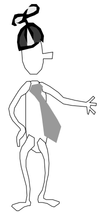
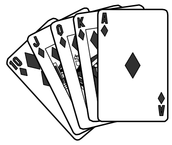
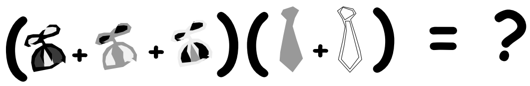
Great Theoretical Ideas In Computer Science
V. Adamchik CS 15-251 Spring 2006
Lecture 6 Feb 2, 2005 Carnegie Mellon University

Counting II: Recurring Problems And Correspondences

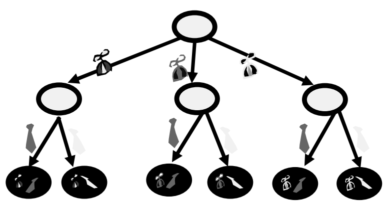





Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Choice Tree



A <u>choice tree</u> is a rooted, directed tree with an object called a “choice” associated with each edge and a label on each leaf.

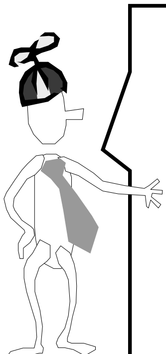
Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN

there are $P_1P_2P_3 \ldots P_n$ objects in S

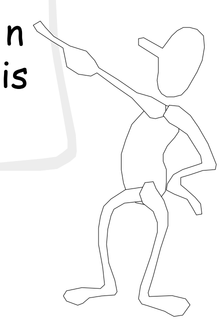
Note, choices must be independent.

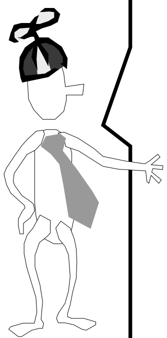


Warm-up Problem

How many distinct five-letters words can be produced by rearranging the letters of the word LUCKY?

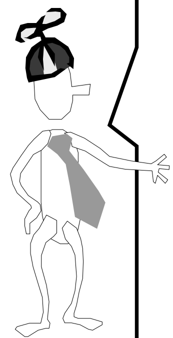
The number of permutations of n distinct objects is n!



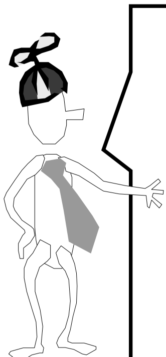


Warm-up Problem

How many four-letters words can be produced by rearranging the letters of the word TOOL?



How many ways to rearrange the letters in the word SYSTEMS?



Are you ready for MISSISSIPPI?

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$

MISSISSIPPI

Arrange n symbols
r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{r_k}{r_k}$$

$$=\frac{n!}{r_1!(n-r_1)!}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!}\cdots 1$$

$$=\frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$

Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} \equiv \begin{cases} 0 \text{ if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

Four ways of choosing

We will choose 2-letter words from the alphabet {L,U,C,K,Y}

1) ??, no repetitions,
 the order does not matter

Four ways of choosing

We will choose 2-letter words from the alphabet (L,U,C,K,Y}

2) ??, no repetitions,
 the order is important

Four ways of choosing

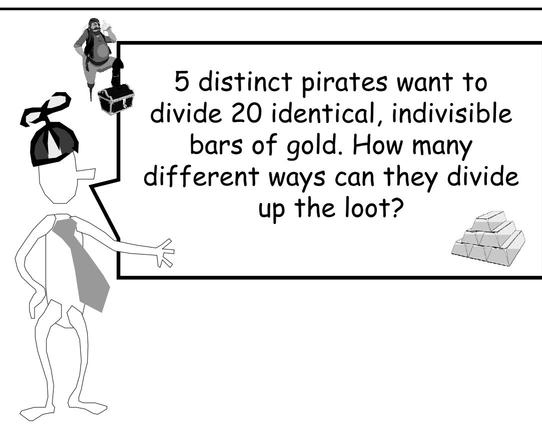
We will choose 2-letter words from the alphabet (L,U,C,K,Y}

3) ?? with repetitions,
 the order is important

Four ways of choosing

We will choose 2-letter words from the alphabet {L,U,C,K,Y}

4) ???? repetitions,
 the order is NOT important
 C(5,2) + {LL,UU,CC,KK,YY}



Sequences with 20 G's and 4 /'s

1st pirate gets 2 bars
2nd and 5th pirate get 1 bar each
3rd gets nothing
4th gets 16 bars

GG/G//GGGGGGGGGGGGGGGG/G
represents the above division among the pirates

Sequences with 20 G's and 4 /'s

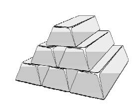
GG/G//GGGGGGGGGGGGGGGG/G

In general, the i^{th} pirate gets the number of G's after the i-1st / and before the i^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

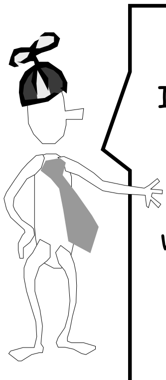
How many different ways to divide up the loot?
Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$



In how many ways 10 items can be chosen from {A,B,C,D,E}?

with repetitions and order does not matter

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

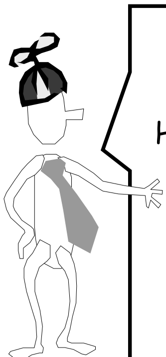
How many nonnegative integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$
$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer positive solutions to the following equations?

