15-251

Great Theoretical Ideas in Computer Science

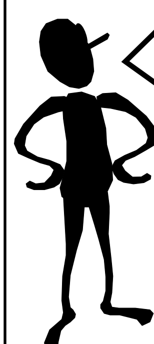
Oh No!

Homework #1 is due today at 11:59pm
Give yourself sufficient time to make PDF

Quiz #1 is next Thursday during lecture

Ancient Wisdom: Unary and Binary

Lecture 3 (January 24, 2006)



Your Ancient Heritage

Let's take a historical view on abstract representations

Mathematical Prehistory 30,000 BC

Paleolithic peoples in Europe record unary numbers on bones

1 represented by 1 mark
2 represented by 2 marks
3 represented by 3 marks

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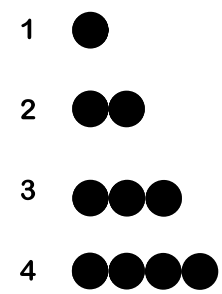
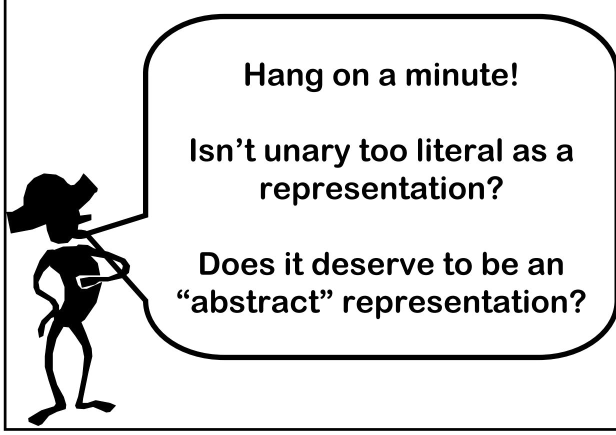
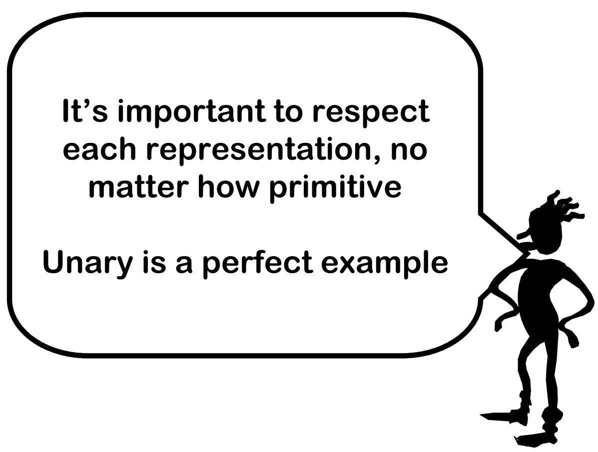
.
.

Prehistoric Unary

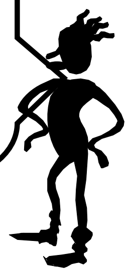






Consider the problem of finding a formula for the sum of the first n numbers

You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$

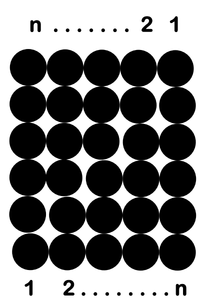


$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \ldots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \ldots & + & 2 & + & 1 & = & S \\
\hline
n+1 & + & n+1 & + & n+1 & + & \ldots & + & n+1 & + & n+1 & = & 2S
\end{array}
$$

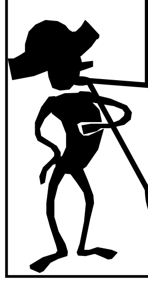
$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \ldots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \ldots & + & 2 & + & 1 & = & S
\end{array}
$$



There are n(n+1) dots in the grid!

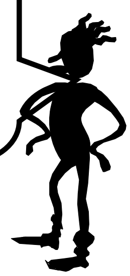


Very convincing!
Unary brings out the geometry of the problem and makes each step look natural

By the way, my name is Bonzo. And you are?

