Great Theoretical Ideas In Computer Science

Anupam Gupta

CS 15-251 Spring 2005

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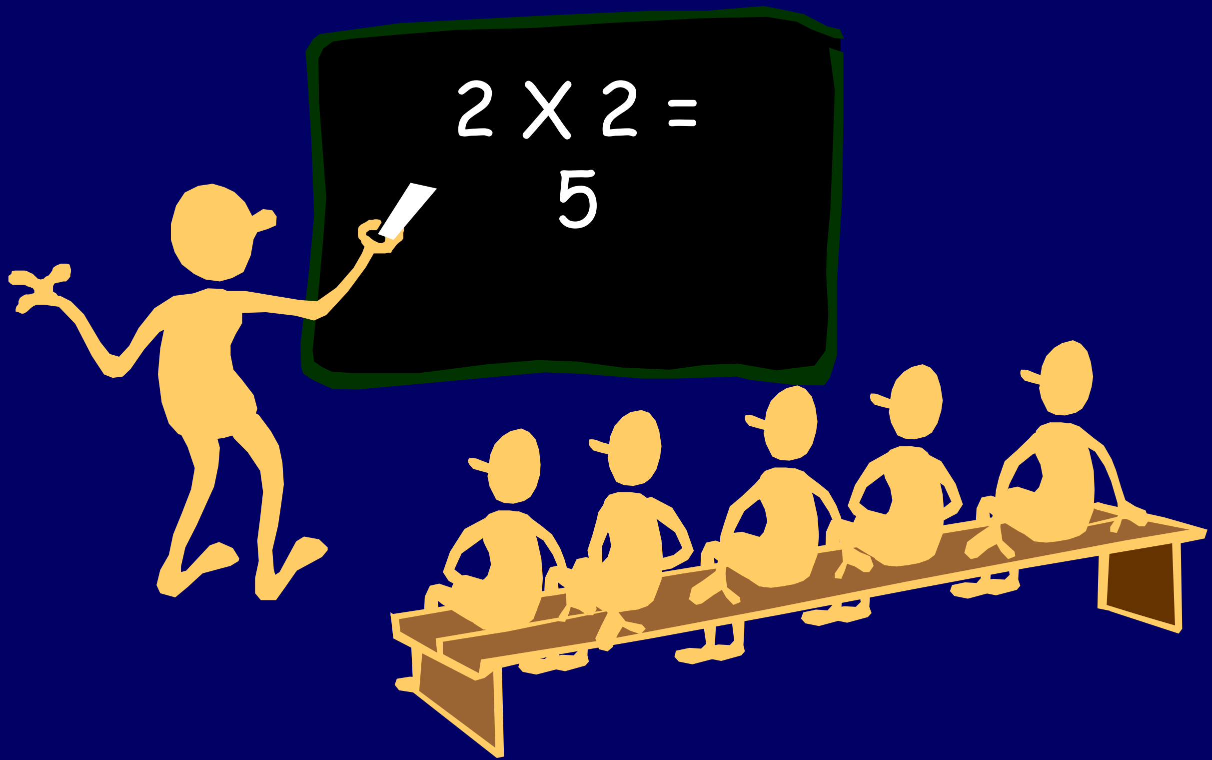
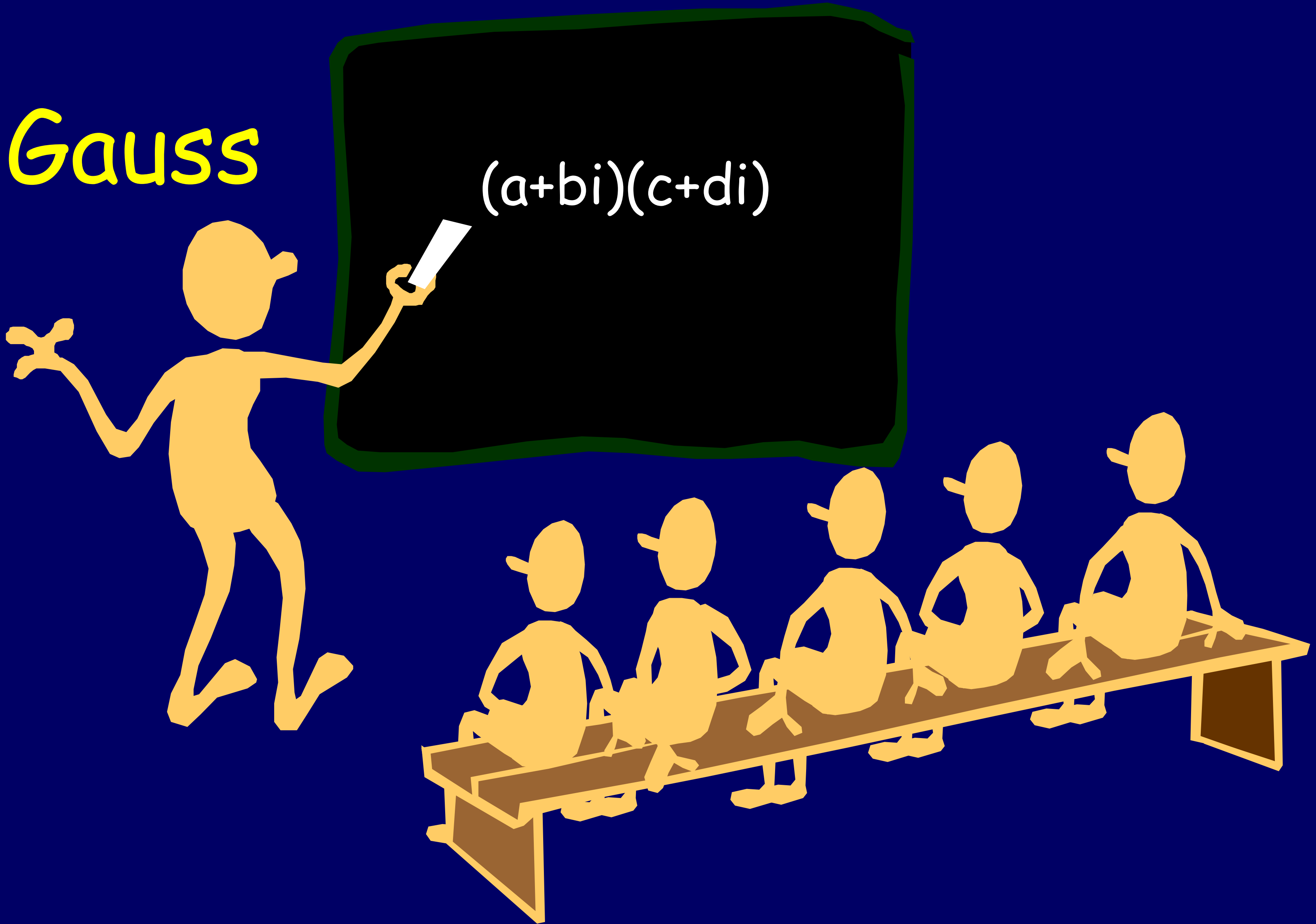
Carnegie Mellon University

