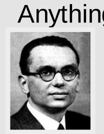
Great Theoretical Ideas In Computer Science		
Steven Rudich		CS 15-251 Spring 2005
Lecture 28	April 26, 2005	Carnegie Mellon University

Gödel's Legacy: The Limits Of Logics

Anything

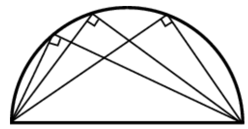


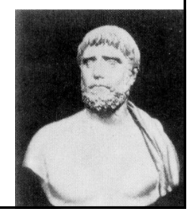
says is false!

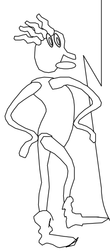
Thales Of Miletus (600 BC) Insisted on Proofs!

"first mathematician"

Most of the starting theorems of geometry. SSS, SAS, ASA, angle sum equals 180, . . .



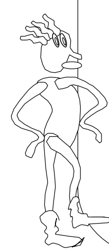




GENERAL PICTURE:

A **decidable** set of "SYNRACTICALLY VALID STATEMENTS S.

A **possibly incomputable** subset of S called $TRUTH_S$ I.e. TRUE STATEMENTS OF S

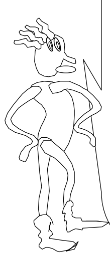


GENERAL PICTURE:

A computable $LOGIC_S$ function $Logic_S(x,y)$ = y does/doesn't follow from x.

$PROVABLE_{S,L}$ =

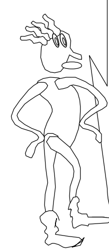
All $Q \in S$ for which there is a valid proof of Q in logic L



PROOF IN $LOGIC_S$.

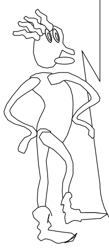
If LOGIC_S(x,y) = y follows x in one step.

Then we say that statement x implies statement y.



We add a "start statement" Δ to the input set of our LOGIC function.

$Logic_S(\Delta,S)$ = Yes will mean that our logic views S as an axiom.

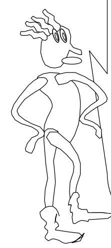


A sequence of statements $s_1, s_2, \ldots, s_n$ is a VALID PROOF of statement Q in $LOGIC_S$ iff

$LOGIC(\Delta, s_1)$ = True

And for n+1> i>1
$LOGIC(s_{i-1}, s_i)$ = True

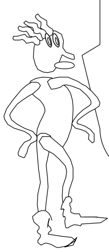
s_n = Q



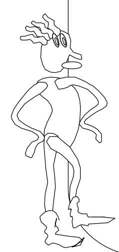
Let S be a set of statements. Let L be a logic function.

$PROVABLE_{S,L}$ =

All $Q \in S$ for which there is a valid proof of Q in logic L



A logic is "sound" for a truth concept if everything it proves is true according to the truth concept.



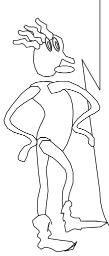
$LOGIC_S$ is SOUND for $TRUTH_S$ if

$LOGIC(\Delta, A)$ = true

$\Rightarrow A \in TRUTH_S$

$LOGIC(B,C)$=true
and $B \in TRUTH_S$

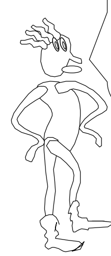
$\Rightarrow TRUTH(C)$=True



If $LOGIC_S$ is sound for $TRUTH_S$

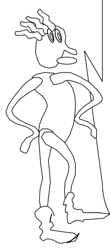
Then

$LOGIC_S$ proves C
$\Rightarrow C \in TRUTH_S$



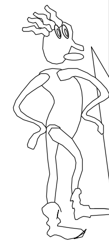
If a $LOGIC_S$ is sound for $TRUTH_S$

it means that L can't prove anything not in $TRUTH_S$.

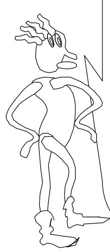


Boolean algebra is SOUND for the truth concept of propositional tautology.

High school algebra is SOUND for the truth concept of algebraic equivalence.

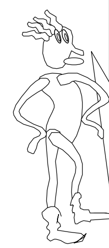


SILLY FOO FOO 3 is SOUND for the truth concept of an even number of ones.



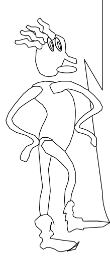
Euclidean Geometry is SOUND for the truth concept of facts about points and lines in the Euclidean plane.

Peano Arithmetic is SOUND for the truth concept of (first order) number facts about Natural numbers.



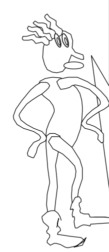
A logic may be SOUND but it still might not be complete.

A logic is "COMPLETE" if it can prove every statement that is True in the truth concept.



SOUND:
$\text{PROVABLE}_{S,L} \subset \text{TRUTH}_S$

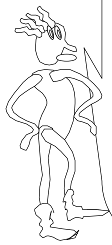
COMPLETE:
$\text{TRUTH}_S \subset \text{PROVABLE}_{S,L}$



SOUND:
$\text{PROVABLE}_{S,L} \subset \text{TRUTH}_S$

COMPLETE:
$\text{TRUTH}_S \subset \text{PROVABLE}_{S,L}$

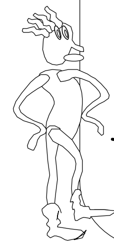
Ex: Axioms of Euclidean Geometry are known to be sound and complete for the truths of line and point in the plane.