Odette

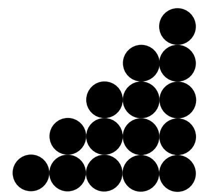
Yes, Bonzo. Let’s take it even further…



n^{th} Triangular Number

$\Delta_n = 1 + 2 + 3 + . . . + n\text{-}1 + n$

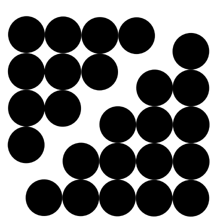
$= n(n+1)/2$

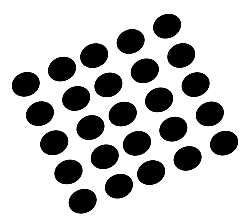


n^{th} Square Number

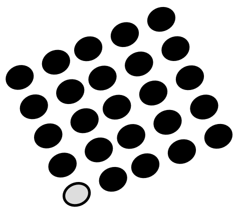
$\square_n = n^2$

$= \Delta_n + \Delta_{n-1}$





Breaking a square up in a new way



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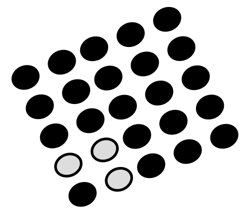
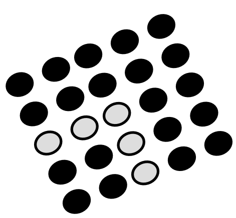
Breaking a square up in a new way



1 + 3

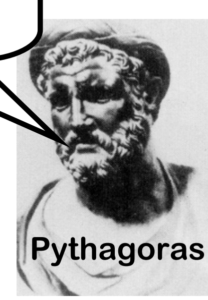
Breaking a square up in a new way



1 + 3 + 5

Breaking a square up in a new way

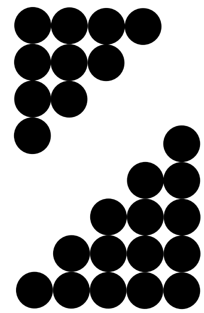
The sum of the first n odd numbers is n^2



n^{th} Square Number

$\square_n = \Delta_n + \Delta_{n-1}$

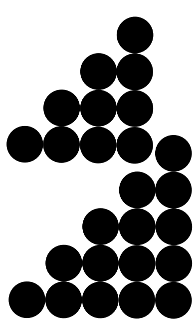
$= n^2$



n^{th} Square Number

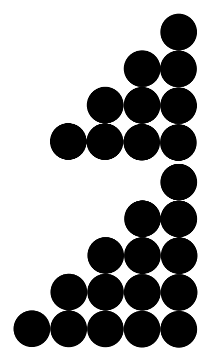
$\square_n = \Delta_n + \Delta_{n-1}$

$= n^2$



n^{th} Square Number

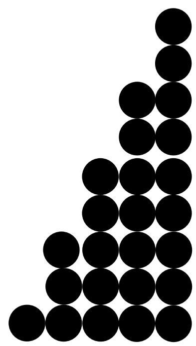
$\square_n = \Delta_n + \Delta_{n-1}$



n^{th} Square Number

$\square_n = \Delta_n + \Delta_{n-1}$

= Sum of first n
odd numbers



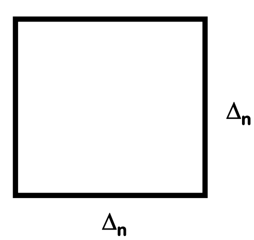
High School Notation

$\Delta_n + \Delta_{n-1} =$

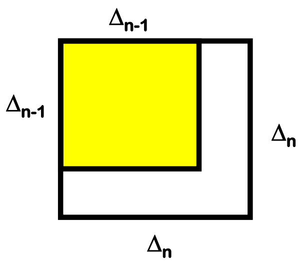
$$\begin{array}{r} 1 + 2 + 3 + 4 \ldots \\ +\ \ 1 + 2 + 3 + 4 + 5 \ldots \\ \hline 1 + 3 + 5 + 7 + 9 \ldots \end{array}$$

