15-251

Great Theoretical Ideas in Computer Science

Homework #1

Make sure your programming assignment was correctly graded!

Look at program_grade.txt in your handin directory

Ancient Wisdom: On Raising A Number To A Power

Lecture 4 (January 26, 2006)

Egyptian Multiplication

The Egyptians used decimal numbers but multiplied and divided in binary

a x b By Repeated Doubling

b has n-bit representation: $b_{n-1}b_{n-2}\ldots b_1b_0$

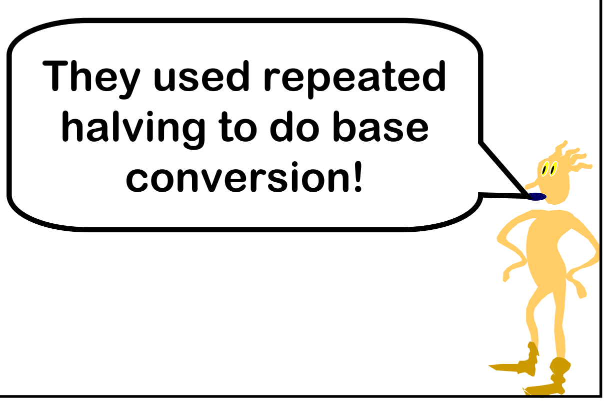
Starting with a,
repeatedly double largest number so far
to obtain: a, 2a, 4a, …., $2^{n-1}a$

Sum together all the $2^k a$ where $b_k = 1$

$$b = b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1}2^{n-1}$$

$$ab = b_0 2^0 a + b_1 2^1 a + b_2 2^2 a + \ldots + b_{n-1}2^{n-1}a$$

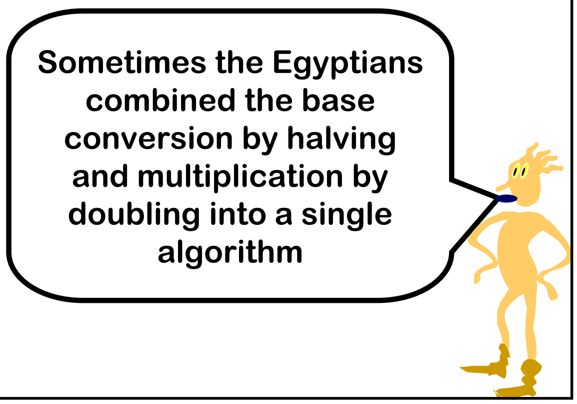
$2^k a$ is in the sum if and only if $b_k = 1$



Egyptian Base Conversion

Output stream will print right to left

```
Input X;
repeat {
    if (X is even)
        then print 0;
    else
        {X := X-1; print 1;}
    X := X/2 ;
} until X=0;
```



70 x 13

Rhind Papyrus [1650 BC]

Doubling	Halving	Odd?	Running Total
70	13	*	70
140	6		
280	3	*	350
560	1	*	910

Binary for 13 is 1101 = $2^3 + 2^2 + 2^0$

$70*13 = 70*2^3 + 70*2^2 + 70*2^0$

30 x 5

Doubling	Halving	Odd?	Running Total
5	30		
10	15	*	10
20	7	*	30
40	3	*	70
80	1	*	150

184 / 17

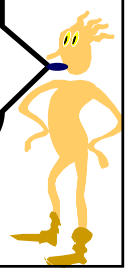
Rhind Papyrus [1650 BC]

Doubling	Powers of 2	Check
17	1	
34	2	*
68	4	
136	8	*

184 = 17*8 + 17*2 + 14

184/17 = 10 with remainder 14

This method is called
“Egyptian Multiplication / Division”
or
“Russian Peasant Multiplication / Division”



Wow! Those Russian peasants were pretty smart

Standard Binary Multiplication = Egyptian Multiplication

```
        * * * * * * * *
 ×                 1101
 ------------------------
        * * * * * * * *

   * * * * * * * *
 * * * * * * * *
   * * * * * * * * * * * *
```

