Great Theoretical Ideas In Computer Science

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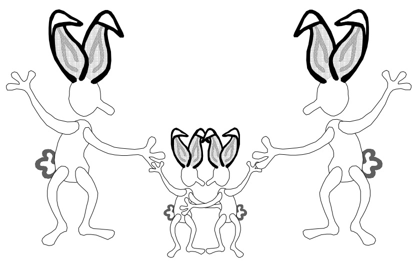
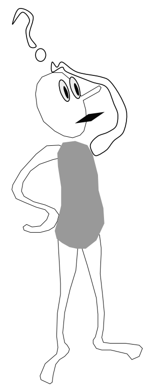
Recurrences, Phi and CF



Warm-up



Solve in integers

$x_1+x_2+x_3=11$
$x_1 \geqslant 0;\ x_2 \leqslant 3;\ x_3 \geqslant 0$

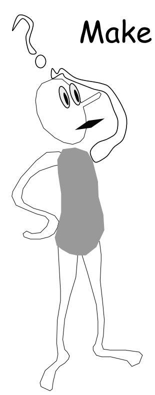
Warm-up

Solve in integers

$x_1+x_2+x_3=11$
$x_1 \geqslant 0;\ x_2 \leqslant 3;\ x_3 \geqslant 0$

$$[X^{11}] \quad \frac{1+x+x^2+x^3}{(1-x)^2}$$

Make Change (interview question)



Find the number of ways to make change for $1 using pennies, nickels, dimes and quarters

Make Change

Find the number of ways to make change for \$1 using pennies, nickels, dimes and quarters

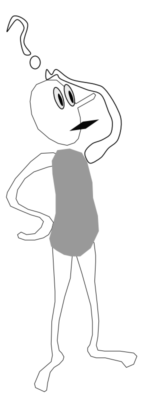
$$[X^{100}]\ \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

Make Change

Find the number of ways to make change for \$1 using pennies, nickels, dimes and quarters

$$x_1 + 5x_2 + 10x_3 + 25x_4 = 100$$

Partitions



Find the number of ways to partition the integer n

3 = 1+1+1=1+2

4=1+1+1+1=1+1+2=1+3=2+2

Partitions

Find the number of ways to partition the integer n

$$x_1 + 2x_2 + 3x_3 + ... + nx_n = n$$

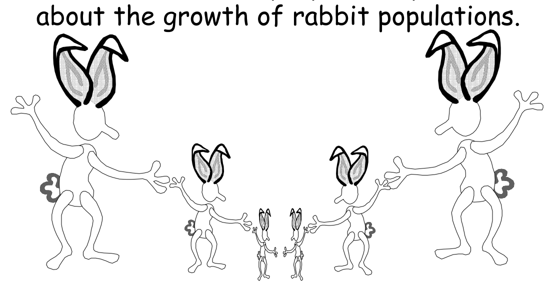
Partitions

Find the number of ways to partition the integer n

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}$$

Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

F_n= # of rabbit pairs at the beginning of the n^{th} month

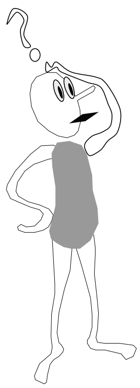
month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

F_n is called the n^{th} Fibonacci number

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

$F_0=0$, $F_1=1$, and $F_n=F_{n-1}+F_{n-2}$ for $n\geq2$

F_n is defined by a recurrence relation.



$F_0=0$, $F_1=1$,

$F_n=F_{n-1}+F_{n-2}$ for $n\geq2$

What is a closed form formula for F_n?

Techniques you have seen so far

$F_n=F_{n-1}+F_{n-2}$

Consider solutions of the form:

$F_n= c^n$

for some constant c

c must satisfy:

$c^n - c^{n-1} - c^{n-2} = 0$

$c^n - c^{n-1} - c^{n-2} = 0$

iff $c^{n-2}(c^2 - c - 1) = 0$

iff $c=0$ or $c^2 - c - 1 = 0$

Iff $c = 0$, $c = \phi$, or $c = -1/\phi$

ϕ ("phi") is the golden ratio

$c = 0$, $c = \phi$, or $c = -(1/\phi)$

So for all these values of c the inductive condition is satisfied:

$c^n - c^{n-1} - c^{n-2} = 0$

Do any of them happen to satisfy the base condition as well?