Sum of odd numbers

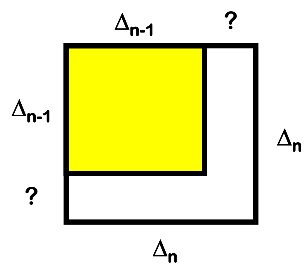
Area of square $= (\Delta_n)^2$



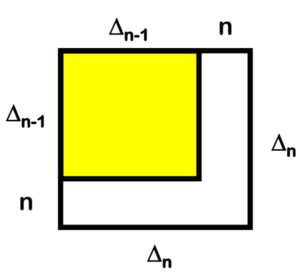
Area of square $= (\Delta_n)^2$



Area of square $= (\Delta_n)^2$



Area of square $= (\Delta_n)^2$



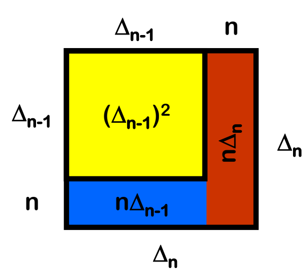
Area of square $= (\Delta_n)^2$

$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$

$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$

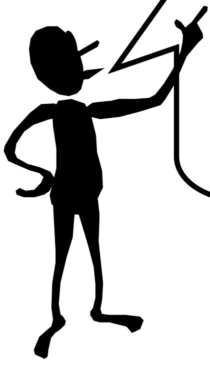
$= (\Delta_{n-1})^2 + n(\square_n)$

$= (\Delta_{n-1})^2 + n^3$



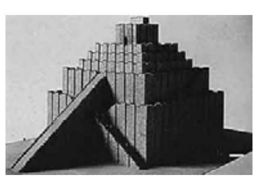
$$
\begin{aligned}
(\Delta_n)^2 &= n^3 + (\Delta_{n-1})^2 \\
&= n^3 + (n-1)^3 + (\Delta_{n-2})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3
\end{aligned}
$$

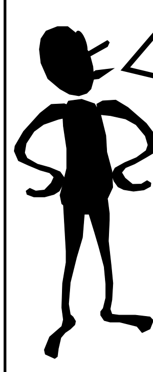
$$
\begin{aligned}
(\Delta_n)^2 &= 1^3 + 2^3 + 3^3 + \ldots + n^3 \\
&= [\, n(n+1)/2 \,]^2
\end{aligned}
$$



Can you find a formula for the sum of the first n squares?

Babylonians needed this sum to compute the number of blocks in their pyramids





Ancients grappled with abstraction in representation

Let’s look back to the dawn of symbols…

Sumerians [modern Iraq]

Sumerians [modern Iraq]

8000 BC Sumerian tokens use multiple symbols to represent numbers

3100 BC Develop Cuneiform writing

2000 BC Sumerian tablet demonstrates base 10 notation (no zero), solving linear equations, simple quadratic equations

Biblical timing: Abraham born in the Sumerian city of Ur in 2000 BC

Babylonians Absorb Sumerians

1900 BC Sumerian/Babylonian Tablet:
Sum of first n numbers
Sum of first n squares
"Pythagorean Theorem"
"Pythagorean Triples"
some bivariate equations

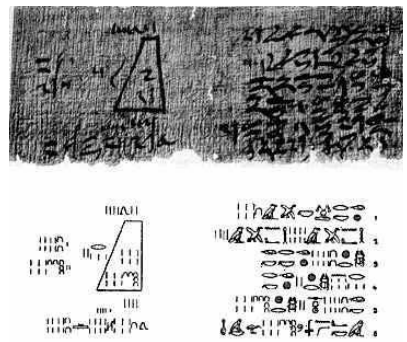
1600 BC Babylonian Tablet:
Take square roots
Solve system of n linear equations

Egyptians

6000 BC	Multiple symbols for numbers
3300 BC	Developed Hieroglyphics
1850 BC	Moscow Papyrus: Volume of truncated pyramid
1650 BC	Rhind Papyrus [Ahmes/Ahmose]: Binary Multiplication/Division Sum of 1 to n Square roots Linear equations

Biblical timing: Joseph Governor is of Egypt

Moscow Papyrus



Harrappans

[Indus Valley Culture] Pakistan/India

3500 BC	Perhaps the first writing system?!
2000 BC	Had a uniform decimal system of weights and measures

China

1200 BC	Independent writing system (Surprisingly late)
1200 BC	I Ching [Book of changes]: Binary system developed to do numerology

Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

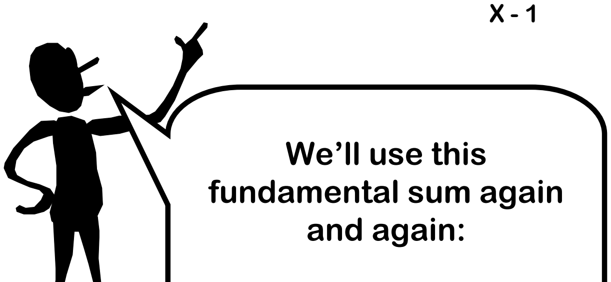
Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



A Frequently Arising Calculation

$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$\begin{array}{rl} = & X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n \\ & -\ 1 - X^1 - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1} \\ \hline = & X^n - 1 \end{array}$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

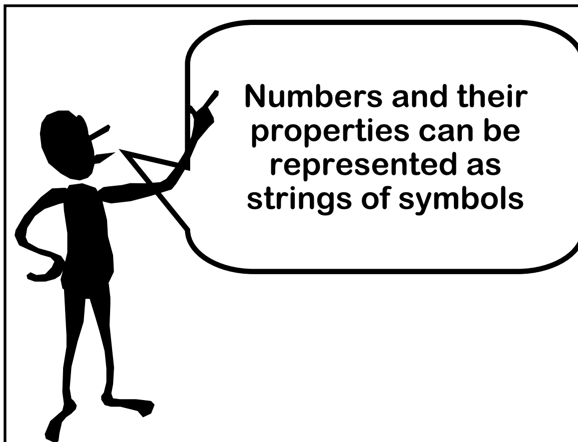
(when $x \neq 1$)

Geometric Series for X=2

$$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)



Strings Of Symbols

We take the idea of symbol and sequence of symbols as primitive

Let Σ be any fixed finite set of symbols.
Σ is called an alphabet, or a set of symbols

Examples:

- $\Sigma = \{0,1,2,3,4\}$
- $\Sigma = \{a,b,c,d, \ldots, z\}$
- Σ = all typewriter symbols
- $\Sigma = \{ a , b , c , d , \ldots, z \}$

Strings Over the Alphabet Σ

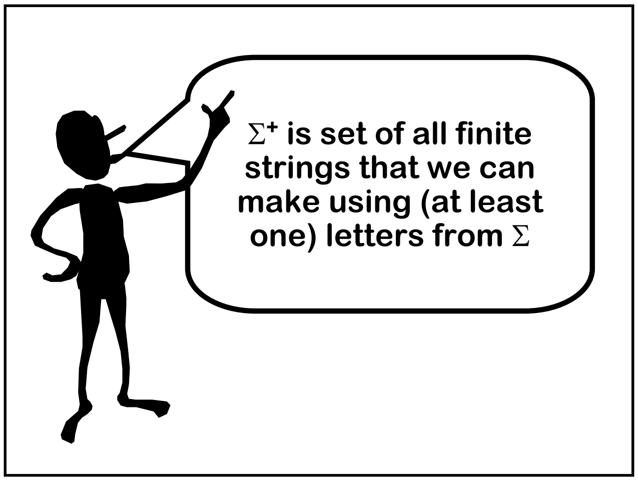
A string is a sequence of symbols from Σ

Let s and t be strings

Then st denotes the concatenation of s and t
 i.e., the string obtained by the string s followed by the string t

Now define Σ^+ by these inductive rules:

$$x \in \Sigma \Rightarrow x \in \Sigma^+$$
$$s,t \in \Sigma^+ \Rightarrow st \in \Sigma^+$$

