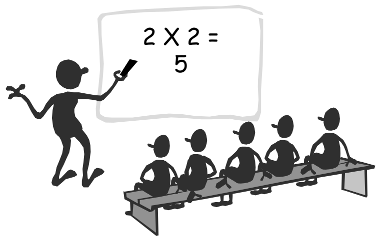
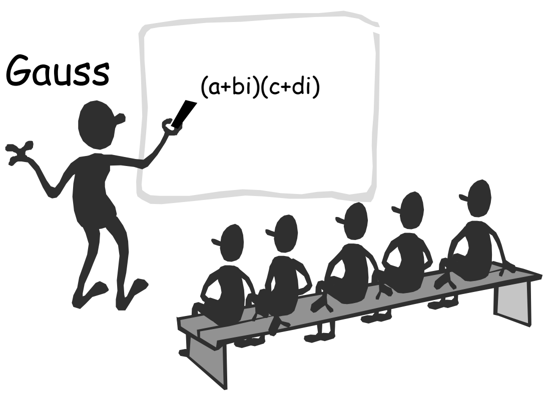
Great Theoretical Ideas In Computer Science		
Anupam Gupta		CS 15-251 Spring 2005
Lecture 18	March 17, 2005	Carnegie Mellon University

Grade School Revisited: How To Multiply Two Numbers



Gauss



Gauss' Complex Puzzle

Remember how to multiply two complex numbers a + bi and c + di?

(a+bi)(c+di) = [ac –bd] + [ad + bc] i

Input: a,b,c,d
Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

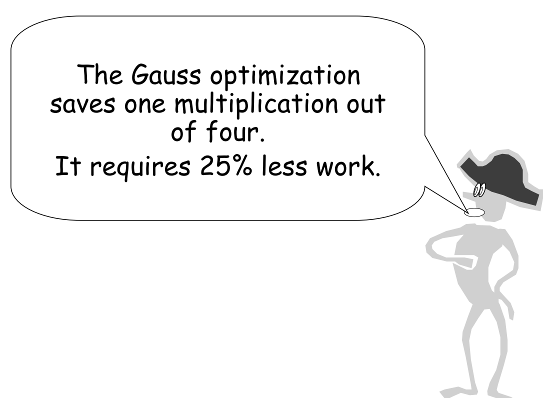
Can you do better than $4.02?

Gauss' $3.05 Method

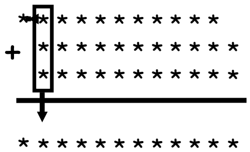
Input: a,b,c,d
Output: ac-bd, ad+bc

(a+bi)(c+di) = [ac –bd] + [ad + bc] i

c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	= ac + ad + bc + bd
$	$X_4 = ac$	
$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	= ac - bd
cc	$X_7 = X_3 - X_4 - X_5$	= bc + ad



Time complexity of grade school addition

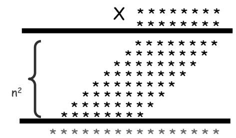


T(n) = The amount of time grade school addition uses to add two n-bit numbers

We saw that T(n) was linear.

$T(n) = \Theta(n)$

Time complexity of grade school multiplication

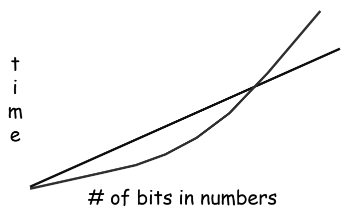


T(n) = The amount of time grade school multiplication uses to add two n-bit numbers

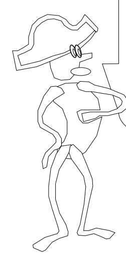
We saw that T(n) was quadratic.

$T(n) = \Theta(n^2)$

Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



No matter how dramatic the difference in the constants the quadratic curve will eventually dominate the linear curve



Grade school addition is linear time.

Is there a sub-linear time method for addition?

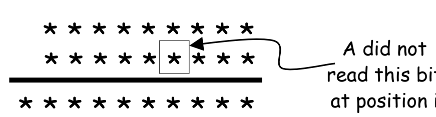
Any addition algorithm takes $\Omega(n)$ time

Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm A that does not examine each bit

Give A a pair of numbers. There must be some unexamined bit position i in one of the numbers

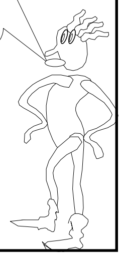
Any addition algorithm takes $\Omega(n)$ time



- If A is not correct on the inputs, we found a bug
- If A is correct, flip the bit at position i and give A the new pair of numbers. A gives the same answer as before, which is now wrong.

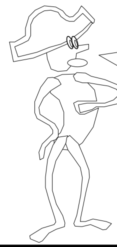
So any algorithm for addition must use time at least linear in the size of the numbers.

Grade school addition can’t be improved upon by more than a constant factor.



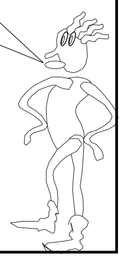
Grade School Addition: Θ(n) time
Furthermore, it is optimal

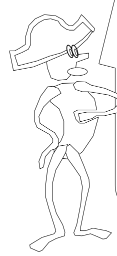
Grade School Multiplication: Θ(n^2) time



Is there a clever algorithm to multiply two numbers in <u>linear</u> time?