Grade School Revisited: How To Multiply Two Numbers




The best way is often far from obvious!

Gauss



Gauss' Complex Puzzle

Remember how to multiply two
complex numbers a + bi and c + di?

(a+bi)(c+di) = [ac –bd] + [ad + bc] i

Input: a,b,c,d
Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

Can you do better than $4.02?

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

(a+bi)(c+di) =
[ac –bd] + [ad + bc] i

c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
$	$X_4 = ac$	
$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	$= ac - bd$
cc	$X_7 = X_3 - X_4 - X_5$	$= bc + ad$

The Gauss optimization saves one multiplication out of four.

It requires 25% less work.

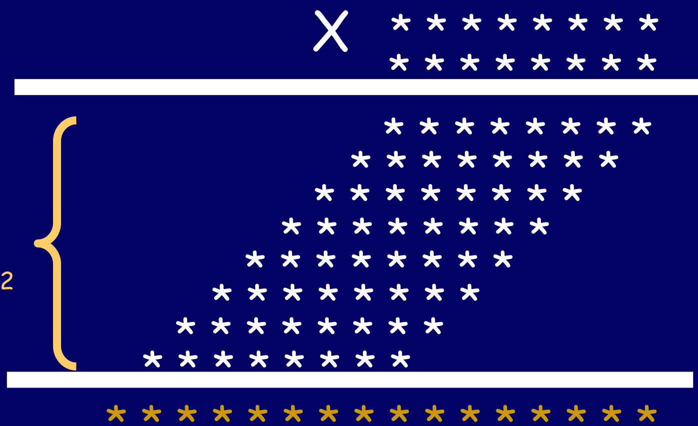
Time complexity of grade school addition

T(n) = The amount of time grade school addition uses to add two n-bit numbers

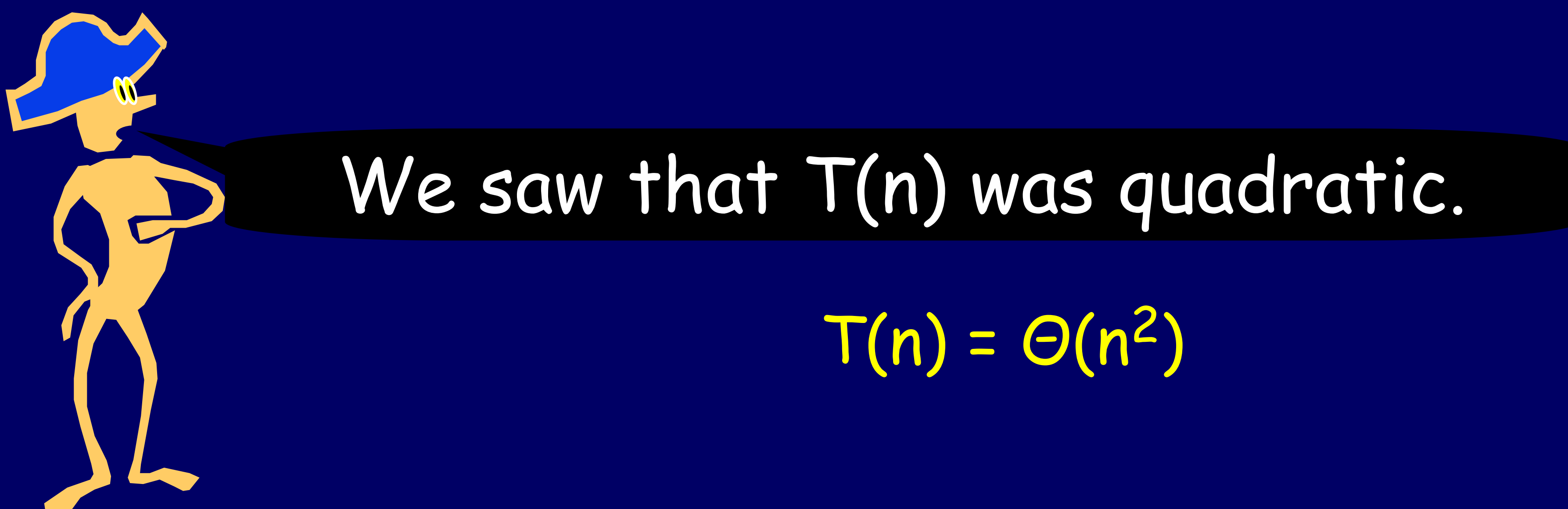
We saw that T(n) was linear.