Our story so far…

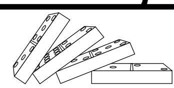
We can view numbers in many different, but corresponding ways

Representation:
Understand the relationship between different representations of the same information or idea

1
2
3

Our story so far…

Induction is how we define and manipulate mathematical ideas

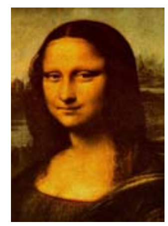


Induction has many guises
Master their interrelationship

- Formal Arguments
- Loop Invariants
- Recursion
- Algorithm Design
- Recurrences

Let's Articulate New One:

Abstractior:
Abstract away the inessential features of a problem or solution



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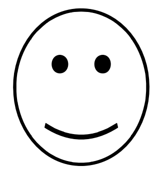


b:=a^8

b:=a*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a

b:=a*a
b:=b*b
b:=b*b

This method costs only 3 multiplications. The savings are significant if b:=a^8 is executed often

Powering By Repeated Multiplication

Input: a,n

Output: Sequence starting with a, ending with a^n, such that each entry other than the first is the product of two previous entries

Example

Input: a,5

Output: a, a^2, a^3, a^4, a^5

or

Output: a, a^2, a^3, a^5

or

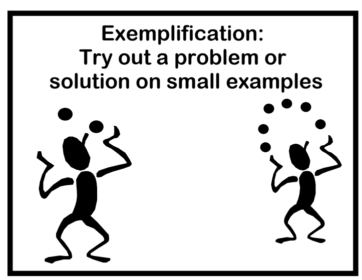
Output: a, a^2, a^4, a^5

Given a constant n,
how do we implement
$b:=a^n$
with the fewest number
of multiplications?

Definition of M(n)

M(n) = Minimum number of multiplications required to produce a^n from a by repeated multiplication

What is M(n)? Can we calculate it exactly? Can we approximate it?



Very Small Examples

What is M(1)?

M(1) = 0 [a]

What is M(0)?

Not clear how to define M(0)

What is M(2)?

M(2) = 1 $[a, a^2]$

M(8) = ?

a, a^2, a^4, a^8 is one way to make a^8 in 3 multiplications

What does this tell us about the value of M(8)?

$M(8) \leq 3$ ← Upper Bound

$? \leq M(8) \leq 3$

$3 \leq M(8)$ by exhaustive search

There are only two sequences with 2 multiplications. Neither of them make 8:

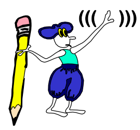
a, a^2, a^3 and a, a^2, a^4

$3 \leq M(8) \leq 3$

Lower Bound Upper Bound

$\boxed{M(8) = 3}$

What is the more essential representation of M(n)?



The "a" is a red herring

$a^x a^y$ is a^{x+y}

Everything besides the exponent is inessential. This should be viewed as a problem of repeated addition, rather than repeated multiplication

Addition Chains

M(n) = Number of stages required to make n, where we start at 1 and in each stage we add two previously constructed numbers

