15-251: GTI Quiz 1 SOLUTIONS

Repeat After Me.

This part is to test your ability to study material that has been labeled as likely to appear on exams. If you do not do well on this section, your study habits could use adjustment.

1. *[5 points]*
How many different numbers can be represented by n digits in base b? Leading zeroes are allowed.

$$b^n$$

2. *[5 points]*
Give a closed form expression for $1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1$

$$n^2$$

3. *[5 points]*
What is the shortest addition chain for 65?

$$1, 2, 4, 8, 16, 32, 64, 65$$

4. *[5 points]*
Give a closed form expression for $\sum_{i=0}^{n} x^i$.

$$\frac{1 - x^{n+1}}{1 - x}$$

5. *[5 points]*
How many nodes are there in the nth pancake graph?

$$n!$$

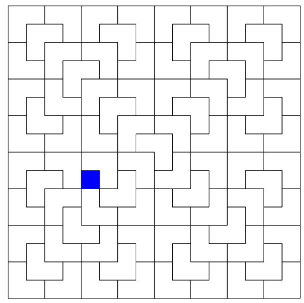
6. *[5 points]*
How many edges are there in the nth pancake graph?

$$n!(n-1)/2$$

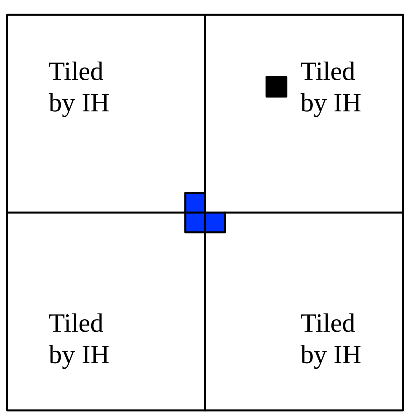
Reading Solutions.

This section tests whether you read the solutions that we hand out.

7. *[10 points]*
Triominoes: *Prove that it is always possible to use L-shaped triominoes to tile a board of size* $2^n \times 2^n$ *with* any *one square removed. Below is one example of such a tiling for a 16 by 16 board.*



We shall prove this by induction on n.
Base case: $n = 0$, and hence the board is 1×1. Since there is only one square, this must be the uncovered square. But it is trivial to cover this board with (no) triominoes such that an arbitrary square is uncovered.
Inductive Hypothesis (IH): It is always possible to cover a board of size $2^n \times 2^n$ with any one square removed.
Inductive Step: Consider a board of size 2^{n+1} by 2^{n+1}. Break the board up into 4 *quadrants*, each of size 2^n by 2^n. Without loss of generality, the square we want uncovered is in the upper right quadrant. (See the figure below.) By IH, we can tile the upper right quadrant with this square uncovered. Also by IH, we can tile the upper left quadrant with the lower right corner removed, the lower left quadrant with the upper right corner removed and the lower right quadrant with the upper left corner removed. Put these 4 quadrants together and we now have an L-shaped hole consisting of 3 pieces that has appeared as the result of the tilings of the upper left, lower left and lower right quadrants. Fill in this hole with a triominoe.



8. *[10 points]*
Choosing Stones: *Start off with a pile of n stones. At each step, you can remove an existing pile of k stones and replace it by two piles (say, with k_1 and k_2 stones, where $k_1 + k_2 = k$). On doing this, you are paid $k_1 \times k_2$.* [Clearly, as long as there is a pile with more than one stone, you can make more money; hence the game stops when there are n "piles" with one stone each.]
How can you maximize the amount of money you make? Prove that this amount is indeed the maximum.

Represent each stone as a node in a graph. Place an edge of the graph between every pair of nodes. Notice that separating the pile of k stones into two piles of k_1 and k_2 stones corresponds to removing some edges in the graph. Further, the number of edges removed is exactly $k_1 \times k_2$. This follows because each of the k_1 nodes in pile 1 had k_2 edges to the nodes in pile 2 removed when the original pile was separated. Thus, the number of edges removed is exactly the same as the payment received for the separation.
Finally, remember that the game ends when there are n piles, which corresponds to every edge in the graph being removed. Thus, the total payment regardless of the choices made is simply the total number of edges in the initial graph: $n(n-1)/2$.

Common Mistakes: A common mistake for this problem was to specify a strategy for creating two piles, and proving that the strategy was optimal for creating two piles. However, it is necessary to show that the strategy is optimal for creating piles *all the way down* until there are n piles of 1 stone each. This requires a non-trivial inductive proof which a few students provided. However, it should be clear that the above argument is perfectly valid and does not require the use of induction. (Yet another example of the power of changing representation.)

Basic Techniques.

This part will test your ability to apply techniques that I have explicitly identified in lecture. You need to have practiced each technique enough to be able to handle small variations in the problems. You do not need to reduce factorials and binomial coefficients to a numeric answer.

9. *[10 points]*
Calculate $GCD(84, 75)$.

We use Euclid's GCD Algorithm:

$$\begin{aligned} 84 \bmod 75 &= 9 \\ 75 \bmod 9 &= 3 \\ 9 \bmod 3 &= 0 \end{aligned}$$

So we see GCD$(84, 75)$=3.

