Great Theoretical Ideas In Computer Science
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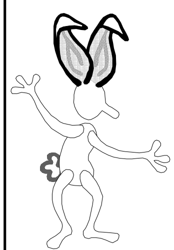
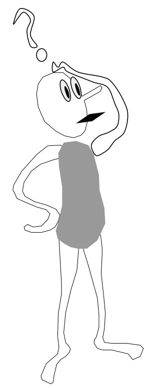
Generating Functions II



$$\frac{x}{1-x-x^2} = \sum_{k=0}^{\infty} F_k x^k$$

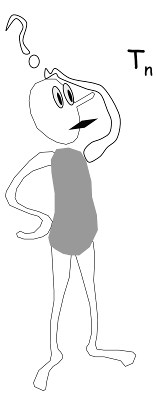


$F_0=0$, $F_1=1$,

$F_n=F_{n-1}+F_{n-2}$ for $n \geq 2$

What is a closed form the Fibonacci power series?

$$\sum_{k=0}^{\infty} F_k x^k = ??$$



$T_0=1$, $T_1=1$,

$T_n = T_0\, T_{n-1} + T_1\, T_{n-2} + \ldots + T_{n-1}\, T_0$

Counting binary trees

$$\sum_{k=0}^{\infty} T_k x^k = ??$$

Solving with Generating Functions

$a_k = a_{k-1} + a_{k-2}$, $k \geq 2$

$a_0=0$, $a_1=1$

We assume that f(x) is a generating function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$a_k = a_{k-1} + a_{k-2}$, $k \geq 2$

What are the generating functions for sequences

a_k, a_{k-1}, a_{k-2}

when k = 2, 3, 4,... ?

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

What is the generating function for a_k , k = 2,3, ... ?

$$\sum_{k=2}^{\infty} a_k x^k = (a_2 x^2 + a_3 x^3 + a_4 x^4 + ...)$$
$$= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...) - (a_0 + a_1 x)$$
$$= f(x) - (a_0 + a_1 x)$$
$$= f(x) - x$$

What is the generating function for a_{k-1}, k = 2,3, ... ?

$$\sum_{k=2}^{\infty} a_{k-1} x^k = (a_1 x^2 + a_2 x^3 + a_3 x^4 + ...)$$
$$= x(a_1 x + a_2 x^2 + a_3 x^3 + ...)$$
$$= x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...)$$
$$= xf(x)$$

What is the generating function for a_{k-2}, k = 0, 1, 2, ... ?

$$\sum_{k=2}^{\infty} a_{k-2} x^k = (a_0 x^2 + a_1 x^3 + a_2 x^4 + ...)$$
$$= x^2 (a_0 + a_1 x + a_2 x^2 + ...)$$
$$= x^2 f(x)$$

Solving with Generating Functions

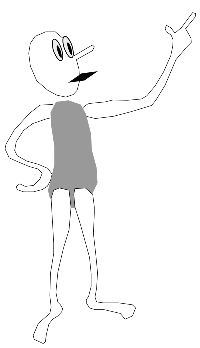
$a_k = a_{k-1} + a_{k-2}$, $k>1$
$a_0=0$, $a_1=1$

In terms of generating functions

$$f(x) - x = xf(x) + x^2 f(x)$$

$a_k = a_{k-1} + a_{k-2}$

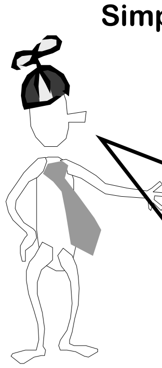
$$f(x) - x = xf(x) + x^2 f(x)$$
$$f(x)(1 - x - x^2) = x$$
$$f(x) = \frac{x}{1 - x - x^2}$$



$$\frac{x}{1-x-x^2} = \sum_{k=0}^{\infty} F_k x^k =$$

$$F_0 + F_1 x + F_2 x^2 + \ldots$$

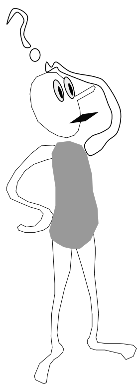
Simple Transformations



If f(x) is a generating function for a_k, then

a_{k-1} -> x f(x)