$T(n) = \Theta(n)$

Time complexity of grade school multiplication

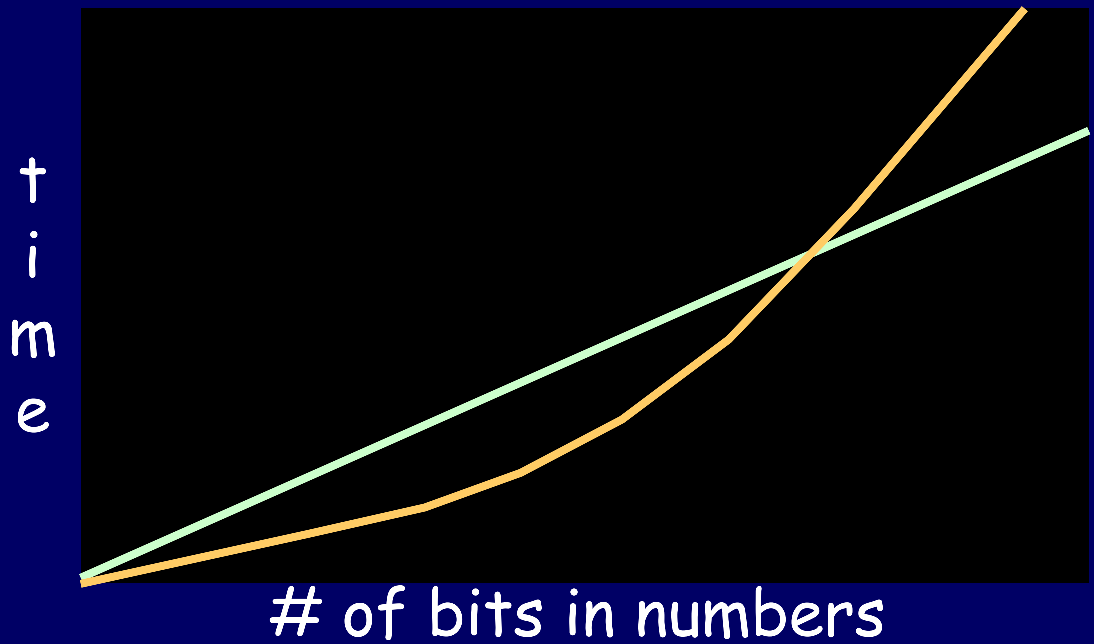


T(n) = The amount of time grade school multiplication uses to add two n-bit numbers



$T(n) = \Theta(n^2)$

Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



No matter how dramatic the difference in the constants the quadratic curve will eventually dominate the linear curve

Grade school addition is linear time.

Is there a sub-linear time method for addition?

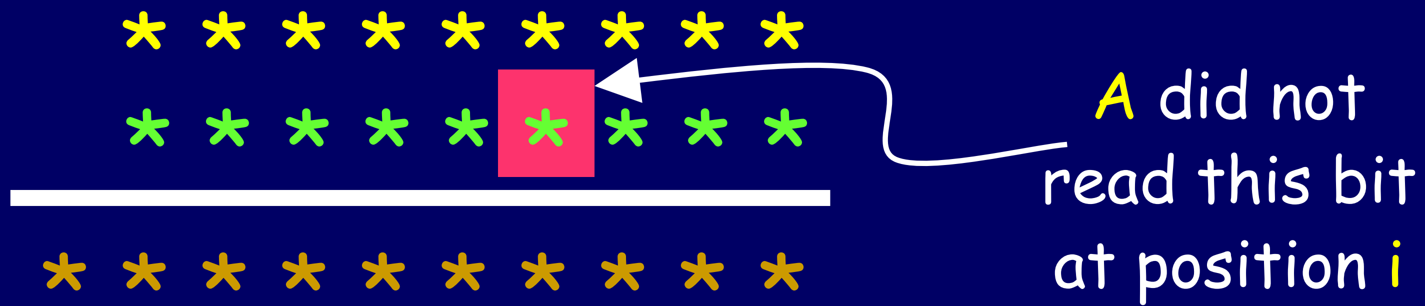
Any addition algorithm takes $\Omega(n)$ time

Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm *A* that does not examine each bit

Give *A* a pair of numbers. There must be some unexamined bit position *i* in one of the numbers