Despite years of research, no one knows! If you resolve this question, Carnegie Mellon will give you a PhD!

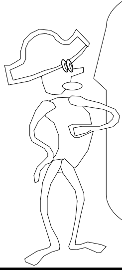




Can we even break the quadratic time barrier?

In other words, can we do something very different than grade school multiplication?

Grade School Multiplication: The Kissing Intuition



Intuition:
Let’s say that each time an algorithm has to multiply a digit from one number with a digit from the other number, we call that a “kiss”.

It seems as if any correct algorithm must kiss at least n^2 times.

Divide And Conquer

An approach to faster algorithms:

1. DIVIDE a problem into smaller subproblems
2. CONQUER them recursively
3. GLUE the answers together so as to obtain the answer to the larger problem

Multiplication of 2 n-bit numbers

n bits

X = | X |

Y = | Y |

n/2 bits   n/2 bits

$X = a\ 2^{n/2} + b \qquad Y = c\ 2^{n/2} + d$

$X \times Y = ac\ 2^n + (ad + bc)\ 2^{n/2} + bd$

Multiplication of 2 n-bit numbers

X = | a | b |

Y = | c | d |

n/2 bits   n/2 bits

$X \times Y = ac\ 2^n + (ad + bc)\ 2^{n/2} + bd$

```
MULT(X,Y):
 If |X| = |Y| = 1 then return XY
 break X into a;b and Y into c;d
 return
   MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) 2^{n/2} + MULT(b,d)
```

Same thing for numbers in decimal!

n digits

X = | a | b |

Y = | c | d |

n/2 digits   n/2 digits

$X = a\ 10^{n/2} + b \qquad Y = c\ 10^{n/2} + d$

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

*4276 5678*2139   5678*4276

X = | a | b |

Y = | c | d |

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

X = | a | b |

Y = | c | d |

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Y = c d

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Y = c d

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Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

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1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

1*2 1*1 2*2 2*1 xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

X = a b

Y = c d

$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

2 1 4 2 xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Hence: 12*21 = $2*10^{2} + (1 + 4)10^{1} + 2 = 252$

X = a b

Y = c d

$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

2 1 4 2 xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Hence: 12*21 = $2*10^{2} + (1 + 4)10^{1} + 2 = 252$

X = a b

Y = c d

$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

1*2 1*1 2*2 2*1 xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Hence: 12*21 = $2*10^{2} + (1 + 4)10^{1} + 2 = 252$

X = a b

Y = c d

$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

1*3 1*9 2*3 2*9 xxxxxxxxxxxxxxxxxxxx

X = a b

Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

242 12*39 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

3 9 6 18 xxxxxxxxxxxxxxxxxxxx

$*10^2 + *10^1 + *10^1 + *1$

X = a b

Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

242 468 34*21 34*39 xxxxxxxxxxxxxxxxxxxxxxxxx

3 9 6 18 xxxxxxxxxxxxxxxxxxxx

$*10^2 + *10^1 + *10^1 + *1$

X = a b

Y = c d

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Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 468 714 1326 xxxxxxxxxxxxxxxxxxxxxxxxx

X = a b

Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 468 714 1326 xxxxxxxxxxxxxxxxxxxxxxxxx

$*10^4 + *10^2 + *10^2 + *1$

X = a b

Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 468 714 1326 xxxxxxxxxxxxxxxxxxxxxxxxx

$*10^4 + *10^2 + *10^2 + *1$

= 2639526

X = a b

Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 1234*4276 5678*2139 5678*4276

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 5276584 12145242 24279128

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 5276584 12145242 24279128
$*10^8$ + $*10^4$ + $*10^4$ + $*1$

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 5276584 12145242 24279128
$*10^8$ + $*10^4$ + $*10^4$ + $*1$

= 264126842539128

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

264126842539128

X = a b
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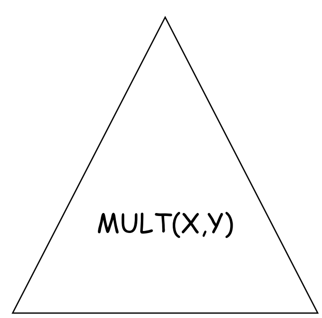
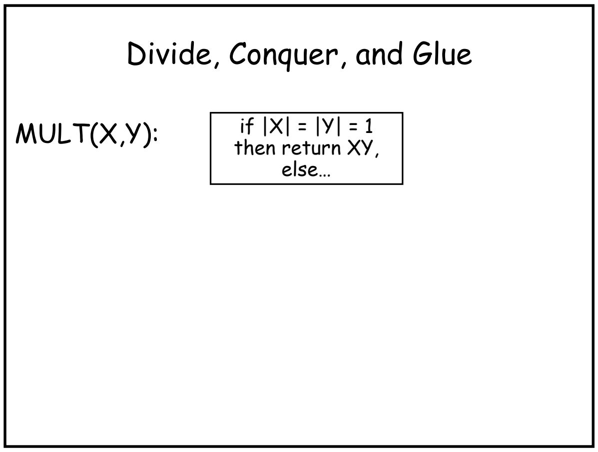
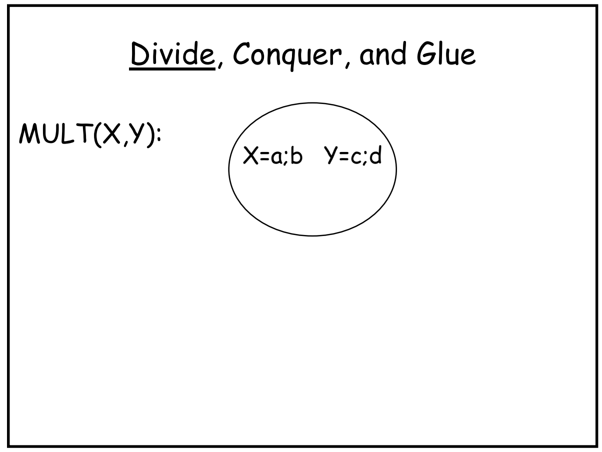
Divide, Conquer, and Glue