10. *[10 points]*
Let F_n denote the nth Fibonacci number. Prove by induction that for any integer k, F_n divides F_{kn}.
Hint: recall that $F_{a+b} = F_a \cdot F_{b+1} + F_{a-1} \cdot F_b$

Base: $F_n | F_n$ since $F_n = 1 \cdot F_n$.
IH: $F_n | F_{kn}$.
Inductive Step:

$$\begin{aligned} F_{(k+1)n} &= F_{kn+n} \\ &= F_n \cdot F_{kn+1} + F_{n-1} \cdot F_{kn} \text{ from our hint} \\ &= F_n \cdot F_{kn+1} + F_{n-1} \cdot F_n \cdot X \text{ by IH} \\ &= F_n \cdot (F_{kn+1} + F_{n-1} \cdot X) \end{aligned}$$

Hence F_n divides $F_{(k+1)n}$.

Common Mistakes: A common mistake for this problem was to induct on n instead of k. (The wording of the problem tried to indicate this.) Indeed, if you do try that, you will notice that you will be required to prove $F_{n+1} | F_{(k(n+1))}$, which is difficult to show from the IH.
A word about a mistake many of you made: remember that if $a|c$ and $b|d$, it does not follow that $(a + b)|(c + d)$. (Note that $1|4$ and $2|4$, but 3 does not divide 8.)

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers.

11. *[20 points]*
Suppose we define "alternating base-Fibonacci" as follows:
Given a sequence of digits $a_n \cdots a_2 a_1$*,* $a_i \in \{-1, 0.1\}$*, we interpret them in alternating base-Fibonacci notation as the number*

$$\sum_{i=1}^{n} a_i F_{2i-1}$$

where we take $F_0 = 1$*,* $F_1 = 1$*,* $F_2 = 2$*, etc.*
Show that all integers in the range $\pm(F_{2k} - 1)$ *can be represented using* k *alternating base-Fibonacci digits.*

We proceed by induction on the number of digits.
Base: $k = 1$. $F_2 = 2$, and since $F_1 = 1$, we can clearly represent 1, 0, and -1 as themselves.
IH: In k digits we can represent all integers in the range $\pm(F_{2k} - 1)$.
Induction Step: $k+1$ digits. If $a_{k+1} = 1$, then we have F_{2k+1} plus some $k-1$ digit number. By IH, we can represent all k-digit numbers down to $-F_{2k} + 1 = -F_{2k-1} - F_{2k-2} - 1$, where, if $k \geq 1$, $F_{2k-2} \geq 1$. In particular, since we can represent down to $-F_{2k-1}$ with the lower k digits, we can represent

$$F_{2k+1} - F_{2k-1} = F_{2k} + F_{2k-1} - F_{2k-1} = F_{2k}$$

Now, by IH, these lower k digits can represent up to $F_{2k} - 1$, so we can represent all smaller numbers if $a_{k+1} = 0$, but moreover, we can represent

$$F_{2k+1} + F_{2k} - 1 = F_{2k} + F_{2k-1} + F_{2k} - 1 = F_{2k} + F_{2k+1} - 1 = F_{2(k+1)} - 1$$

so clearly can represent all positive numbers up to $F_{2(k+1)} - 1$. Similarly, by flipping the sign of each digit, it is clear that we can also represent every integer down to $-(F_{2k+1} - 1)$ in these $k + 1$ digits.

12. *[20 points]*
Now suppose we define "sparse base-Fibonacci" as follows:
Given a sequence of digits $a_n \cdots a_1$, $a_i \in \{0,1\}$, with the additional constraint that we do not allow a_i and a_{i+1} to both be 1 for any i, we interpret them in sparse base-Fibonacci as the number

$$\sum_{i=1}^{n} a_i F_i$$

where we again take $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, etc.
Can all natural numbers be represented in this notation? Are all sparse base-Fibonacci numbers uniquely represented?

Yes and yes. We give a proof by strong induction on the number of digits, k.
Base: We can represent 1 in one digit as $1 \cdot F_1$ and 0 as $0 \cdot F_1$
IH: We can represent all natural numbers up to $F_{i+1} - 1$ in i digits, for every $1 \leq i \leq k$.
Induction Step: In $k+1$ digits, we can still, by IH, represent all of the natural numbers up to $F_{k+1} - 1$ uniquely when a_{k+1} is 0. Since all of the digits are positive, we cannot represent any number smaller than F_{k+1} when $a_{k+1} = 1$, so the representation of these smaller numbers is still unique.
When $a_{k+1} = 1$, we know that a_k cannot be 1, as this would violate our constraint on no consecutve 1's. We also know that every natural number 0–$F_k - 1$ can be uniquely represented by the lower $k-2$ digits. Therefore, we can also uniquely represent the numbers $0 + F_{k+1} = F_{k+1}$ up to $F_{k+1} + F_k - 1 = F_{k+2} - 1$ by letting $a_{k+1} = 1$, $a_k = 0$, and the lower $k-2$ digits be the unique representation of the corresponding number.