Great Theoretical Ideas In Computer Science		
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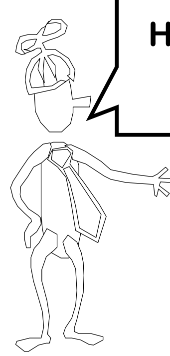
Lecture 5	Jan 31, 2006	Carnegie Mellon University

Counting I: Bijection and Choice Trees

Not everything that can be counted counts, and not everything that counts can be counted.

A. Einstein

If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?



How many seats in this auditorium?

Hint:
Count without counting!

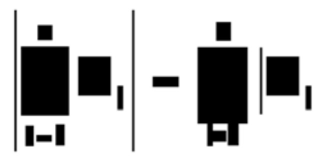
Addition Rule

Let A and B be two <u>disjoint</u> finite sets.

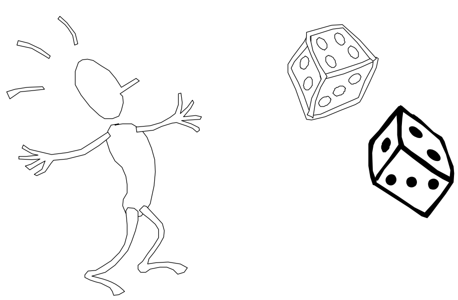
The size of $A \cup B$ is the sum of
the size of A and the size of B.

Corollary (by induction)

Let $A_1, A_2, A_3, \ldots, A_n$ be disjoint, finite sets.



Suppose I roll a white die and a black die.



S ≡ Set of all outcomes where the dice show different values.
$|S| = ?$

T ≡ set of outcomes where the two dice agree.

Two ways of counting…

$|S| =$

S ≡ Set of all outcomes where the dice show different values.
|S| = ?

T ≡ set of outcomes where the two dice agree.

[illegible]

[illegible]

[illegible]

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. |S| = ?

A_i ≡ set of outcomes where the black die says i and the white die says something larger.

[illegible]

[illegible]

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. |S| = ?

L ≡ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

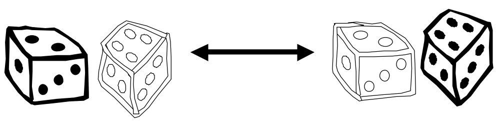
It is clear by symmetry that |S| = |L|.

Therefore $|S| = 15$

It is <u>clear</u> by symmetry that |S| = |L|.

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.



S L

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.

Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: $|S| = |L|$.

Let $f{:}A \to B$ be a function from a set A to a set B.

What is a function???

Read 21-127's textbook, ch. 6

Let $f{:}A \to B$ be a function from a set A to a set B.

f is 1-1 (or injective) if and only if

$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$

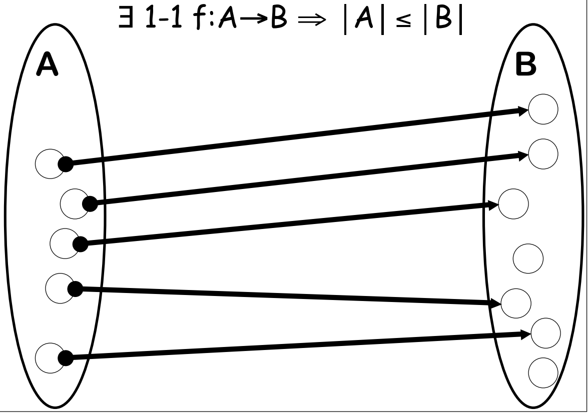
f is onto (or surjective) if and only if