Divide, Conquer, and Glue
MULT(X,Y):
if |X| = |Y| = 1
then return XY,
else…


Divide, Conquer, and Glue
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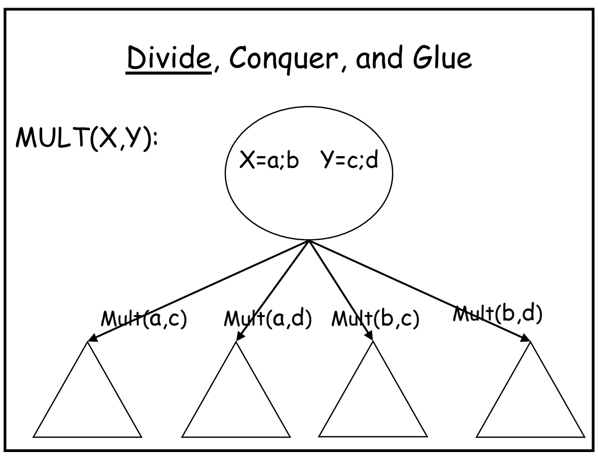
X=a;b Y=c;d


Divide, Conquer, and Glue
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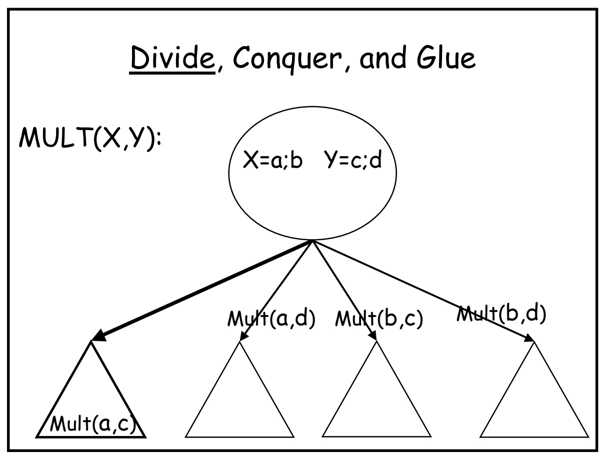
X=a;b Y=c;d
Mult(a,c)
Mult(a,d)
Mult(b,c)
Mult(b,d)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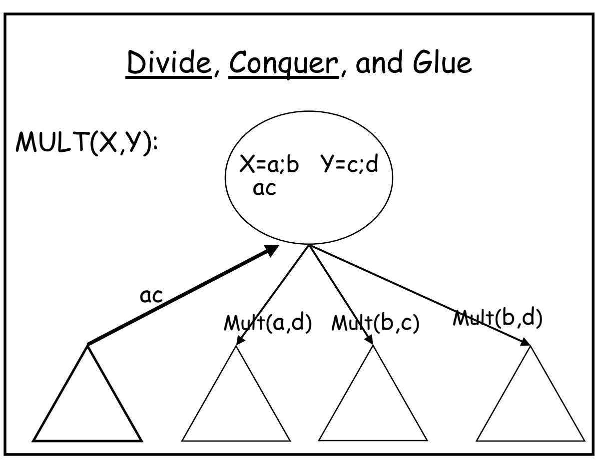
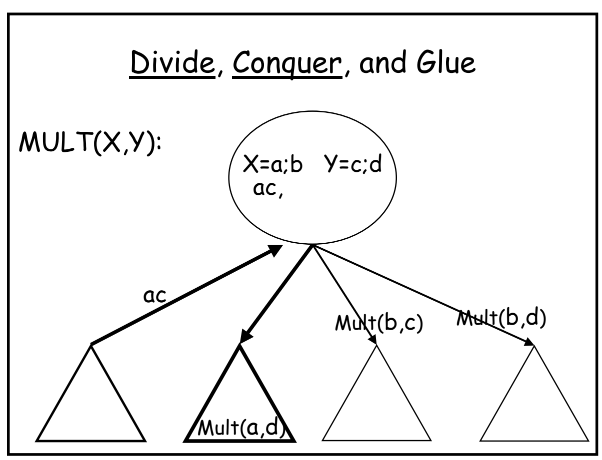
Mult(b,c)
Mult(b,d)
Mult(a,c)


Divide, Conquer, and Glue
MULT(X,Y):
X=a;b Y=c;d
ac
ac
Mult(a,d)
Mult(b,c)
Mult(b,d)