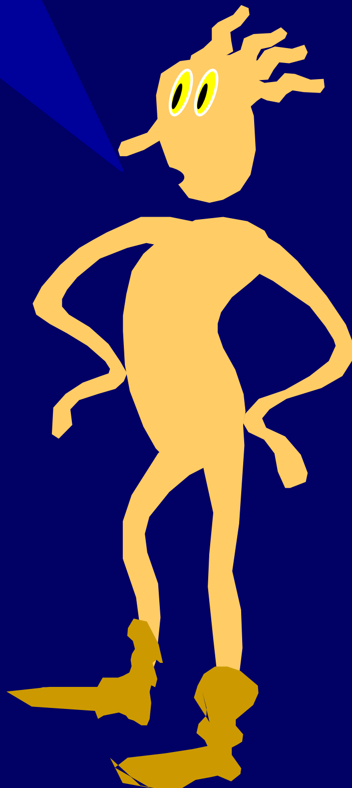
Any addition algorithm takes $\Omega(n)$ time



- If A is not correct on the inputs, we found a bug
- If A is correct, flip the bit at position i and give A the new pair of numbers. A gives the same answer as before, which is now wrong.

So any algorithm for addition must use time at least linear in the size of the numbers.

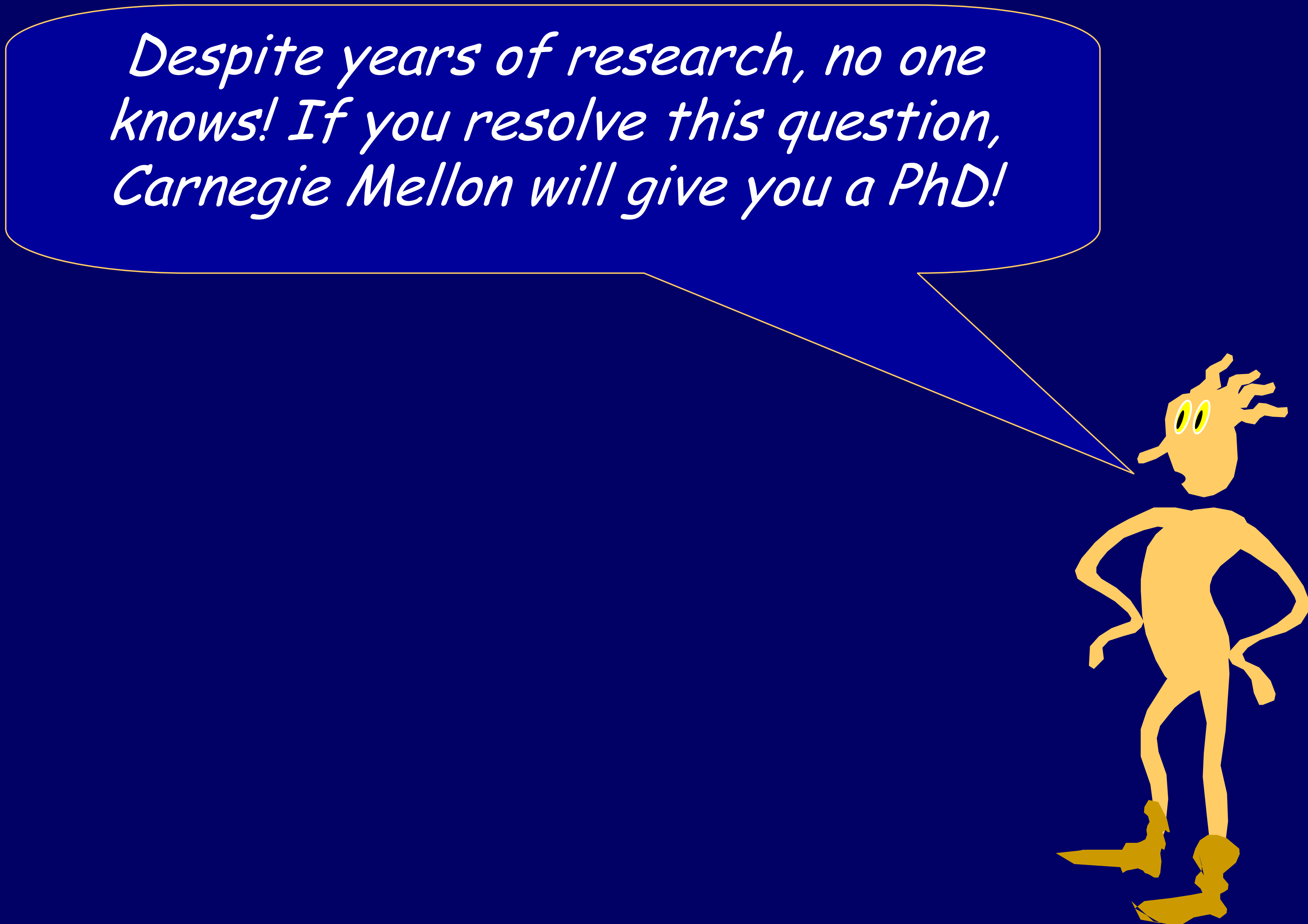
Grade school addition can't be improved upon by more than a constant factor.