$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$

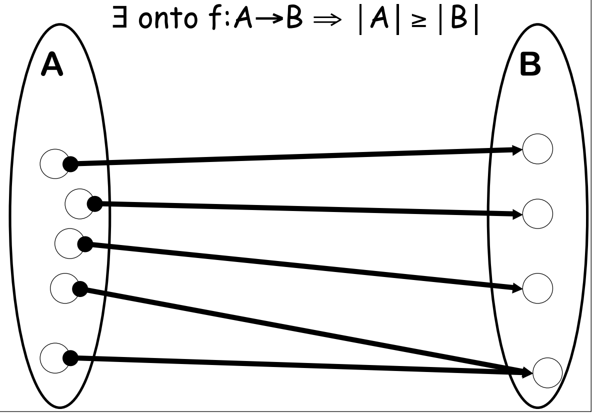
There Exists

For Every

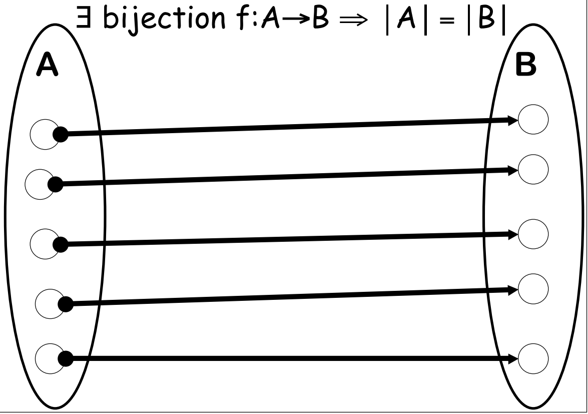
∃ 1-1 $f:A\rightarrow B \Rightarrow |A| \le |B|$



∃ onto $f:A\rightarrow B \Rightarrow |A| \ge |B|$



∃ bijection $f:A\rightarrow B \Rightarrow |A| = |B|$



1-1 Onto Correspondence
(just "correspondence" for short)

Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

000000	←→	0
000001	←→	1
000010	←→	2
000011	←→	3
	…	
1…11111	←→	2^n-1

2^n sequences

S = {a,b,c,d,e} has many subsets.

{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, …

The empty set is a set with all the rights and privileges pertaining thereto.

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

$\{b \quad c \qquad\qquad e\}$

1 means "TAKE IT"
0 means "LEAVE IT"

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

$S = \{a_1, a_2, a_3, \ldots, a_n\}$

$b = b_1 b_2 b_3 \ldots b_n$

a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

$$f(b) = \{a_i \mid b_i = 1\}$$

a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

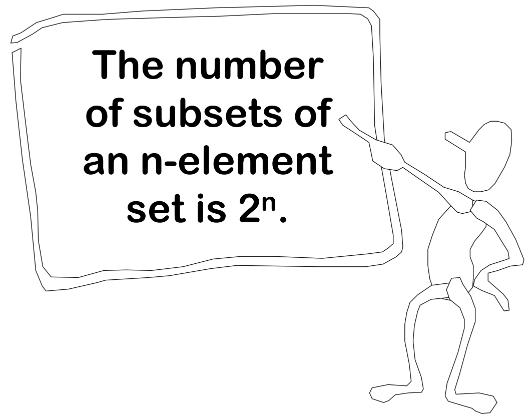
$$f(b) = \{a_i \mid b_i = 1\}$$

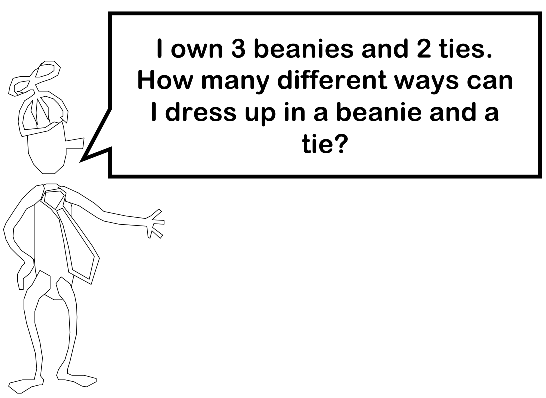
f is 1-1: Any two distinct binary sequences b and b' have a position i at which they differ. Hence, f(b) is not equal to f(b') because they disagree on element a_i.

