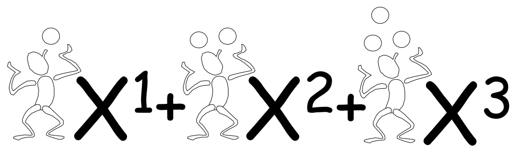
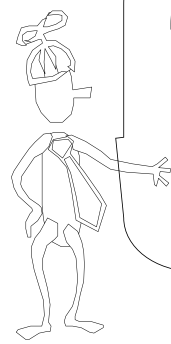
Great Theoretical Ideas In Computer Science		
S. Rudich V. Adamchik		CS 15-251 Spring 2006
Lecture 7	Feb. 07, 2006	Carnegie Mellon University

Counting III: Pigeonhole principal, Pascal’s Triangle and more





Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Application:
two people in Pitt must have the same number of hairs on their heads

Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Problem:
among any n integer numbers, there are some whose sum is divisible by n.

Pigeonhole Principle

Problem:
among any n integer numbers, there are some whose sum is divisible by n.

Proof:
Consider $s_i=x_1+\ldots+x_i$ modulo n.
Remainders are either zero or {1, 2, …, n-1}.
Exist $s_i = s_k$ (mod n). Take $s_i - s_k$

Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Problem:
Prove that among any subset of n+1 integers selected from the set {1,2,3,…,2n} there are two elements a and b such that a is a multiple of b.

Pigeonhole Principle

Problem:
Prove that among any subset of n+1 integers selected from the set {1,2,3,…,2n} there are two elements a and b such that a is a multiple of b.

Proof.
Let S = $\{x_1, x_2, \ldots, x_{n+1}\}$ be a set of n+1 integers

$$x_k = 2^{r(k)} * y_k$$

where y_k is odd.

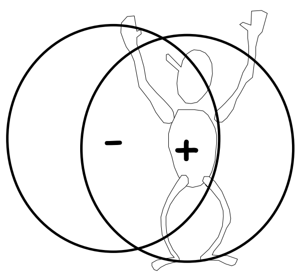
Pigeonhole Principle

Problem:
Prove that among any subset of n+1 integers selected from the set {1,2,3…,2n} there are two elements a and b such that a is a multiple of b.

Proof. $x_k = 2^{r(k)} * y_k$

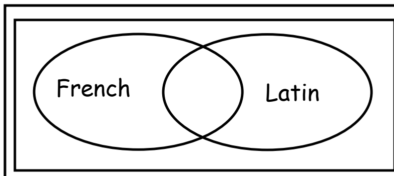
The set {1,2,3…,2n} has n odd integers.
Therefore, exist $y_k = y_i$.

To Exclude Or Not To Exclude?



1. A school has 100 students. 50 take French, 40 take Latin, and 20 take both. How many students take neither language?

2. How many positive integers less than 70 are relatively prime to 70? (70=2•5•7)



F ≡ French students
L ≡ Latin students

$|F|=50$ $|L|=40$

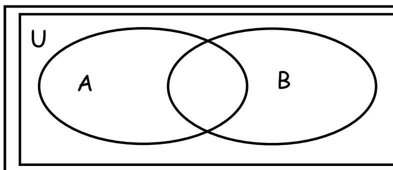
French AND Latin students:
$|F \cap L| = 20$

French OR Latin students:

$|F \cup L| = |F| + |L| - |F \cap L|$
$= 50 + 40 - 20 = 70$

Neither language:

100 – 70 = $\boxed{30}$



U ≡ universe of elements

$\overline{A} = U - A$
$\overline{B} = U - B$

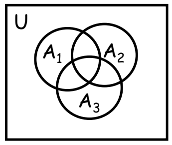
Lesson:

$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$

$$|\overline{A} \cap \overline{B}| = |U| - (|A| + |B|) + |A \cap B|$$

2. How many positive integers less than 70 are relatively prime to 70?

70 = 2•5•7 U = [1..70]



$A_1 \equiv$ integers in U divisible by 2
$A_2 \equiv$ integers in U divisible by 5
$A_3 \equiv$ integers in U divisible by 7

$|A_1| = 35$ $|A_2| = 14$ $|A_3| = 10$

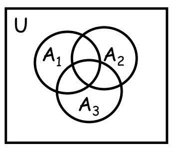
$|A_1 \cap A_2| = 7$
$|A_2 \cap A_3| = 2$
$|A_1 \cap A_3| = 5$
$|A_1 \cap A_2 \cap A_3| = 1$