Examples

Addition Chain for 8:
1 2 3 5 8

Minimal Addition Chain for 8:
1 2 4 8

Addition Chains Are a Simpler Way To Represent The Original Problem

Abstractior:
Abstract away the inessential features of a problem or solution

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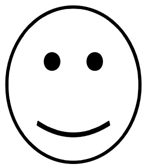
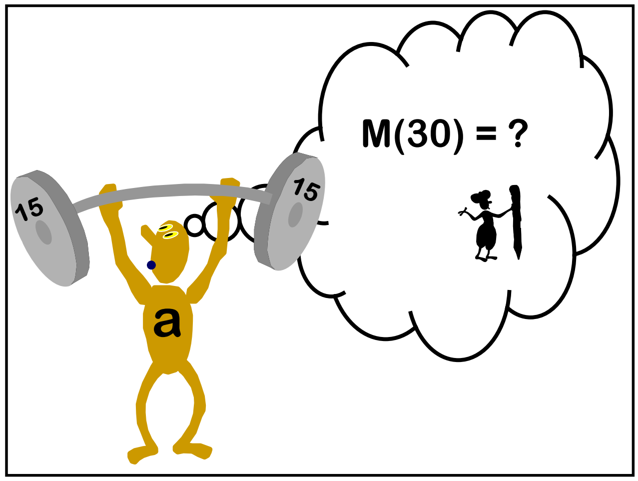
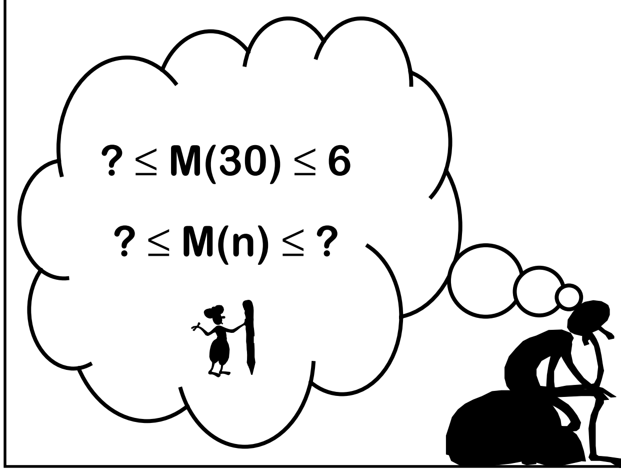




Addition Chains For 30

1	2	4	8	16	24	28	30
1	2	4	5	10	20	30	
1	2	3	5	10	15	30	
1	2	4	8	10	20	30	



Binary Representation

Let B_n be the number of 1s in the binary representation of n

E.g.: $B_5 = 2$ since $5 = (101)_2$

Proposition: $B_n \leq \lfloor \log_2 (n) \rfloor + 1$

(It is at most the number of bits in the binary representation of n)

Binary Method

(Repeated Doubling Method)

Phase I (Repeated Doubling)

For $\lfloor \log_2 (n) \rfloor$ stages:

Add largest so far to itself

(1, 2, 4, 8, 16, . . .)

Phase II (Make n from bits and pieces)

Expand n in binary to see how n is the sum of B_n powers of 2. Use $B_n - 1$ stages to make n from the powers of 2 created in phase I

Total cost: $\lfloor \log_2 n \rfloor + B_n - 1$

Binary Method

Applied To 30

Phase I

1, 2, 4, 8, 16 Cost: 4 additions

Phase II

$30 = (11110)_2$

2 + 4 = 6

6 + 8 = 14

14 + 16 = 30 Cost: 3 additions

Rhind Papyrus [1650 BC]

What is 30 x 5?

Addition chain for 30 → | | ← Start at 5 and perform same additions as chain for 30

1	5
2	10
4	20
8	40
16	80
24	120
28	140
30	150

Repeated doubling is the same as the Egyptian binary multiplication

Rhind Papyrus [1650 BC]