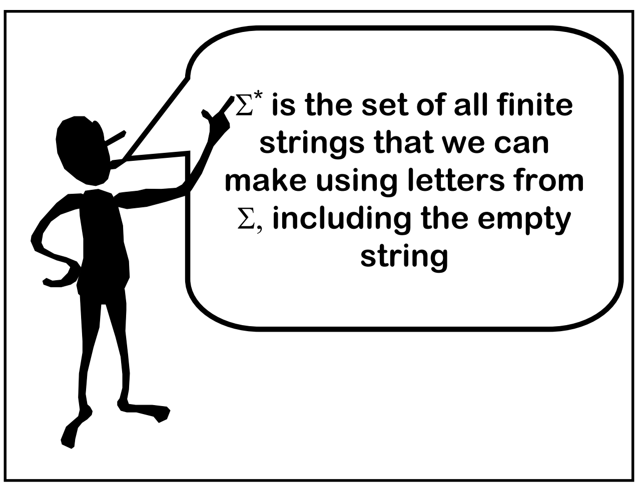


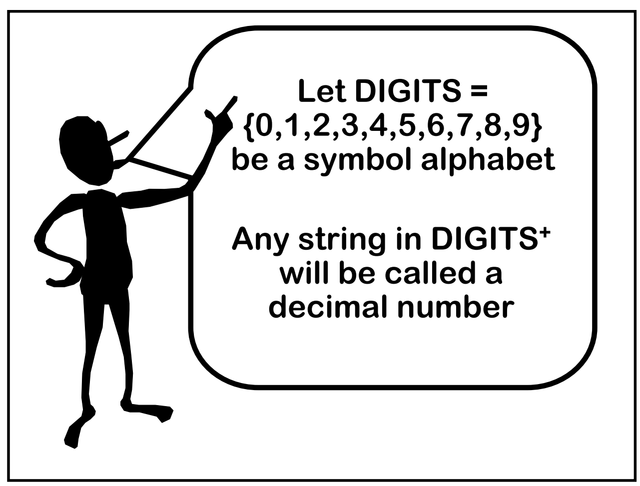
The Set Σ^*

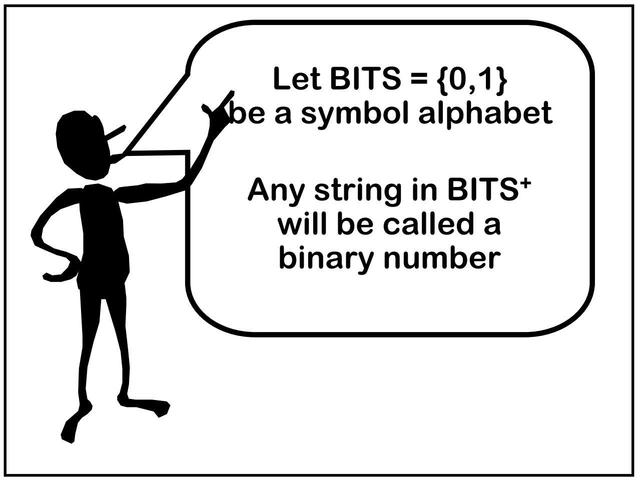
Define ε to be the empty string
i.e., $X\varepsilon Y = XY$ for all strings X and Y

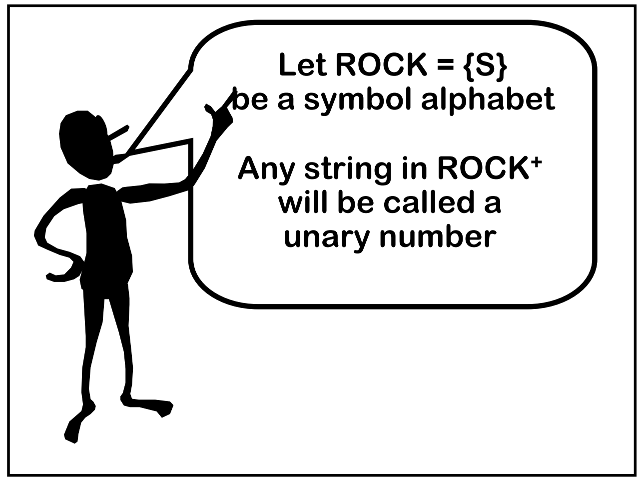
ε is also called the string of length 0

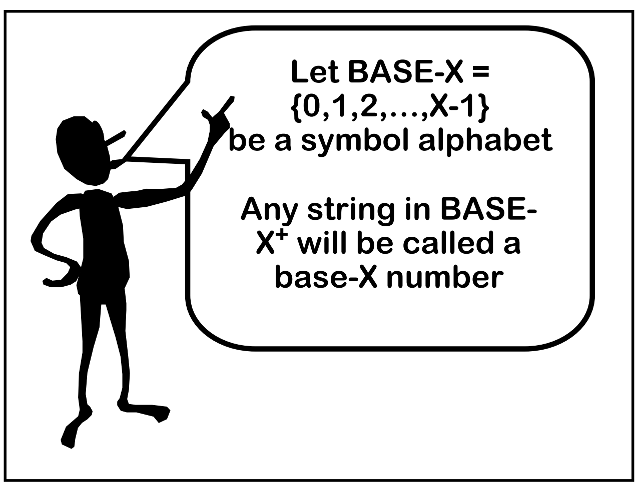
Define $\Sigma^* = \Sigma^+ \cup \{\varepsilon\}$