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= [|A_1| + |A_2| + |A_3|] \\ &-[|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|] \\ &+[|A_1 \cap A_2 \cap A_3|]\end{aligned}$$

$$|A_1| + |A_2| + |A_3| = 35 + 14 + 10 = 59$$
$$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = 7 + 2 + 5 = 14$$
$$|A_1 \cap A_2 \cap A_3| = 1$$

$$\begin{aligned}|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |U| - |A_1 \cup A_2 \cup A_3| \\ &= 70 - 59 + 14 - 1 \\ &= 24\end{aligned}$$

Lesson:



Let S_k be the sum of the sizes of All k-tuple intersections of the A_i's.

$$S_1 = |A_1| + |A_2| + |A_3|$$
$$S_2 = |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|$$
$$S_3 = |A_1 \cap A_2 \cap A_3|$$

$$\boxed{|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3}$$

$$\boxed{|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| = |U| - S_1 + S_2 - S_3}$$

The Principle of Inclusion and Exclusion

Let $A_1, A_2, \ldots, A_n$ be sets in a universe U. Let S_k denote the sum of size of all k-tuple intersections of A_i's.

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| =$$
$$S_1 - S_2 + S_3 - S_4 + S_5 - \cdots + (-1)^{n-1} S_n$$

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \cdots \cap \overline{A}_n| =$$
$$|U| - S_1 + S_2 - S_3 + S_4 - S_5 + \cdots + (-1)^n S_n$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n-1} S_n$$

Let $x \in A_1 \cup \ldots \cup A_n$ be an element appearing in m of the A_i's.

x gets counted m times by S_1
" " " $\binom{m}{2}$ times by S_2
" " " $\binom{m}{3}$ times by S_3
$\vdots$
" " " $\binom{m}{n}$ times by S_n

The formula counts x

$$m - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \binom{m}{5} - \cdots (-1)^{m-1}\binom{m}{m} =$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n-1} S_n$$

The formula counts x

$$m - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \binom{m}{5} - \cdots (-1)^{m-1}\binom{m}{m} = \quad 1 \text{ time.}$$

$$(1+x)^m = 1 + mx + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m \qquad x = -1$$

$$1 = m - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \cdots$$

The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Closed form" or "Generating form"

"Power series" or "Expanded form"

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}_{\text{The number of subsets of an } n\text{-element set}}$$

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

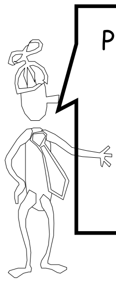
Let *x= -1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

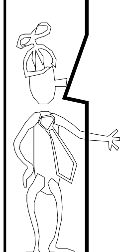
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A <u>combinatorial</u> proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function for odd n. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

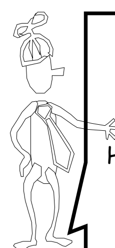
Uh oh...

A correspondence that works for all n

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

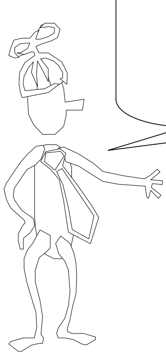
0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010



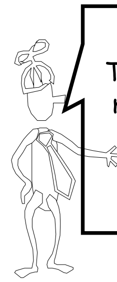
Problem. Consider all words of length n made from {a,b,c,d}.

How many such words have an <u>even</u> number of a's?



Pascal's Triangle

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge...

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pacal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

$(1+X)^0 = 1$

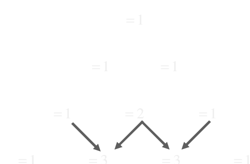
$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

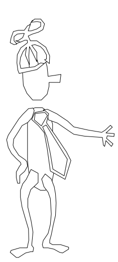
$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"



Blaise Pascal 1654

Pascal's Triangle



"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	=1
1 + 1	=2
1 + 2 + 1	=4
1 + 3 + 3 + 1	=8
1 + 4 + 6 + 4 + 1	=16
1 + 5 + 10 + 10 + 5 + 1	=32
1 + 6 + 15 + 20 + 15 + 6 + 1	=64

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

6+20+6 = 1+15+15+1

Summing on 1^{st} Avenue

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

Summing on k^{th} Avenue

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

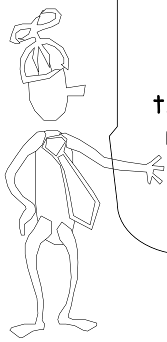
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

```
            1
          1   1   =2
                  =3
        1   2   1   =5
                    =8
      1   3   3   1   =13
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

```
            1
          1   1
        1²  2²  1²
      1²  3²  3²  1²
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Al-Karaji Squares

```
                  1                    =0
              1       1                =1
          1     2 + 2*1                =4
        1    3 + 2*3     1             =9
      1    4 + 2*6    4     1         =16
    1    5 + 2*10    10    5    1     =25
  1    6 + 2*15    20    15    6    1=36
```

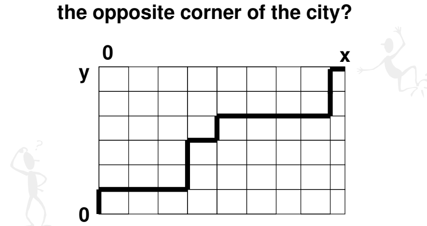


All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients.