Divide, Conquer, and Glue
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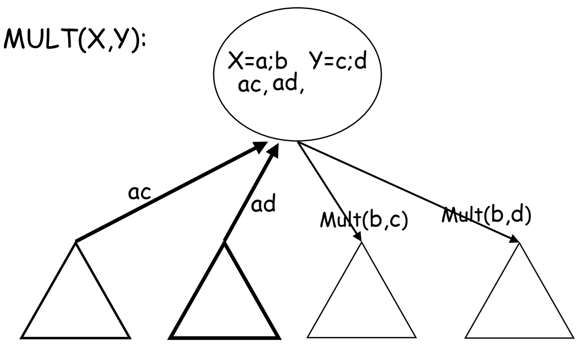
ac,
ac
Mult(b,c)
Mult(b,d)
Mult(a,d)

Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

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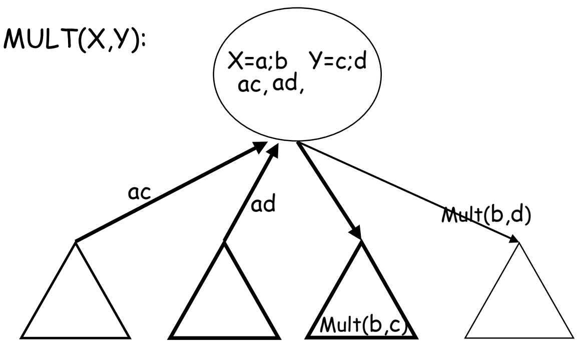
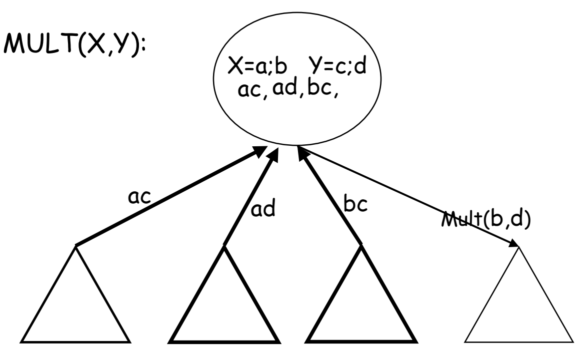


Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

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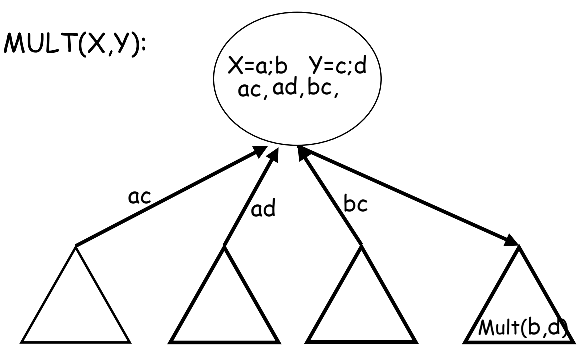
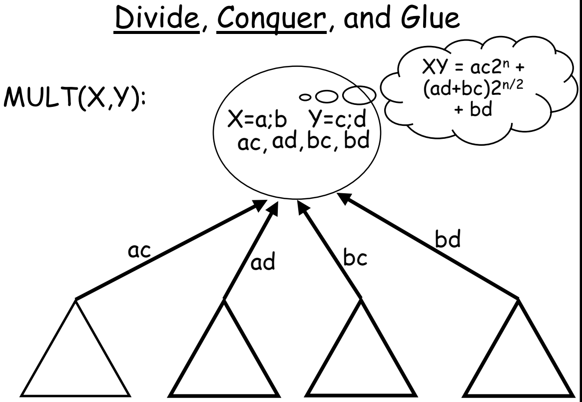


Divide, Conquer, and Glue

MULT(X,Y):

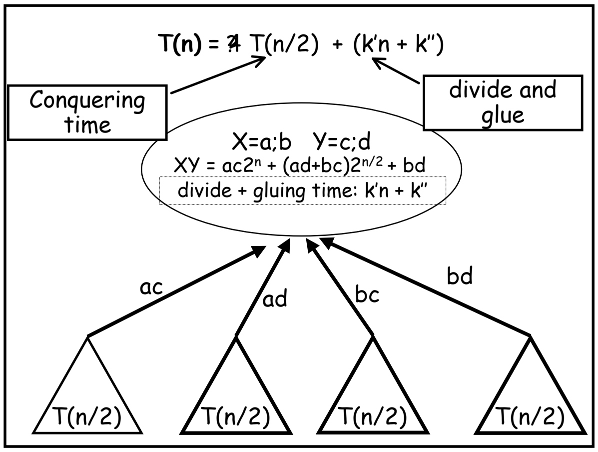


Time required by MULT

T(n) = time taken by MULT on two n-bit numbers

What is T(n)? What is its growth rate?

Big Question: Is it $\Theta(n^2)$?



Recurrence Relation

$T(1) = k$ for some constant k

$T(n) = 4\,T(n/2) + k'n + k''$ for constants k' and k''

```
MULT(X,Y):
If |X| = |Y| = 1 then return XY
break X into a;b and Y into c;d
return
  MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Let's be concrete and keep it simple

$T(1) =$ ~~k~~ 1 for some constant k

$T(n) = 4\,T(n/2) +$ ~~k'~~$n +$ ~~k''~~ for constants k' and k''

```
MULT(X,Y):
If |X| = |Y| = 1 then return XY
break X into a;b and Y into c;d
return
  MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Let's be concrete and keep it simple

$T(1) = 1$

$T(n) = 4\,T(n/2) + n$

(Notice that T(n) is inductively defined only for positive powers of 2.)