SOUND:
$PROVABLE_{S,L} \subset TRUTH_S$

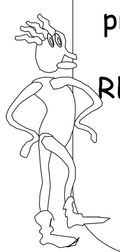
COMPLETE:
$TRUTH_S \subset PROVABLE_{S,L}$

SILLY FOO FOO 3 is sound and complete for the truth concept of strings having an even number of 1s.

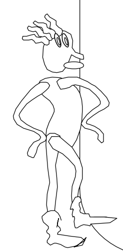


GENRALLY SPEAKING A LOGIC WILL NOT BE ABLE TO KEEP UP WITH TRUTH!

THE PROVABLE CONSEQUENCES OF ANY LOGIC ARE RECURSIVELY ENUMERABLE. THE SET OF TRUE STATEMENTS ABOUT HALT/NON-HALTING PROGRAMS IS NOT.



We have seen that the set of programs which do not halt on themselves – IS NOT RECURSIVELY ENUMERABLE.



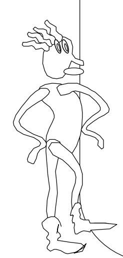
Given any LOGIC, we can enumerate all of its provable consequences.

Listing $PROVABLE_{LOGIC}$

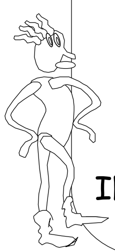
```
k;=0;

For sum = 0 to forever do

{Let PROOF loop through all strings of length k do
        {Let STATEMENT loop through strings of length <k do
         If proofcheck(STATEMENT, PROOF) = valid, output STATEMENT
        k++
        }
}
```

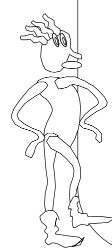


Let S be a language and $TRUTH_S$ be a truth concept. We say that "$TRUTH_S$ EXPRESSES THE HALTING PROBLEM" iff there exists a *computable* function r such that
$r(x) \in TRUTH_S$ exactly when $x \in K$.



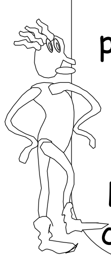
Let S be a language, L be a logic, and $TRUTH_S$ be a truth concept that expresses the halting problem.

THEOREM: If L is sound for $TRUTH_S$, then L is INCOMPLETE for $TRUTH_S$.



THEOREM: If L is sound for $TRUTH_S$, then L is INCOMPLETE for $TRUTH_S$.

L proves r(x) <-> x(x) doesn't halt. Thus, we can run x(x) and list theorems of L – one of them will tell us if x(x) halts.



FACT: Truth's of first order number theory (for every natural, for all naturals, plus, times, propositional logic) express the halting problem.

INCOMPLETNESS: No LOGIC for number theory can be sound and complete.

Hilbert's Question [1900]

Is there a foundation for mathematics that would, in principle, allow us to decide the truth of any mathematical proposition? Such a foundation would have to give us a clear procedure (algorithm) for making the decision.

GÖDEL'S
INCOMPLETENESS THEOREM

In 1931, Gödel stunned the world by proving that for any consistent axioms F there is a true statement of first order number theory that is not provable or disprovable by F. I.e., a true statement that can be made using 0, 1, plus, times, for every, there exists, AND, OR, NOT, parentheses, and variables that refer to natural numbers.

GÖDEL'S
INCOMPLETENESS THEOREM

Commit to any sound LOGIC F for first order number theory. Construct a *true* statement $GODEL_F$ that is not provable in your logic F.

YOU WILL EVEN BE ABLE TO FOLLOW THE CONTRUCTION AND ADMIT THAT $GODEL_F$ is a true statement that is missing from the consequences of F.

$\text{CONFUSE}_F(P)$

Loop though all sequences of symbols S

If S is a valid F-proof of "P halts", then LOOP_FOR_EVER

If S is a valid F-proof of "P never halts", then HALT

GODEL_F

GODEL_F= AUTO_CANNIBAL_MAKER(CONFUSE_F)

Thus, when we run GODEL_F it will do the same thing as:

$\text{CONFUSE}_F(\text{GODEL}_F)$

GODEL_F

Can F prove GODEL_F halts?

Yes -> $\text{CONFUSE}_F(\text{GODEL}_F)$ does not halt
Contradiction

Can F prove GODEL_F does not halt?

Yes -> $\text{CONFUSE}_F(\text{GODEL}_F)$ halts
Contradiction

GODEL_F

F can't prove or disprove that GODEL_F halts.

$\text{GODEL}_F = \text{CONFUSE}_F(\text{GODEL}_F)$
Loop though all sequences of symbols S

If S is a valid F-proof of "GODEL_F halts", then LOOP_FOR_EVER

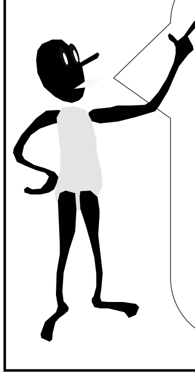
If S is a valid F-proof of "GODEL_F never halts", then HALT

GODEL_F

F can't prove or disprove that GODEL_F halts.

Thus $\text{CONFUSE}_F(\text{GODEL}_F) = \text{GODEL}_F$ will not halt. Thus, we have just proved what F can't.

F can't prove something that we know is true. It is not a complete foundation for mathematics.



So what is mathematics?

THE DEFINING INGREDIENT OF MATHEMATICS IS HAVING A SOUND LOGIC – self-consistent for some notion of truth.

ENDNOTE

You might think that Gödel's theorem proves that people are mathematically capable in ways that computers are not. This would show that the Church-Turing Thesis is wrong.

Gödel's theorem proves no such thing!

We can talk about this over coffee.