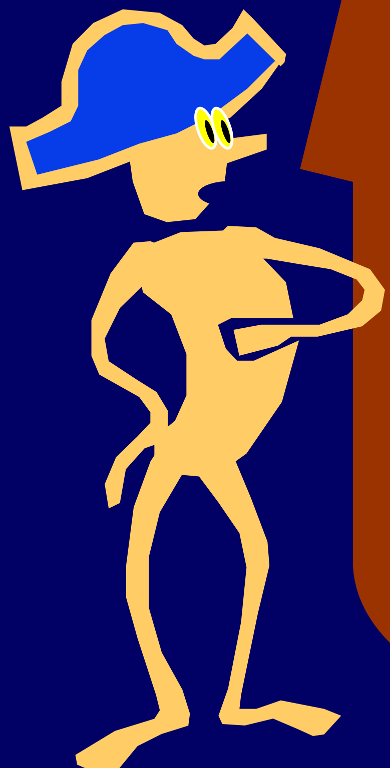


Grade School Addition: $\Theta(n)$ time
Furthermore, it is optimal

Grade School Multiplication: $\Theta(n^2)$ time

Is there a clever algorithm to multiply two numbers in <u>linear</u> time?

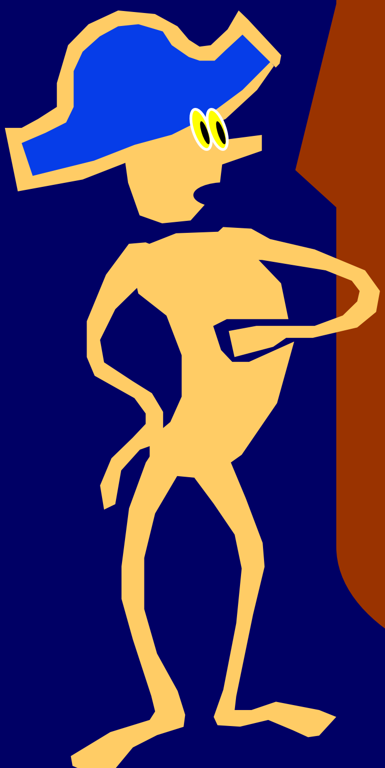

Despite years of research, no one knows! If you resolve this question, Carnegie Mellon will give you a PhD!



Can we even break the quadratic time barrier?

In other words, can we do something very different than grade school multiplication?

Grade School Multiplication: The Kissing Intuition



Intuition:
Let's say that each time an algorithm has to multiply a digit from one number with a digit from the other number, we call that a "kiss".

It seems as if any correct algorithm must kiss at least n^2 times.

Divide And Conquer

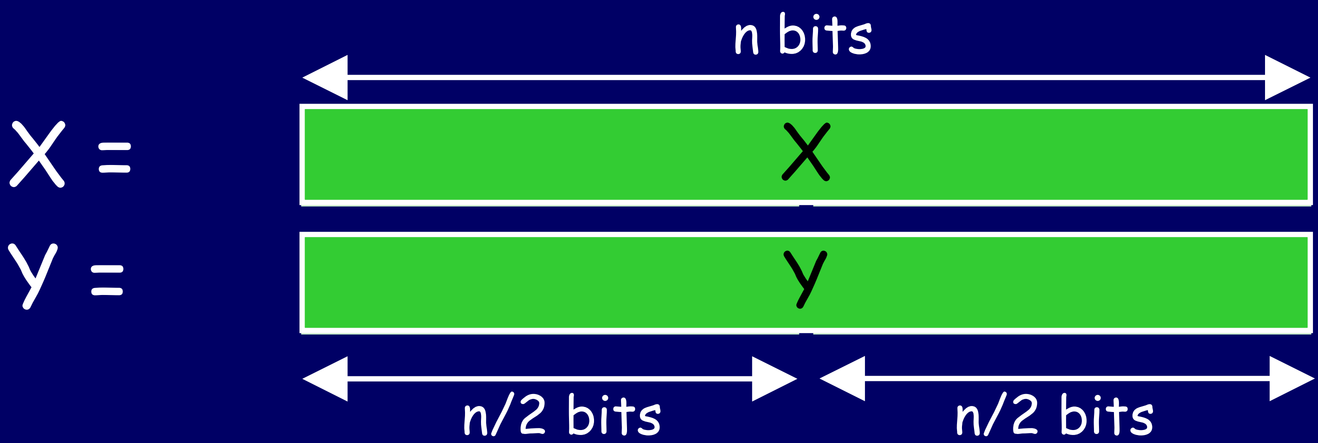
An approach to faster algorithms:

1. DIVIDE a problem into smaller subproblems

2. CONQUER them recursively

3. GLUE the answers together so as to obtain the answer to the larger problem

Multiplication of 2 n-bit numbers



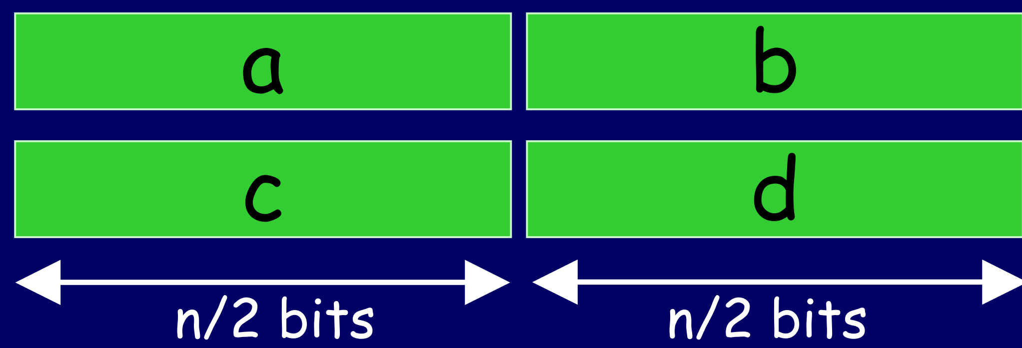
$$X = a\,2^{n/2} + b \qquad Y = c\,2^{n/2} + d$$