$F_0=0$, $F_1=1$

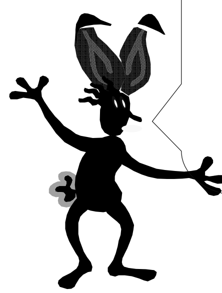
$\forall$a,b   a ϕ^n + b $(-1/\phi)^n$ satisfies the inductive condition

Adjust a and b to fit the base conditions.

n=0: a+b = 0

n=1: a ϕ^1 + b $(-1/\phi)^1$ = 1

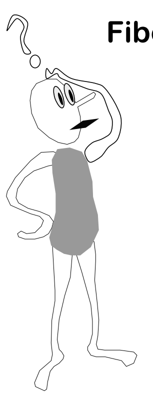
a= $1/\sqrt{5}$ b= $-1/\sqrt{5}$



Leonhard Euler (1765)

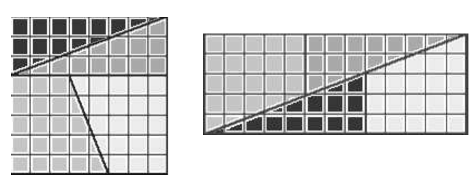
$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

Fibonacci Power Series



$$\sum_{k=0}^{\infty} F_k x^k = ??$$

Fibonacci Bamboozlement



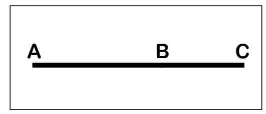
Cassini's Identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

We dissect $F_n x F_n$ square and rearrange pieces into $F_{n+1} x F_{n-1}$ square

Golden Ratio
Divine Proportion

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.



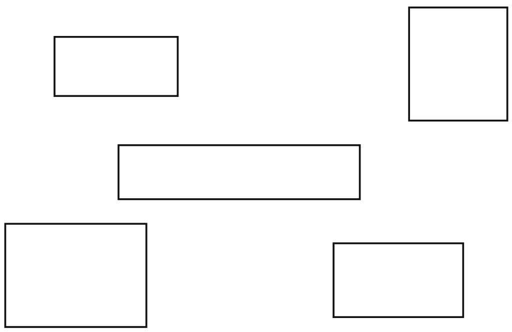
Ratio of height of the person to height of a person's navel

Aesthetics

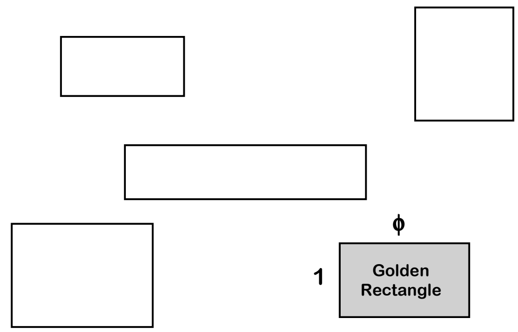
ϕ plays a central role in renaissance art and architecture.

After measuring the dimensions of pictures, cards, books, snuff boxes, writing paper, windows, and such, psychologist Gustav Fechner claimed that the preferred rectangle had sides in the golden ratio (1871).

Which is the most attractive rectangle?



Which is the most attractive rectangle?



Divina Proportione
Luca Pacioli (1509)

Pacioli devoted an entire book to the marvelous properties of ϕ. The book was illustrated by a friend of his named:

Leonardo Da Vinci

Table of contents

- The first considerable effect
- The second essential effect
- The third singular effect
- The fourth ineffable effect
- The fifth admirable effect
- The sixth inexpressible effect
- The seventh inestimable effect
- The ninth most excellent effect
- The twelfth incomparable effect
- The thirteenth most distinguished effect

Divina Proportione
Luca Pacioli (1509)

"Ninth Most Excellent Effect"

two diagonals of a regular pentagon divide each other in the Divine Proportion.

Expanding Recursively

$$\phi^2 - \phi - 1 = 0$$

$$\phi = 1 + \frac{1}{\phi}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

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Expanding Recursively

$$\phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}}}}}$$

A (Simple) Continued Fraction Is Any Expression Of The Form:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\cfrac{1}{g+\cfrac{1}{h+\cfrac{1}{i+\cfrac{1}{j+\ldots}}}}}}}}}$$

where a, b, c, … are whole numbers.

A Continued Fraction can have a finite or infinite number of terms.

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\cfrac{1}{g+\cfrac{1}{h+\cfrac{1}{i+\cfrac{1}{j+\ldots}}}}}}}}}$$

We also denote this fraction by [a,b,c,d,e,f,…]

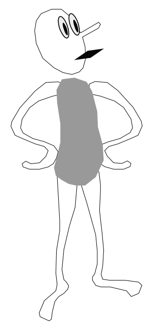
Continued Fraction Representation

$$\frac{8}{5} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}$$

= [1,1,1,1,1,0,0,0,…]

Recursively Defined Form For CF

$$\text{CF} = \text{whole number, or}$$
$$= \text{whole number} + \frac{1}{\text{CF}}$$



Proposition: Any finite continued fraction evaluates to a rational.

Converse: Any rational has a finite continued fraction representation.