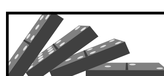
What is the growth rate of T(n)?

Technique 1: Guess and Verify

Guess: $G(n) = 2n^2 - n$

Verify: $G(1) = 1$ and $G(n) = 4\,G(n/2) + n$

$$
\begin{aligned}
& 4\,[2(n/2)^2 - n/2] + n \\
= \;& 2n^2 - 2n + n \\
= \;& 2n^2 - n \\
= \;& G(n)
\end{aligned}
$$



Technique 1: Guess and Verify

Guess: $G(n) = 2n^2 - n$

Verify: $G(1) = 1$ and $G(n) = 4\,G(n/2) + n$

Similarly: $T(1) = 1$ and $T(n) = 4\,T(n/2) + n$

So $T(n) = G(n) = \Theta(n^2)$

Technique 2: Guess Form and Calculate Coefficients

Guess: $T(n) = an^2 + bn + c$ for some a,b,c

Calculate: $T(1) = 1 \Rightarrow a + b + c = 1$

$$T(n) = 4\,T(n/2) + n$$

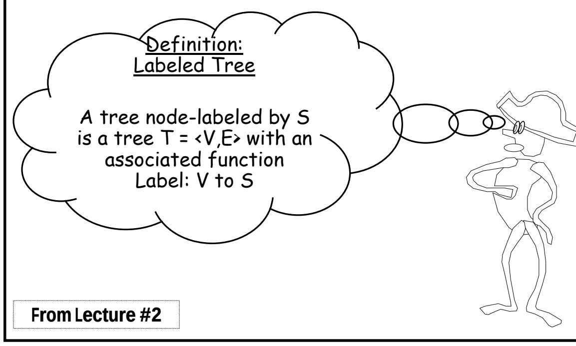
$\Rightarrow\ an^2 + bn + c = 4\,[a(n/2)^2 + b(n/2) + c] + n$

$= an^2 + 2bn + 4c + n$

$\Rightarrow\ (b+1)n + 3c = 0$

Therefore: $b=-1 \quad c=0 \quad a=2$

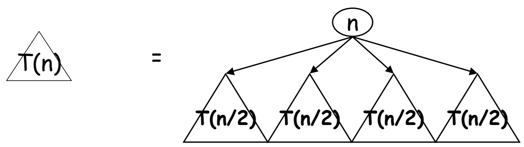
Technique 3: Labeled Tree Representation



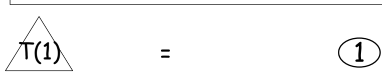
From Lecture #2

Technique 3: Labeled Tree Representation

T(n)	=	n + 4 T(n/2)

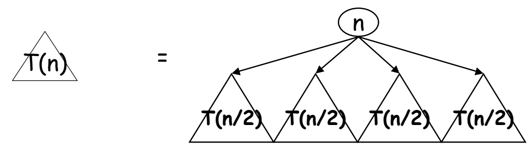


T(1)	=	1

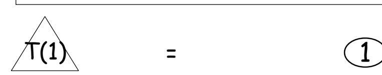


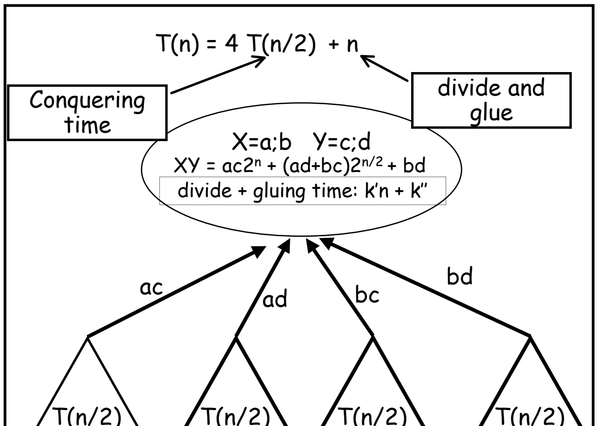
Node labels: time not spent conquering

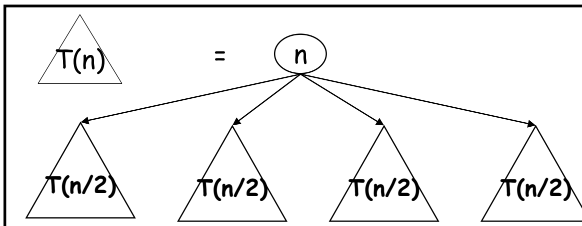
T(n)	=	n + 4 T(n/2)