How to count pathways

Meandering in a nameless modern metropolis

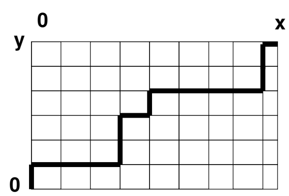
Scenario: You're in a city where all the streets, numbered 0 through x, run north-south, and all the avenues, numbered 0 through y, run east-west. How many [sensible] ways are there to walk from the corner of 0th St. and 0th avenue to the opposite corner of the city?



Meandering in a nameless modern metropolis

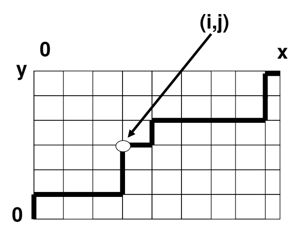
- **All paths require exactly x+y steps:**
- **x steps east, y steps north**
- **Counting paths is the same as counting which of the x+y steps are northward steps:**

$$\binom{x+y}{y}$$

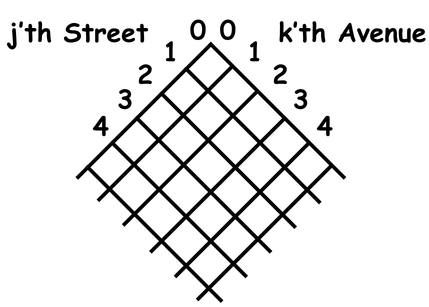


Meandering in a nameless modern metropolis

Now, what if we add the constraint that the path must go through a certain intersection, call it (i,j)?

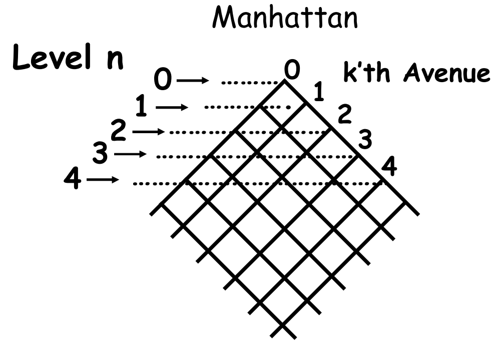


Manhattan

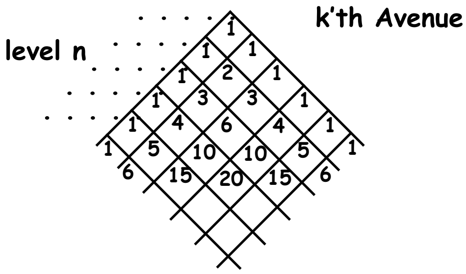


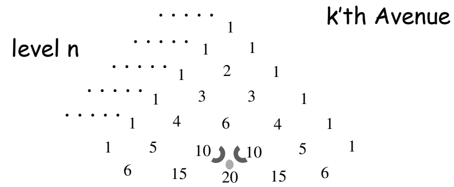
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

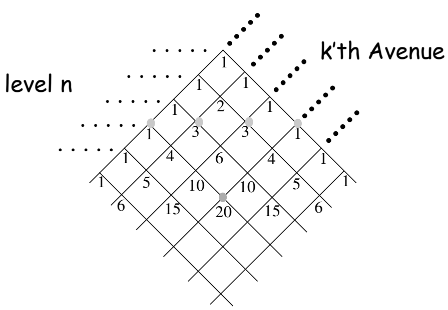


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

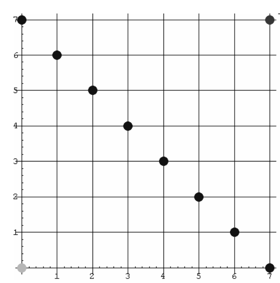




$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



Any path from the green point to the red point must go through one of the blue points.

Let a blue point has coordinates (i, n-i)

There are C(n-i+i, i) paths from (0,0) to this blue point

On the other hand, there are C(n,i) paths from (i,n-i) to the red point.

Study Bee

- Pigeonhole principal
- In- Exclusion principal
- Combinatorial proofs of binomial identities