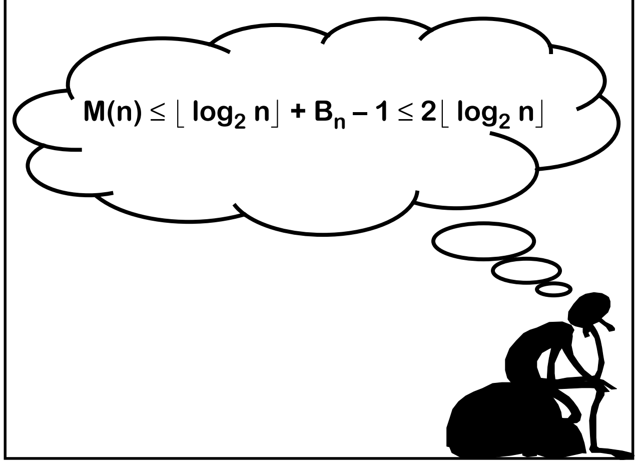
Actually used faster chain for 30*5

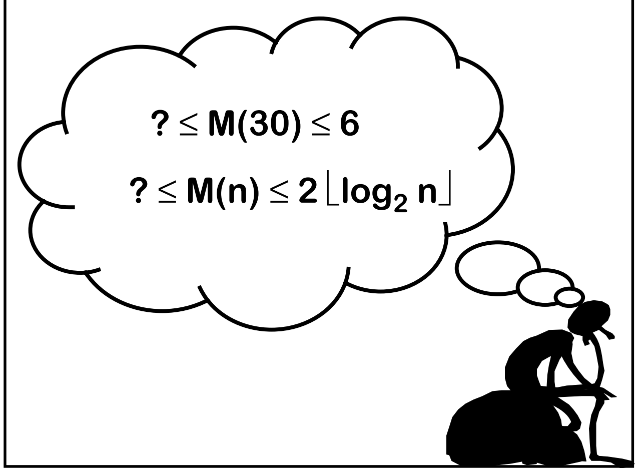
1	5
2	10
4	20
8	40
10	50
20	100
30	150

The Egyptian Connection

A shortest addition chain for n gives a shortest method for the Egyptian approach to multiplying by the number n

The fastest scribes would seek to know M(n) for commonly arising values of n





A Lower Bound Idea

You can't make any number bigger than 2^n in n steps

1 2 4 8 16 32 64 . . .

Let S_k be the statement that no k stage addition chain contains a number greater than 2^k

Base case: k=0. S_0 is true since no chain can exceed 2^0 after 0 stages

$\forall\, k > 0, \quad S_k \Rightarrow S_{k+1}$

At stage k+1 we add two numbers from the previous stage

From S_k we know that they both are bounded by 2^k

Hence, their sum is bounded by 2^{k+1}. No number greater than 2^{k+1} can be present by stage k+1

Change Of Variable

All numbers obtainable in m stages are bounded by 2^m. Let m = $\log_2(n)$

Thus, all numbers obtainable in $\log_2(n)$ stages are bounded by n

$$M(n) \geq \lceil \log_2 n \rceil$$

$? \leq M(30) \leq 6$

$\lceil \log_2 n \rceil \leq M(n) \leq 2 \lfloor \log_2 n \rfloor$

Theorem: 2^i is the largest number that can be made in i stages, and can only be made by repeated doubling

Proof by Induction:

Base i = 0 is clear

To make anything as big as 2^i requires having some X as big as 2^{i-1} in i-1 stages

By I.H., we must have all the powers of 2 up to 2^{i-1} at stage i-1. Hence, we can only double 2^{i-1} at stage i

5 < M(30)

Suppose that M(30)=5

At the last stage, we added two numbers x_1 and x_2 to get 30

Without loss of generality (WLOG), we assume that $x_1 \geq x_2$

Thus, $x_1 \geq 15$

By doubling bound, $x_1 \leq 16$

But $x_1 \neq 16$ since there is only one way to make 16 in 4 stages and it does not make 14 along the way. Thus, $x_1 = 15$ and M(15)=4

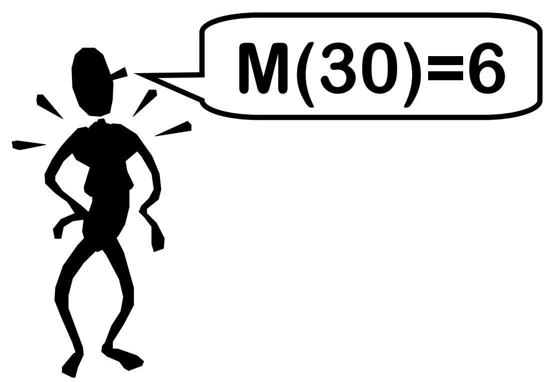
Suppose M(15) = 4

At stage 3, a number bigger than 7.5, but not more than 8 must have existed

There is only one sequence that gets 8 in 3 additions: 1 2 4 8

That sequence does not make 7 along the way and hence there is nothing to add to 8 to make 15 at the next stage