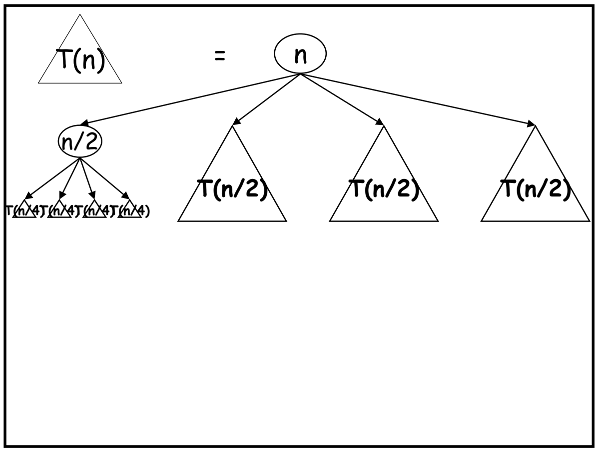


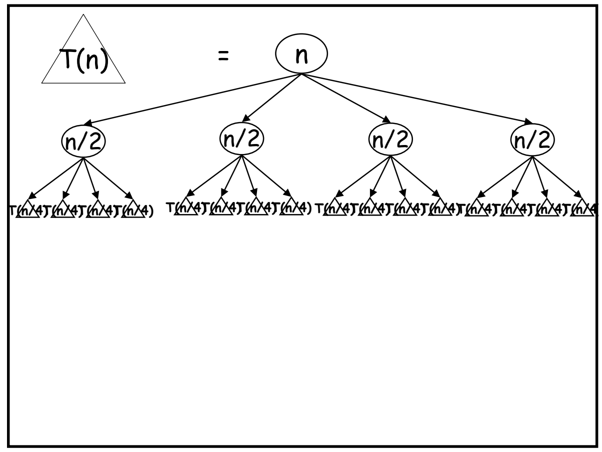
T(1)	=	1

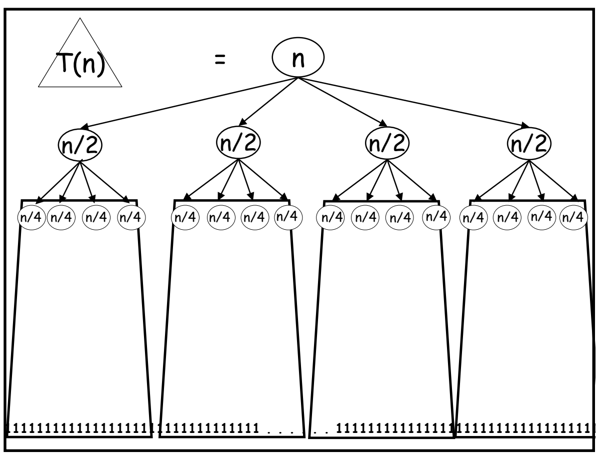


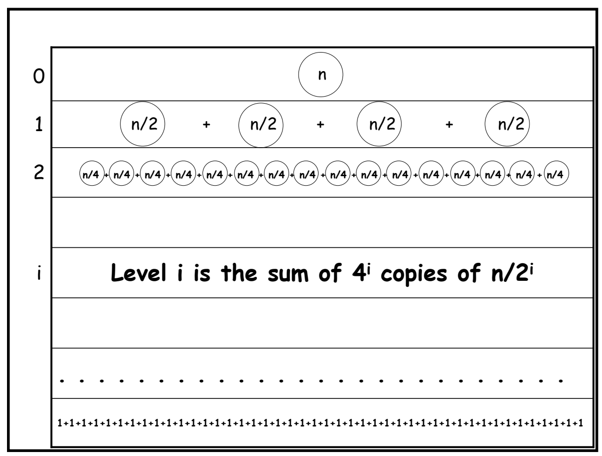


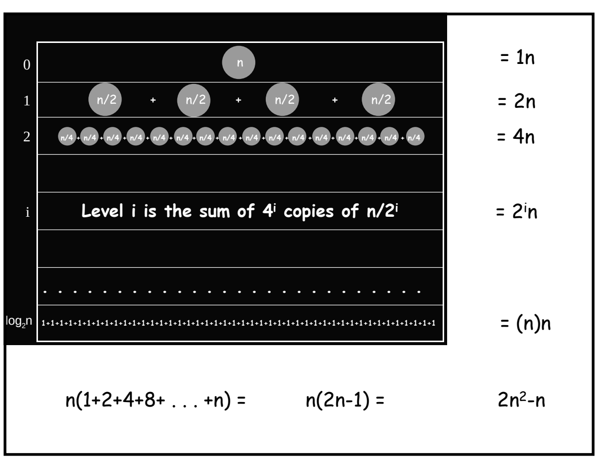




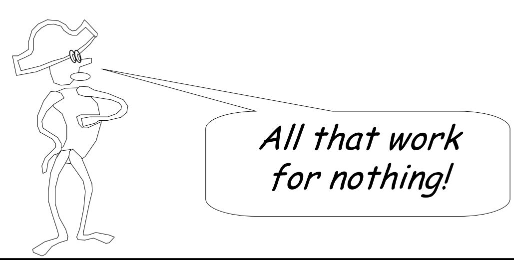




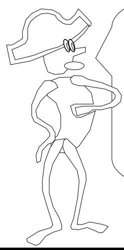




Divide and Conquer MULT: $\Theta(n^2)$ time
Grade School Multiplication: $\Theta(n^2)$ time



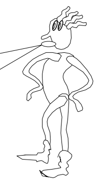
Divide and Conquer MULT: $\Theta(n^2)$ time
Grade School Multiplication: $\Theta(n^2)$ time



In retrospect, it is obvious that the kissing number for Divide and Conquer MULT is n^2, since the leaves are in correspondence with the kisses.