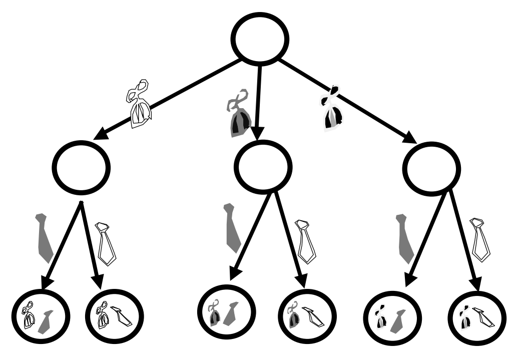
a_1	a_2	a_3	…	a_n
b_1	b_2	b_3	…	b_n

$$f(b) = \{a_i \mid b_i=1\}$$

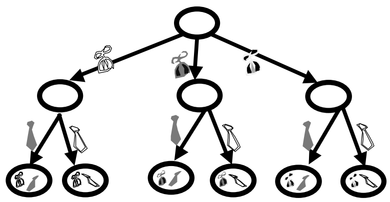
f is onto: Let S be a subset of $\{a_1,\ldots,a_n\}$. Let $b_k = 1$ if a_k in S; $b_k = 0$ otherwise. $f(b_1b_2\ldots b_n) = S$.



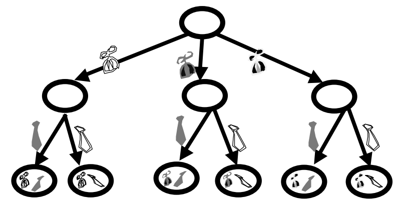




Choice Tree



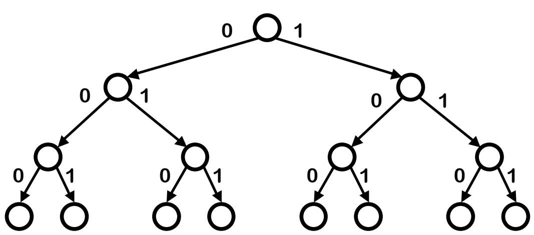
A <u>choice tree</u> is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a "choice tree representation" of a set S, if

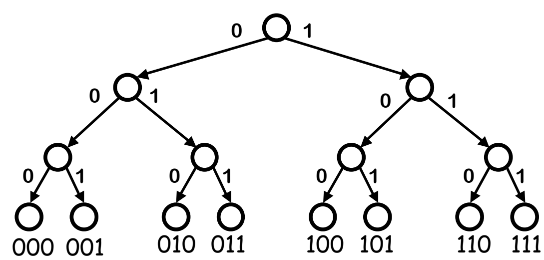
1) Each leaf label is in S
2) No two leaf labels are the same

Choice Tree for 2^n n-bit sequences

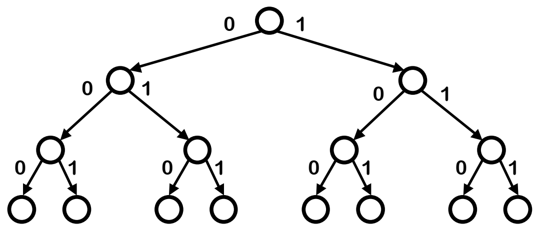


We can use a "choice tree" to represent the construction of objects of the desired type.

2^n n-bit sequences



Label each leaf with the object constructed by the choices along the path to the leaf.



2 choices for first bit
X 2 choices for second bit
X 2 choices for third bit
…
X 2 choices for the n^{th}

We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN
there are $P_1P_2P_3 \ldots P_n$ objects in S

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF
1) Each sequence of choices constructs an object of type S
AND
2) No two different sequences create the same object

THEN
there are $P_1P_2P_3 \ldots P_n$ objects of type S.

How many different orderings of deck with 52 cards?

What type of object are we making?

- Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

50 possible choices for the third card;

…

1 possible choice for the 52cond card.

How many different orderings of deck with 52 cards?

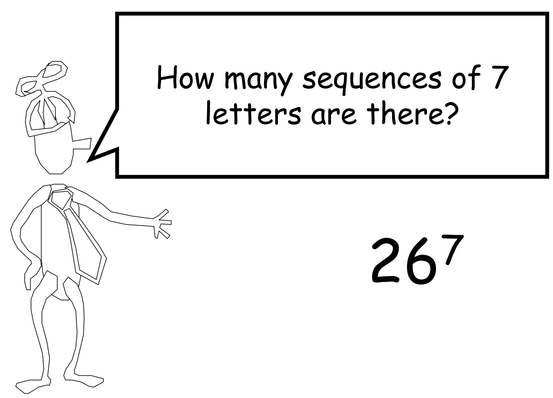
By the product rule:

52 * 51 * 50 * … * 3 * 2 * 1 = 52!