Thus, M(15) > 4   CONTRADICTION



$M(30) = 6$

$\lceil \log_2 n \rceil \leq M(n) \leq 2 \lfloor \log_2 n \rfloor$

Factoring Bound

$M(a \times b) \leq M(a) + M(b)$

Proof:

Construct a in M(a) additions

Using a as a unit follow a construction method for b using M(b) additions.
In other words, each time the construction of b refers to a number y, use the number ay instead

Example

$45 = 5 \times 9$

$M(5)=3$	[1 2 4 5]
$M(9)=4$	[1 2 4 8 9]
$M(45) \leq 3 + 4$	[1 2 4 5 10 20 40 45]

Corollary (Using Induction)

$M(a_1 a_2 a_3 \ldots a_n) \leq M(a_1)+M(a_2)+\ldots+M(a_n)$

Proof:

For n = 1 the bound clearly holds

Assume it has been shown for up to n-1

Now apply previous theorem using
$A = a_1 a_2 a_3 \ldots a_{n-1}$ **and** $b = a_n$ **to obtain:**
$M(a_1 a_2 a_3 \ldots a_n) \leq M(a_1 a_2 a_3 \ldots a_{n-1}) + M(a_n)$

By inductive assumption,
$M(a_1 a_2 a_3 \ldots a_{n-1}) \leq M(a_1) + M(a_2) + \ldots + M(a_{n-1})$

More Corollaries

Corollary: $M(a^k) \leq kM(a)$

Corollary: $M(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}) \leq \alpha_1 M(p_1) + \alpha_2 M(p_2) + \ldots + \alpha_n M(p_n)$

Does equality hold?

M(33) < M(3) + M(11)

M(3) = 2 [1 2 3]

M(11)= 5 [1 2 3 5 10 11]

M(3) + M(11) = 7

M(33) = 6 [1 2 4 8 16 32 33]

The conjecture of equality fails!

Conjecture: M(2n) = M(n) +1

(A. Goulard)

A fastest way to an even number is to make half that number and then double it

Proof given in 1895 by E. de Jonquieres in L'Intermediere Des Mathematiques, volume 2, pages 125-126

Conjecture: M(2n) = M(n) +1

(A. Goulard)

A fastest way to an even number is to make half that number and then double it

Proof given in 1895 b[illegible] E. de Jonquieres in L'In[illegible]s, vo[illegible]

FALSE! M(191)=M(382)=11

Furthermore, there are infinitely many such examples

Open Problem

Is there an n such that:
$M(2n) < M(n)$

Conjecture

Each stage might as well consist of adding the largest number so far to one of the other numbers

First Counter-example: 12,509
[1 2 4 8 16 17 32 64 128 256 512 1024 1041 2082 4164 8328 8345 12509]

Open Problem

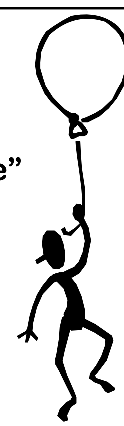
Prove or disprove the Scholz-Brauer Conjecture:

$M(2^n-1) \leq n - 1 + B_n$

(The bound that follows from this lecture is too weak: $M(2^n-1) \leq 2n - 1$)

High Level Point

Don't underestimate "simple" problems. Some "simple" mysteries have endured for thousand of years



Study Bee

Egyptian Multiplication

Raising To A Power
Minimal Addition Chain
Lower and Upper Bounds

Repeated doubling method