MULT revisited

```
MULT(X,Y):
If |X| = |Y| = 1 then return XY
break X into a;b and Y into c;d
return
  MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) 2^(n/2) + MULT(b,d)
```



MULT calls itself 4 times. Can you see a way to reduce the number of calls?

Gauss' optimization

Input: a,b,c,d
Output: ac-bd, ad+bc

(a+bi)(c+di) =
[ac –bd] + [ad + bc] i

$X_1 = a + b$
$X_2 = c + d$
$X_3 = X_1 X_2 \quad = ac + ad + bc + bd$
$X_4 = ac$
$X_5 = bd$
$X_6 = X_4 - X_5 \quad = ac - bd$
$X_7 = X_3 - X_4 - X_5 = bc + ad$

Karatsuba, Anatolii Alexeevich (1937-)

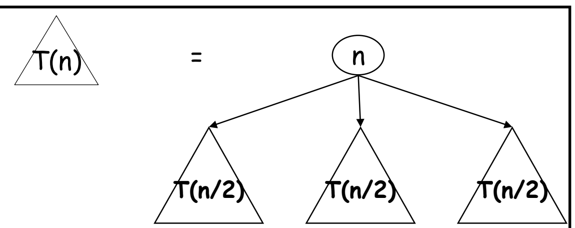
Sometime in the late 1950's Karatsuba had formulated the first algorithm to break the kissing barrier!

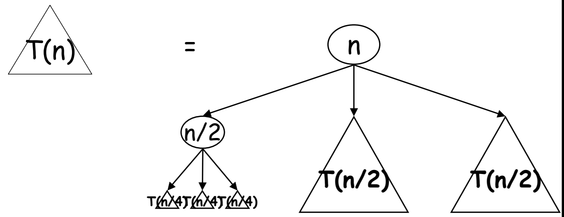
Gaussified MULT (Karatsuba 1962)

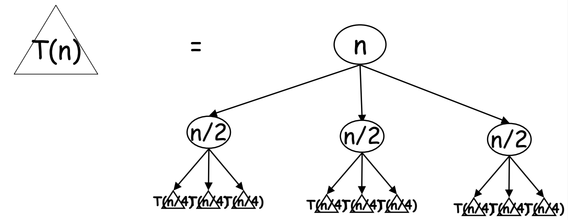
```
MULT(X,Y):
If |X| = |Y| = 1 then return XY
break X into a;b and Y into c;d
e = MULT(a,c)  and f = MULT(b,d)
return
  e 2^n + (MULT(a+c,b+d) – e - f) 2^(n/2) + f
```

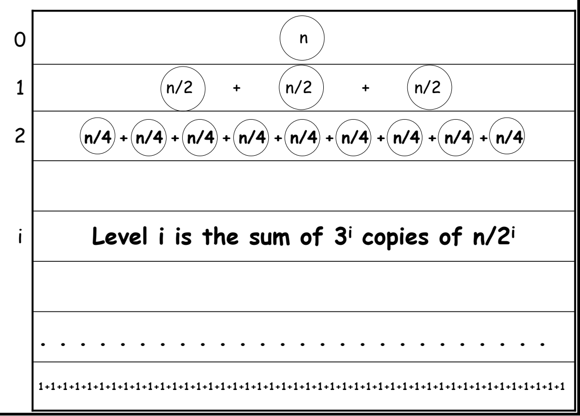
$T(n) = 3\,T(n/2) + n$

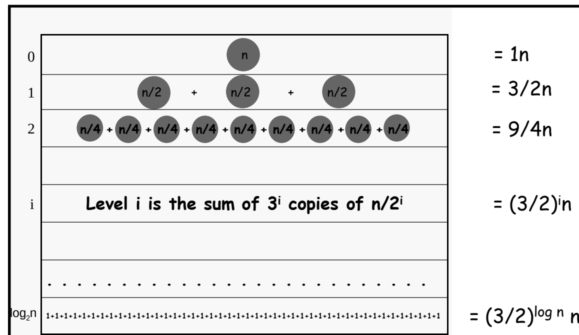
Actually: $T(n) = 2\,T(n/2) + T(n/2 + 1) + kn$





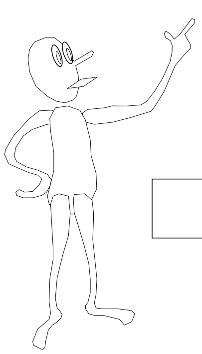






$$n(1+3/2+(3/2)^2+ \ldots + (3/2)^{\log_2 n}) \quad = \quad 3n^{1.58\ldots} - 2n$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

Substituting into our formula….