Euclid's GCD = Continued Fractions

$$\frac{A}{B} = \left\lfloor \frac{A}{B} \right\rfloor + \frac{1}{\dfrac{B}{A \bmod B}}$$

Euclid(A,B) = Euclid(B, A mod B)
Stop when B=0

A Pattern for ϕ

Let r_1 = [1,0,0,0,…] = 1
r_2 = [1,1,0,0,0,…] = 2/1
r_3 = [1,1,1,0,0,0…] = 3/2
r_4 = [1,1,1,1,0,0,0…] = 5/3
and so on.

Theorem:

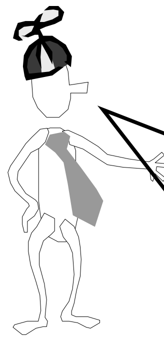
$r_n = F_{n+1}/F_n$

Divine Proportion

$$\frac{F_n}{F_{n-1}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} = \frac{\phi^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} + \frac{-\left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}}$$

$$\boxed{\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi}$$

Heads-on



How to convert kilometers into miles?

Magic conversion

50 km = 34 + 13 + 3

50 = $F_9 + F_7 + F_4$

$F_8 + F_6 + F_3$ = 31 miles

Quadratic Equations

$X^2 - 3x - 1 = 0$

$$X = \frac{3+\sqrt{13}}{2}$$

$X^2 = 3X + 1$

$X = 3 + 1/X$

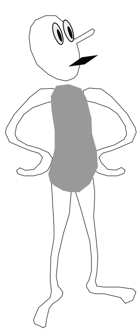
$X = 3 + 1/X = 3 + 1/[3 + 1/X] = \ldots$

A Periodic CF

$$\frac{3+\sqrt{13}}{2} = 3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\dots}}}}}}}}}$$

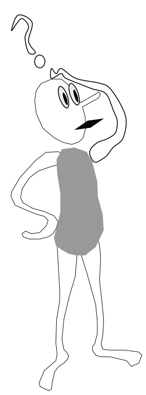
A period-2 CF

$$\frac{1+\sqrt{2}}{3} = \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+\dots}}}}}}$$



Proposition: Any quadratic solution has a periodic continued fraction.

Converse: Any periodic continued fraction is the solution of a quadratic equation



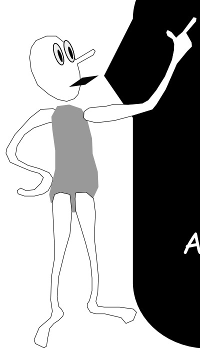
What about those non-periodic continued fractions?

Non-periodic CFs

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+....}}}}}}}}}$$

What is the pattern?

$$\pi=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+....}}}}}}}}}$$

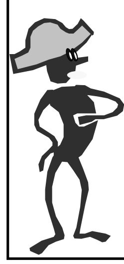


What a cool representation!

Finite CF: Rationals

Periodic CF: Quadratic roots

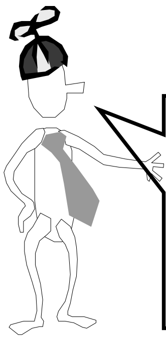
And some numbers reveal hidden regularity.



More recurrences!!

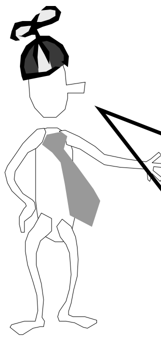
Let us embark now on the Catalan numbers

Counting Binary Trees



Count the number of binary trees with n nodes.

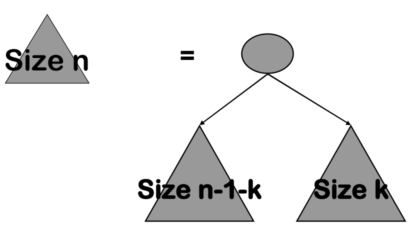
Counting Binary Trees



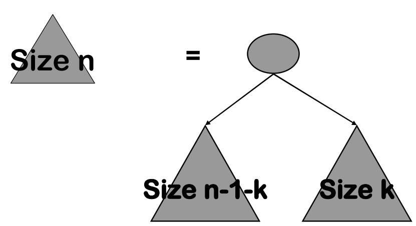
Let T_n represent the number of trees with n nodes

$T_1=1$, $T_2=2$

Counting Binary Trees

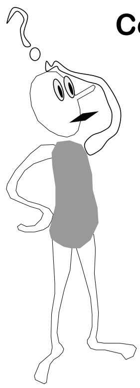


Counting Binary Trees



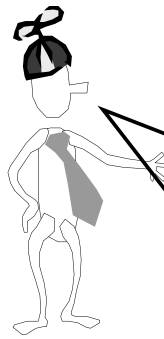
$$T_n = T_0\, T_{n-1} + T_1\, T_{n-2} + \ldots + T_{n-1}\, T_0$$

Counting Binary Trees



$$\sum_{k=0}^{\infty} T_k x^k = ??$$

Triangulation



Count the number of ways to divide a convex n-gon into triangles with noncrossing diagonals.

Study Bee

- Review GCD algorithm
- Recurrences, Phi and CF
- The Catalan numbers