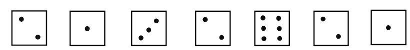
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$
$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n > 0$$



How many integer solutions
x + y + z = 20
x>=1, y>=2, z>=3?

Identical/Distinct Dice

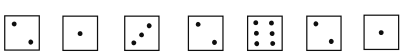
Suppose that we roll seven dice.



How many different outcomes are there, if order matters?

What if order doesn't matter?
(E.g., Yahtzee)

7 Identical Dice



How many different outcomes?

Corresponds to 6 pirates and 7 bars of gold!

$$\binom{6+7-1}{7}$$

Multisets

A <u>multiset</u> is a set of elements, each of which has a *multiplicity*.

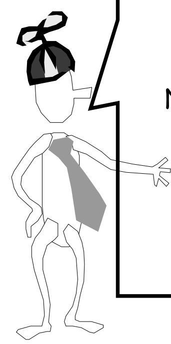
The <u>size</u> of the multiset is the sum of the multiplicities of all the elements.

Example:
{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

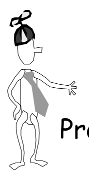
Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose a multiset of size k from n types of elements

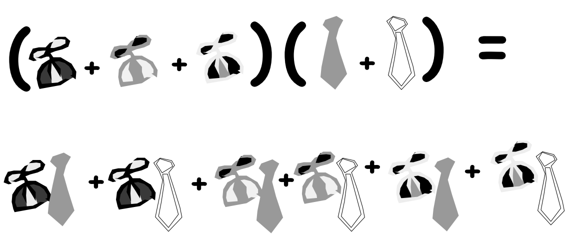


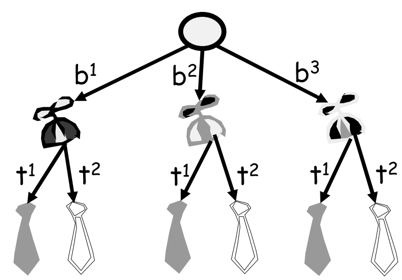
Now, something completely different…

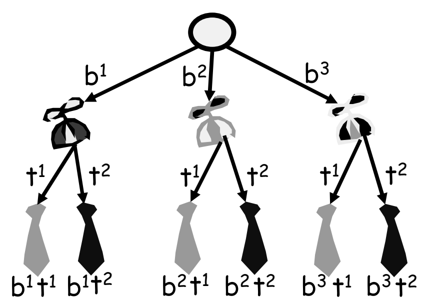


POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

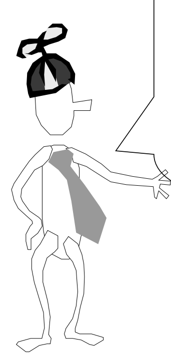
Products of Sum = Sums of Products



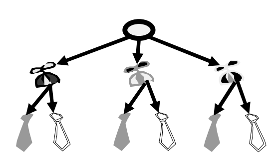




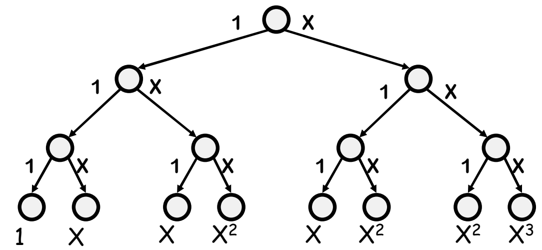
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$



There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

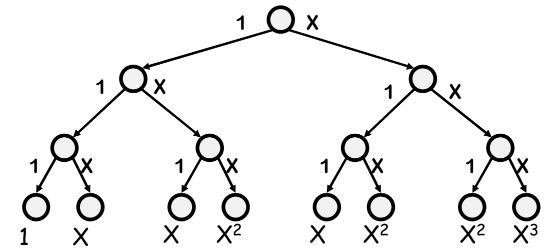


Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

$(1+X)^3 = 1 + 3X + 3X^2 + X^3$



What is the combinatorial meaning of those coefficients?

What is a closed form expression for c_k?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

What is a closed form expression for c_n?

$$(1+X)^n = \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{\text{n times}}$$

After multiplying things out, but *before combining like terms*, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with *exactly k X's*.