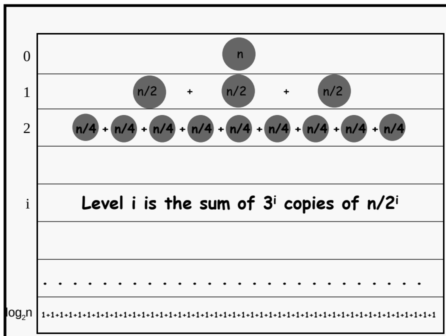
$$1 + X^1 + X^2 + X^3 + \ldots + X^{k-1} + X^k = \frac{X^{k+1} - 1}{X - 1}$$

We have: $X = 3/2$ $\quad$ $k = \log_2 n$

$$\frac{(3/2) \times (3/2)^{\log_2 n} - 1}{\frac{1}{2}} = 3 \times (3^{\log_2 n}/2^{\log_2 n}) - 2$$

$$= 3 \times (3^{\log_2 n}/n) - 2$$

$$= \frac{3\, n^{(\log_2 3)}}{n} - 2$$



= 1n

= 3/2n

= 9/4n

$= (3/2)^i n$

$= (3/2)^{\log n}\, n$

$$n\left(\frac{3n^{\log_2 3}}{n} - 2\right) = 3n^{\log_2 3} - 2n$$

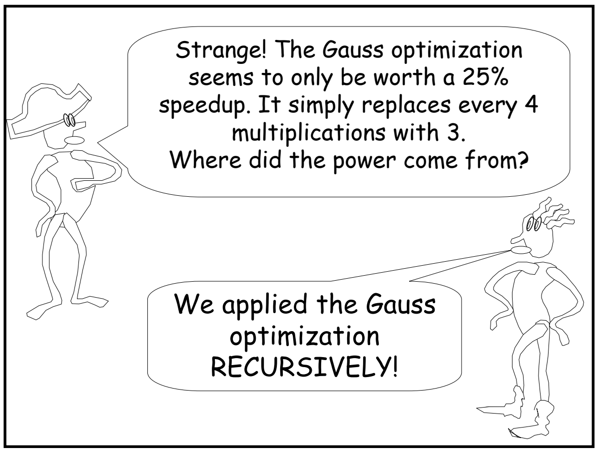
Dramatic improvement for large n

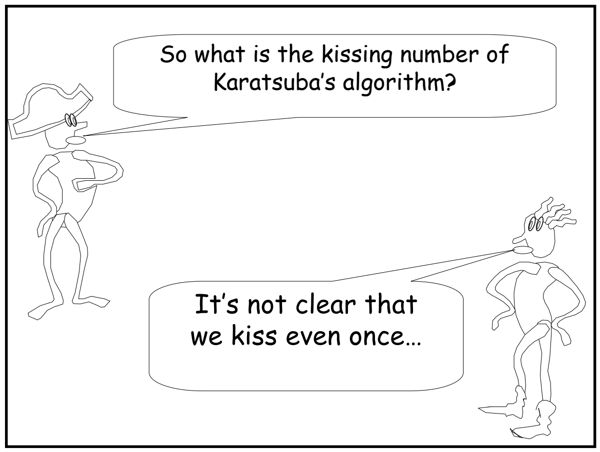
$T(n) = 3n^{\log_2 3} - 2n$

$= \Theta(n^{\log_2 3})$

$= \Theta(n^{1.58\ldots})$

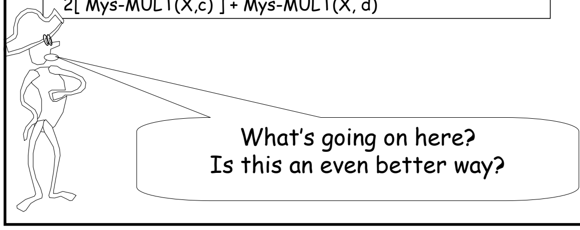
A huge savings over $\Theta(n^2)$ when n gets large.

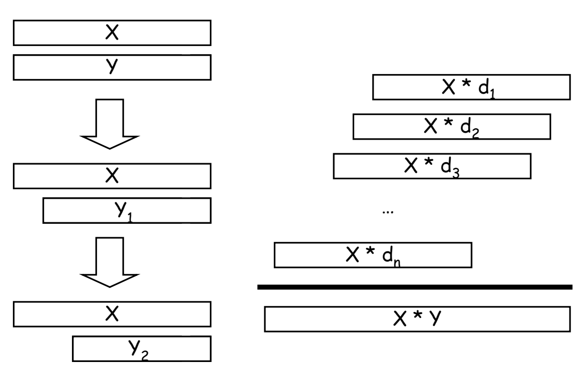


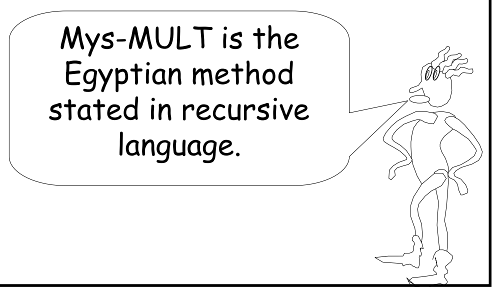


Mystery MULT

```
Mys-MULT(X,Y):
 If |Y| = 1 then return X × Y
 break Y into c;d where |d| = 1
 return
   2[ Mys-MULT(X,c) ] + Mys-MULT(X, d)
```







Multiplication Algorithms

Kindergarten	$n2^n$
Grade School	n^2
Karatsuba	$n^{1.58...}$
Fastest Known	n logn loglogn

REFERENCES

Karatsuba, A., and Ofman, Y. *Multiplication of multidigit numbers on automata*. Sov. Phys. Dokl. 7 (1962), 595--596.