Spanning Trees

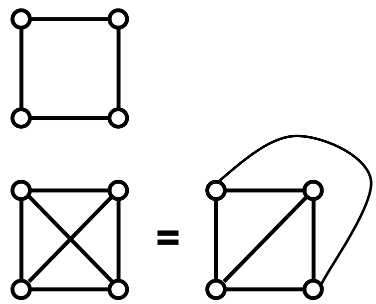
A spanning tree of a graph G is a tree that touches every node of G and uses only edges from G



Every connected graph has a spanning tree

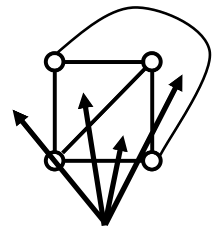


Examples of Planar Graphs



Faces

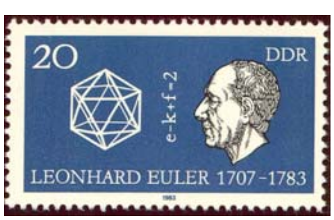
A planar graph splits the plane into disjoint faces



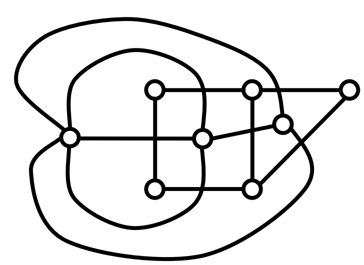
4 faces

Euler’s Formula

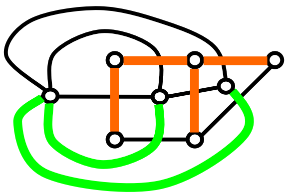
If G is a connected planar graph with n vertices, e edges and f faces, then $n - e + f = 2$



Rather than using induction, we’ll use the important notion of the dual graph



Dual = put a node in every face, and an edge between every adjacent face



Let G* be the dual graph of G

Let T be a spanning tree of G

Let T* be the graph where there is an edge in dual graph for each edge in G – T

Then T* is a spanning tree for G*

$n = e_T + 1$

$f = e_{T^*} + 1$

$n + f = e_T + e_{T^*} + 2$
$= e + 2$

Corollary: Let G be a simple planar graph with $n > 2$ vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most $3n - 6$ edges

Proof of 1:

In any graph, (sum of degrees) = 2e

Assume all vertices have degree ≥ 6

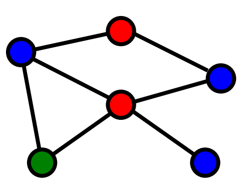
Then $e \geq 3n$

Furthermore, since G is simple, $3f \leq 2e$

So $3n + 3f \leq 3e$ and $3e + 6 \leq 3e$

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color



Graph Coloring

Arises surprisingly often in CS

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register

Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored

Assume G has n vertices

Since G is planar, it has some node v with degree at most 5

Remove v and color by Induction Hypothesis

Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree ≤ 5

4-color theorem remains challenging!

Graph Spectra

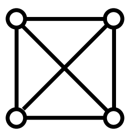
We now move to a different representation of graphs that is extremely powerful, and useful in many areas of computer science: AI, information retrieval, computer vision, machine learning, CS theory,…

Adjacency Matrix

Suppose we have a graph G with n vertices. The adjacency matrix is the n x n matrix $A=[a_{ij}]$ with:

$a_{ij} = 1$ if (i,j) is an edge
$a_{ij} = 0$ if (i,j) is not an edge

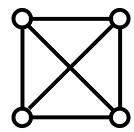
Example



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Counting Paths

The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k



$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

Eigenvalues

Vector x is an eigenvector of A with eigenvalue λ if

$$Ax = \lambda x$$

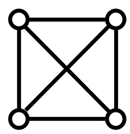
A symmetric n x n matrix has at most n distinct real eigenvalues

Characteristic Polynomial

The characteristic polynomial of A is the polynomial :

$$P(x) = (x - \lambda_1)\,(x - \lambda_2)\ldots\,(x - \lambda_n)$$

Example: K_4



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$$P(x) = (x - 3)(x+1)^3$$

If graph G has adjacency matrix A with characteristic polynomial

$$p_A(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n$$

Then:

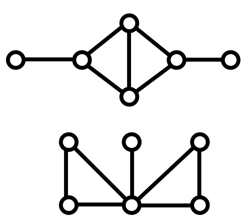
$c_1 = 0$

$-c_2$ = # of edges in G

$-c_3$ = twice # of triangles in G

Two different graphs with the same spectrum

$$p_A(x) = x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1$$



Study Bee

Trees
- Counting Trees
- Different Characterizations

Planar Graphs
- Definition
- Euler's Theorem
- Coloring Planar Graphs

Adjacency Matrix
- Definition
- Useful for counting