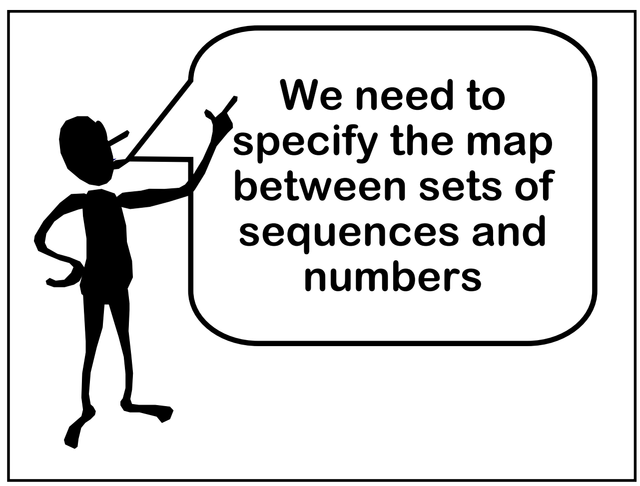


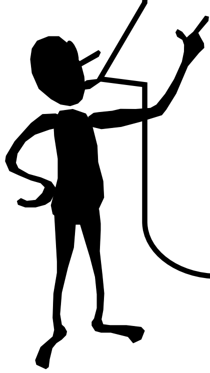



Let BITS = {0,1}
be a symbol alphabet
Any string in BITS$^+$
will be called a
binary number


Let ROCK = {S}
be a symbol alphabet
Any string in ROCK$^+$
will be called a
unary number

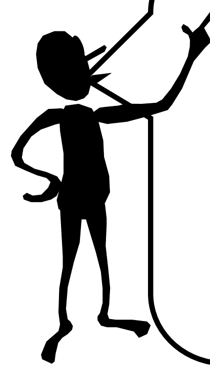

Let BASE-X =
{0,1,2,…,X-1}
be a symbol alphabet
Any string in BASE-
X$^+$ will be called a
base-X number


We need to
specify the map
between sets of
sequences and
numbers



Inductively defined function
f : ROCK⁺ → ℕ

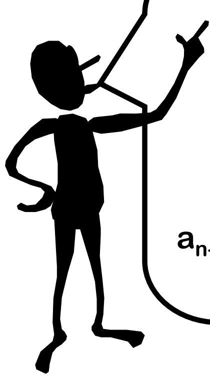
$f(S) = 1$
$f(SX) = f(X) + 1$



Inductively defined function
f : BITS⁺ → ℕ

$f(0) = 0$; $f(1) = 1$

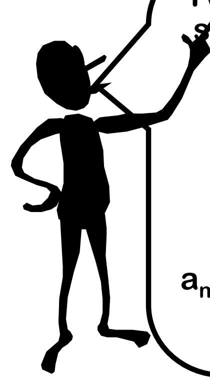
If $|W| > 1$ then $W = Xb$
(b is a bit)
$f(Xb) = 2f(X) + b$



Non-inductive representation of f:

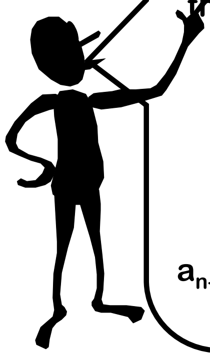
$$f(a_{n-1}\ a_{n-2} \ldots a_0) =$$

$$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$$



Two identical maps from sequences to numbers:

$f(0) = 0$; $f(1) = 1$
$f(Xb) = 2f(X) + b$
and
$$f(a_{n-1}\ a_{n-2} \ldots a_0) = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$$



The symbol a_0 is called the Least Significant Bit or the Parity Bit

$a_0 = 0$

iff

$f(a_{n-1}a_{n-2}\ldots a_0) =$

$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$

is an even number

Theorem: Each natural has a binary representation

Base Case: 0 and 1 do

Induction hypothesis: Suppose all natural numbers less than n have a binary representation