$$X \times Y = ac\,2^{n} + (ad + bc)\,2^{n/2} + bd$$

Multiplication of 2 n-bit numbers

X =

Y =



$$X \times Y = ac\ 2^n + (ad + bc)\ 2^{n/2} + bd$$

<u>MULT(X,Y):</u>

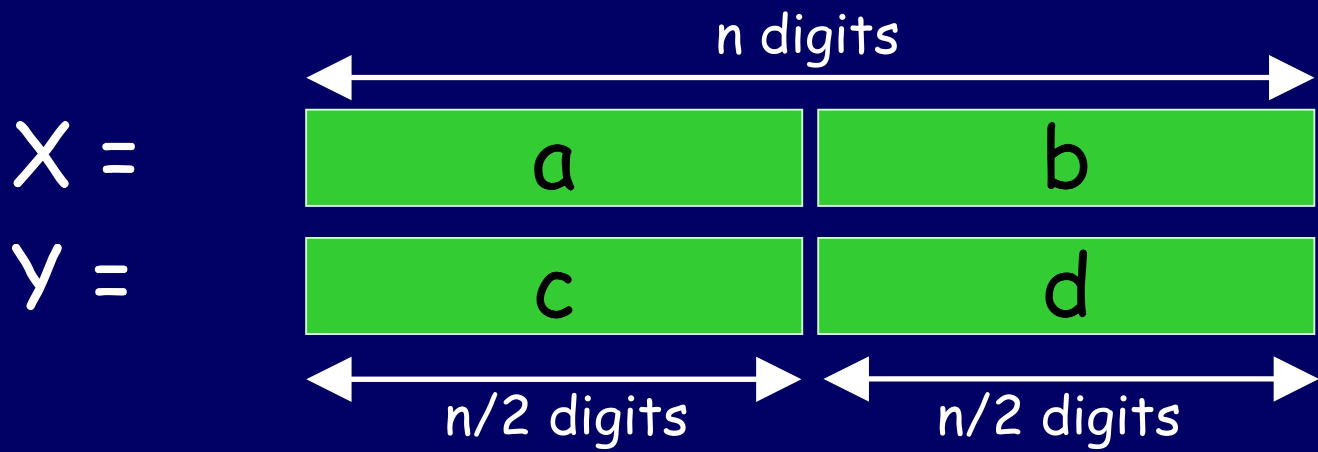
If |X| = |Y| = 1 then return XY

break X into a;b and Y into c;d

return

MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) $2^{n/2}$ + MULT(b,d)

Same thing for numbers in decimal!



$$X = a\,10^{n/2} + b \qquad Y = c\,10^{n/2} + d$$

$$X \times Y = ac\,10^{n} + (ad + bc)\,10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

12*21

X = a | b

Y = c | d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39

X = a b

Y = c d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

12*21   12*39   34*21

X = a b

Y = c d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

X = a b

Y = c d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

12*21   12*39   34*21   34*39

1*2   1*1   2*2   2*1

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

2 1 4 2

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

252 12*39 34*21 34*39

2 1 4 2

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

252   12*39   34*21   34*39

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

252   12*39   34*21   34*39

1*3   1*9   2*3   2*9

X =	a	b
Y =	c	d

$$X \times Y = ac\,10^{n} + (ad + bc)\,10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

242 12*39 34*21 34*39

3 9 6 18

$*10^2 + *10^1 + *10^1 + *1$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

242   468   34*21   34*39

3   9   6   18

$*10^2 + *10^1 + *10^1 + *1$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

252   468   714   1326

X = | a | b |

Y = | c | d |

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

252   468   714   1326

$*10^4 + *10^2 + *10^2 + *1$

X = | a | b |

Y = | c | d |

$$X \times Y = ac\,10^n + (ad + bc)\,10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

252 $*10^4$ + 468 $*10^2$ + 714 $*10^2$ + 1326 $*1$

= 2639526

X = | a | b |

Y = | c | d |

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 1234*4276 5678*2139 5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 5276584 12145242 24279128

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526	5276584	12145242	24279128
$*10^8$ +	$*10^4$ +	$*10^4$ +	$*1$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 $*10^8$ + 5276584 $*10^4$ + 12145242 $*10^4$ + 24279128 $*1$