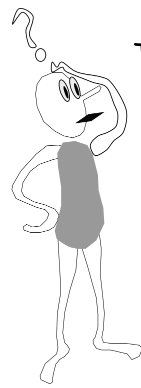
a_{k-2} -> x^2 f(x)



Find a GF for

$a_0=2,\ a_1=4, a_2=31$
$a_n=4a_{n-1}+3a_{n-2}-18a_{n-3}$

$$\sum_{k=0}^{\infty} a_k x^k = ??$$



$T_0=1,\ T_1=1,$

$T_n = T_0\ T_{n-1} + T_1\ T_{n-2}+\ldots+T_{n-1}\ T_0$

Counting binary trees

$$\sum_{k=0}^{\infty} T_k x^k = ??$$

$T_0=1$, $T_1=1$,
$T_n = T_0\ T_{n-1} + T_1\ T_{n-2}+\ldots+T_{n-1}\ T_0$

Let f(x) be a generating function

$$f(x) = \sum_{n=0}^{\infty} T_n x^n$$

Then summing-up the above recurrence, yields

$$f(x) = 1 + \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n-1} T_k T_{n-1-k}\Big) x^n$$

Power series multiplication

Let us multiply formal power series

$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots$
$q(x) = b_0 + b_1 x + b_2 x^2 + \ldots$

$p(x)q(x) = a_0b_0 + (a_0b_1+a_1b_0)\ x +$
$(a_0b_2+a_1b_1+a_2b_0)\ x^2 +\ldots$

Power series multiplication

If

$$p(x) = \sum_{n=0}^{\infty} a_n x^n \qquad q(x) = \sum_{n=0}^{\infty} b_n x^n$$

then

$$p(x)q(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

$T_0=1$, $T_1=1$, $T_n = T_0 T_{n-1} + T_1 T_{n-2} + \ldots + T_{n-1} T_0$

$$f(x) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} T_k T_{n-1-k}\right) x^n$$

$$f(x) = 1 + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} T_k T_{n-k}\right) x^{n+1}$$

$$f(x) = 1 + x\left(\sum_{n=0}^{\infty} T_k x^n\right)\left(\sum_{n=0}^{\infty} T_k x^n\right)$$

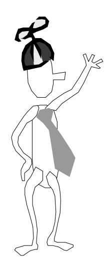
$$f(x) = 1 + x\ f(x)^2$$

$T_0=1$, $T_1=1$, $T_n = T_0 T_{n-1} + T_1 T_{n-2} + \ldots + T_{n-1} T_0$

$$f(x) = 1 + x\ f(x)^2$$

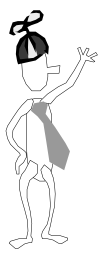
$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Counting Binary Trees



$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} T_n x^n$$

Applying some calculus…

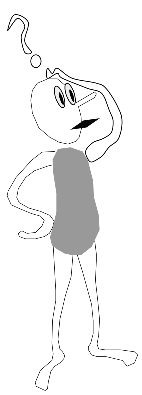


$$\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{n+1}x^n$$

Catalan Numbers

Newton's Binomial Theorem

$$\sqrt{1-4x}=\left(1+(-4x)\right)^{1/2}=\sum_{k=0}^{\infty}\binom{1/2}{k}(-4)^k$$

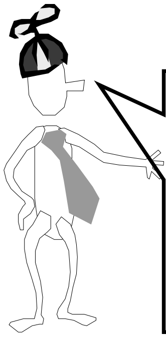


$$\binom{1/2}{1}=?$$

Newton's Binomial Theorem

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n(n-1)\ldots(n-k+1)}{k!}$$

Proving Identities



Use generating functions to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Consider the right hand side

$$\binom{2n}{n} = [x^n](1+x)^{2n}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Use the binomial theorem to obtain

$$(1+x)^{2n} = (\binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n)^2$$

What is the coefficient by x^n?

Study Bee

- Solving recurrences via GFs
- Power series manipulations
- Proving identities via GFs