Induction Step: Note that n = 2m+b for some m < n, with b = 0 or 1

Represent n as the left-shifted sequence for m concatenated with the symbol for b

No Leading Zero Binary
(NLZB)

A binary string that is either 0 or 1,

Or has length > 1, and does not have a leading zero

1	Is in NLZB
000001101001	Is NOT in NLZB
0	Is in NLZB
01	Is NOT in NLZB
10000001	Is in NLZB

Theorem: Each natural has a unique NLZB representation

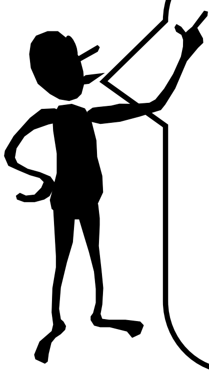
Base Case: 0 and 1 do

Induction hypothesis: Suppose all natural numbers less than n have a unique NLZB representation

Induction Step: Suppose n = 2m+b has 2 NLZB representations

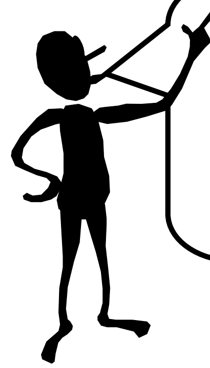
Their parity bit b must be identical

Hence, m also has two distinct NLZB representations, which contradicts the induction hypothesis. So n must have a unique representation



Inductive definition is great for showing UNIQUE representation:
f(Xb) = 2f(X) + b

Let n be the smallest number reprinted by two different binary sequences. They must have the same parity bit, thus we can make a smaller number that has distinct representations



Each natural number has a unique representation as a (No Leading Zeroes) Binary number!

BASE X Representation

S = $a_{n-1}\ a_{n-2} \ldots a_1\ a_0$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$$

Base 2 [Binary Notation]
101 represents: $1\ (2)^2 + 0\ (2^1) + 1\ (2^0)$

= ●●●●●

Base 7
015 represents: $0\ (7)^2 + 1\ (7^1) + 5\ (7^0)$

= ●●●●●●●●●●●●

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE X Representation

S = (a_{n-1} a_{n-2} … a_1 a_0 $)_X$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

Largest number representable in base-X with n "digits"

= $(X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1 \ldots X\text{-}1)_X$

= $(X-1)(X^{n-1} + X^{n-2} + \ldots + X^0)$

= $(X^n - 1)$

Fundamental Theorem For Binary

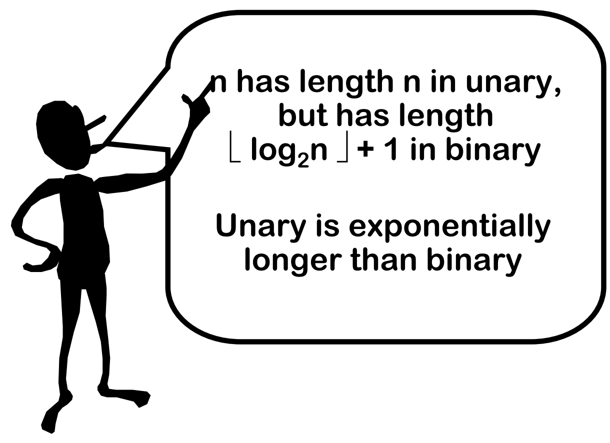
Each of the numbers from 0 to 2^{n-1} is uniquely represented by an n-bit number in binary

k uses $\lfloor \log_2 k \rfloor + 1$ digits in base 2

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^{n-1} is uniquely represented by an n-"digit" number in base X

k uses $\lfloor \log_X k \rfloor + 1$ digits in base X



Other Representations: Egyptian Base 3

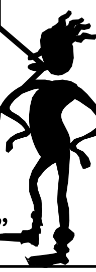
Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

Example: 1 -1 -1 = 9 - 3 - 1 = 5

We can prove a unique representation theorem

How could this be Egyptian? Historically, negative numbers first appear in the writings of the Hindu mathematician Brahmagupta (628 AD)



One weight for each power of 3
Left = “negative”. Right = “positive”

Study Bee

Unary and Binary
Triangular Numbers
Dot proofs

$(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$

Base-X representations
unique binary representations
proof for no-leading zero binary

k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ digits in base 2

Largest length n number in base X