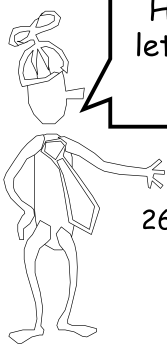
52 “factorial” orderings

A permutation or arrangement of n objects is an ordering of the objects.

The number of permutations of n distinct objects is n!



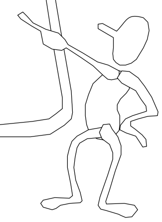
26^7



How many sequences of 7 letters contain at least two of the same letter?

$26^7 - 26*25*24*23*22*21*20$

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.



Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed?

How many unordered pairs?

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

- 52 * 51

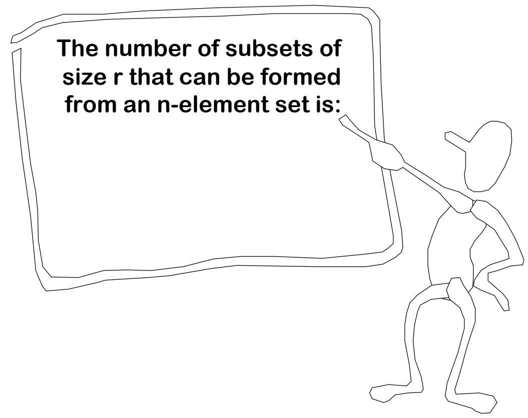
How many unordered 5 card hands? pairs?

- 52*51 / 2 ← divide by overcount

A <u>combination</u> or <u>choice</u> of r out of n objects is an (unordered) set of r of the n objects.

The number of r combinations of n objects:

n choose r



How many 8 bit sequences have two 0's and six 1's?

1) Choose the set of 2 positions to put the 0's. The 1's are forced.

2) Choose the set of 6 positions to put the 1's. The 0's are forced.

How many 8 bit sequences have two 0's and six 1's?

Tempting, but incorrect:

8 ways to place first 0 times

7 ways to place second 0

Violates condition 2 of product rule! Choosing position i for the first 0 and then position j for the second 0 gives the same sequence as choosing position j for the first 0 and position i for the second.

How many hands have at least 3 aces?

$\binom{4}{3} = 4$ ways of picking 3 of the 4 aces.

$\binom{49}{2} = 1176$ ways of picking 2 cards from the remaining 49 cards.

$4 \times 1176 = 4704$

How many hands have at least 3 aces?

How many hands have exactly 3 aces?

$\binom{4}{3} = 4$ ways of picking 3 of the 4 aces.

$\binom{48}{2} = 1128$ ways of picking 2 cards non − ace cards.

$4 \times 1128 = 4512$

How many hands have exactly 4 aces?

$\binom{4}{4} = 1$ way of picking 4 of the 4 aces.

48 ways of picking one of the remaining cards

4512 + 48 = 4560

4704 ≠ 4560

At least one of the two counting arguments is not correct.

Four different sequences of choices produce the same hand

$\binom{4}{3} = 4$ ways of picking 3 of the 4 aces.

$\binom{49}{2} = 1176$ ways of picking 2 cards from the remaining 49 cards.

$4 \times 1176 = 4704$

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

1) Choose 3 of 4 aces
2) Choose 2 non-ace cards

A♣ Q♠ A♦ A♥ K♦

The aces came from the first choice and the non-aces came from the second choice.

1) Choose 4 of 4 aces
2) Choose 1 non-ace

A♣ A♠ A♦ A♥ K♦

The aces came from the first choice and the non-ace came from the second choice.

Study Bee

- Correspondence Principle
  If two finite sets can be placed into 1-1 onto correspondence, then they have the same size
- Choice Tree
- Product Rule
  two conditions
- Counting by complementing
  it's sometimes easier to count the "opposite" of something
- Binomial coefficient
  Number of r sets of an n set