= 264126842539128

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

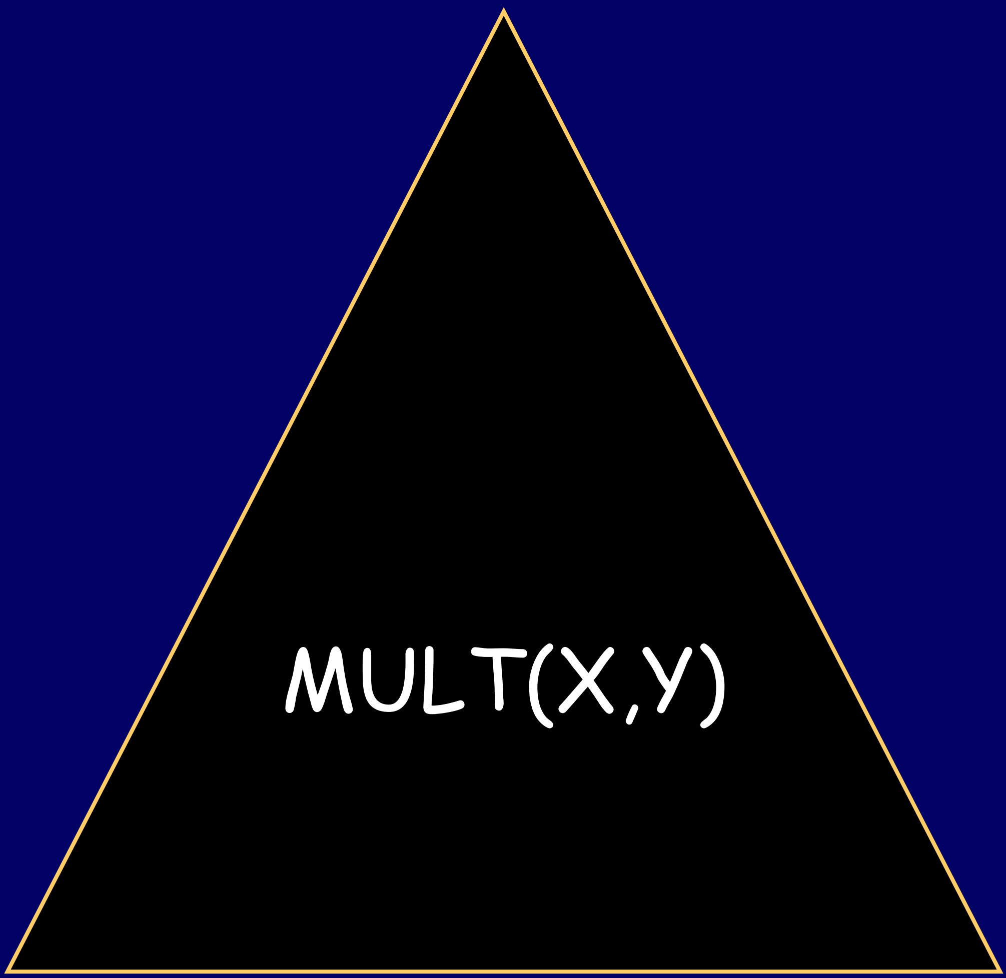
Multiplying (Divide & Conquer style)

264126842539128

X =	a	b
Y =	c	d

$$X \times Y = ac\,10^n + (ad + bc)\,10^{n/2} + bd$$

Divide, Conquer, and Glue



Divide, Conquer, and Glue

MULT(X,Y):