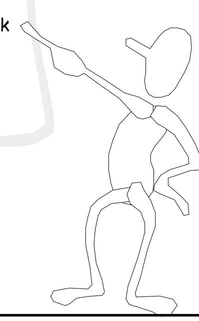
$$c_k = \binom{n}{k}$$

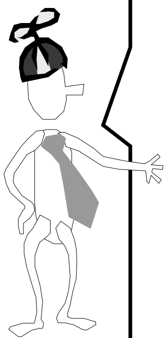
The Binomial Theorem

$$\underbrace{(1+X)^n}_{\text{binomial expression}} = \underbrace{\binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n}_{\text{Binomial Coefficients}}$$

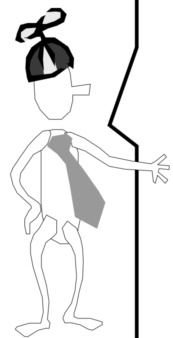
The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{k=n} \binom{n}{k} X^k Y^{n-k}$$

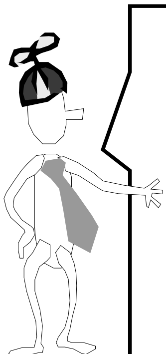




What is the coefficient of EMPTY in the expansion of $(E + M + P + T + Y)^5$?



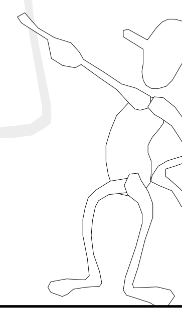
What is the coefficient of EMP^3TY in the expansion of $(E + M + P + T + Y)^7$?



What is the coefficient of $M^2P^2T^3$ in the expansion of $(E + M + P + T + Y)^7$?

What is the coefficient of $X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$ in the expansion of $(X_1+X_2+\ldots+X_k)^n$?

$$\boxed{\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}}$$

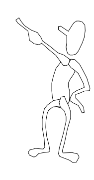


Multinomial Coefficients

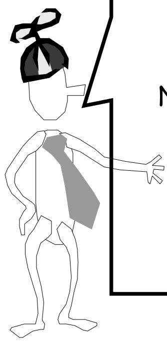
$$\binom{n}{r_1;r_2;\ldots;r_k} \equiv \begin{cases} 0 \text{ if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula

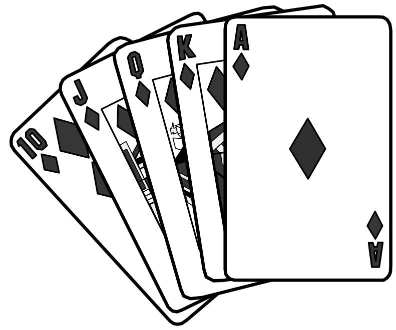


$$(X_1+X_2+\ldots+X_k)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i = n}} \binom{n}{r_1;r_2;\ldots;r_k} X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$$



Now, something completely different…

Storing Poker Hands…



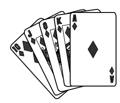
52 Card Deck
5 card hands

4 possible suits:

- ♥♦♣♠

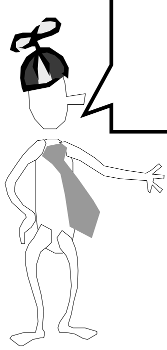
13 possible ranks:

- 2,3,4,5,6,7,8,9,10,J,Q,K,A

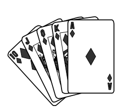


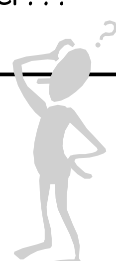
Storing Poker Hands
How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient).



30 bits
Can we do better???
How???





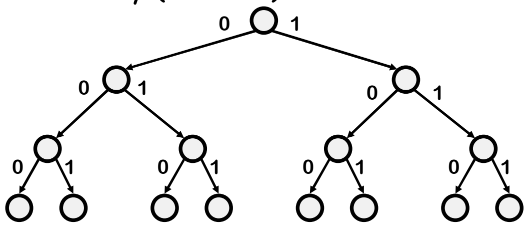
Order all Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2{,}598{,}560) \rceil$=22 bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.
.
.

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2. Usually the choices will be labeled 0 and 1.

22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2,598,560$

A binary choice tree of depth 21 can have at most 2^{21} leaves. Hence, there are not enough leaves for

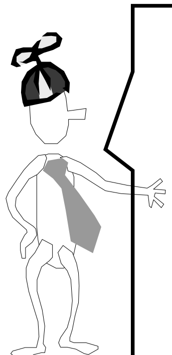
Hence, you can't have a leaf for each hand.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

Furthermore, any representation of the set will have some string of that length.

Information Counting Principle:

If each element of a set can be represented using k bits, the size of the set is bounded by 2^k



ONGOING MEDITATION:
Let S be any set and T be a binary choice tree representation of S. We can think of each element of S being encoded by the binary sequences of choices that lead to its leaf. We can also start with a binary encoding of a set and make a corresponding binary choice tree.

Study Bee

Go Steelers!!!