if $|X| = |Y| = 1$
then return XY,
else…

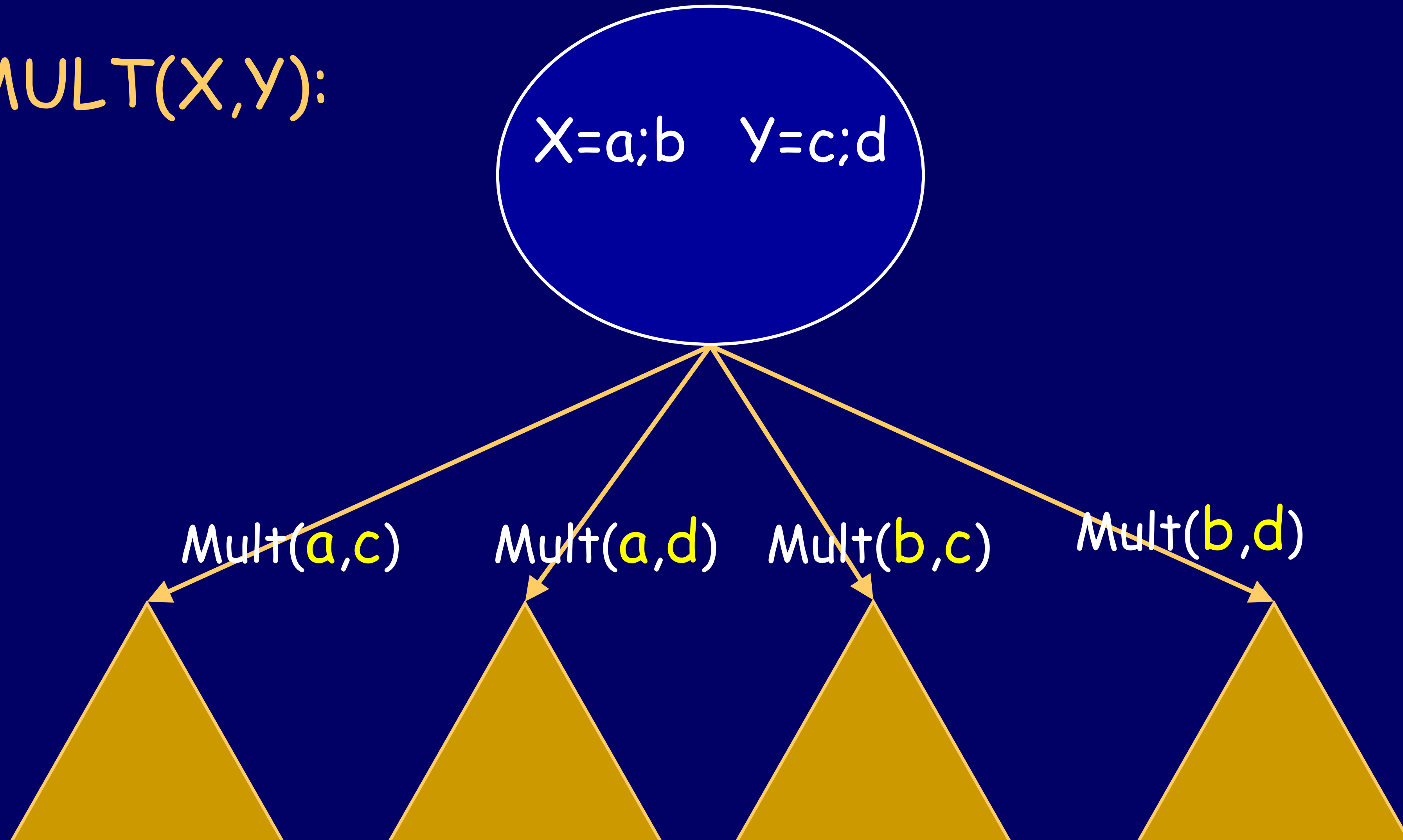
Divide, Conquer, and Glue

MULT(X,Y):

X=a;b   Y=c;d

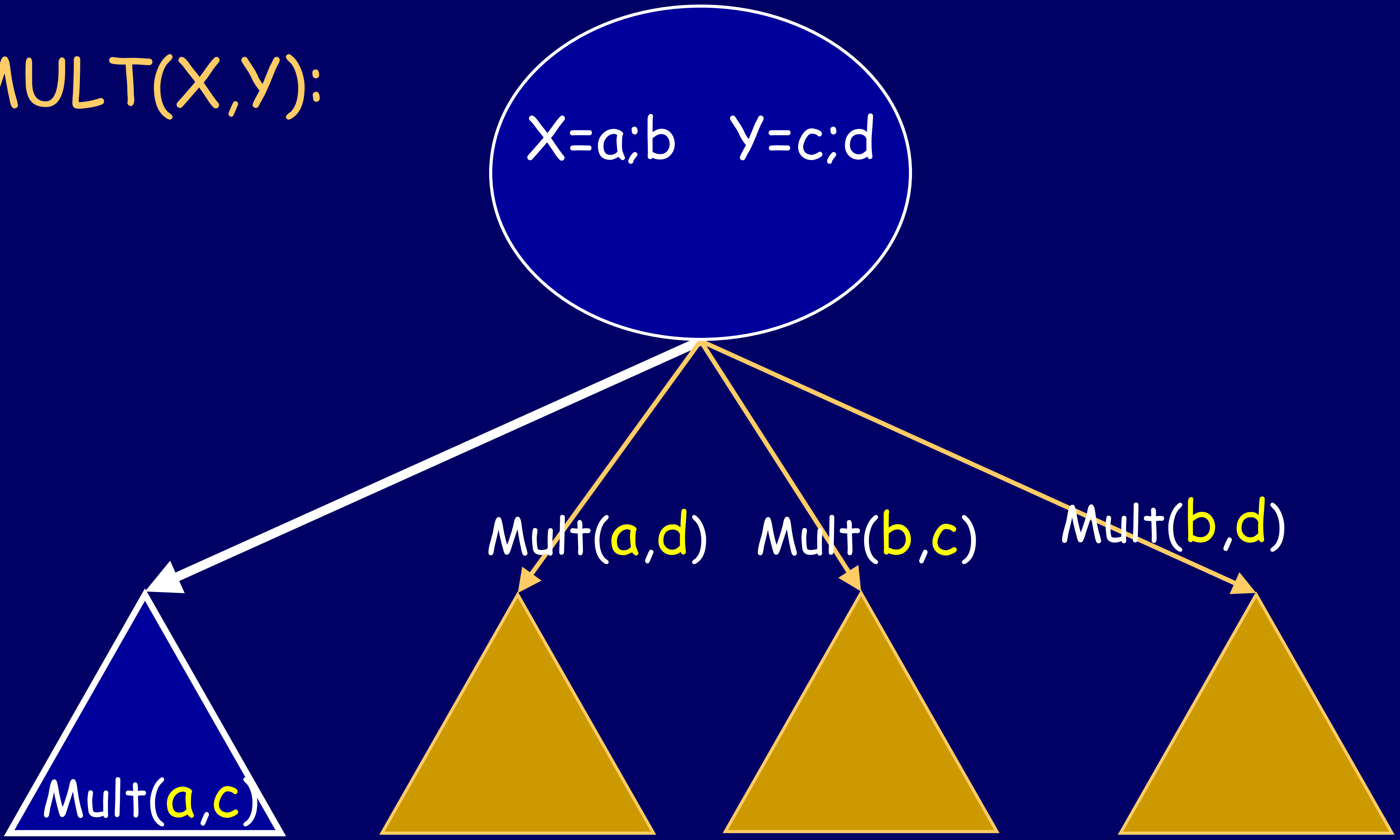
Divide, Conquer, and Glue

MULT(X,Y):



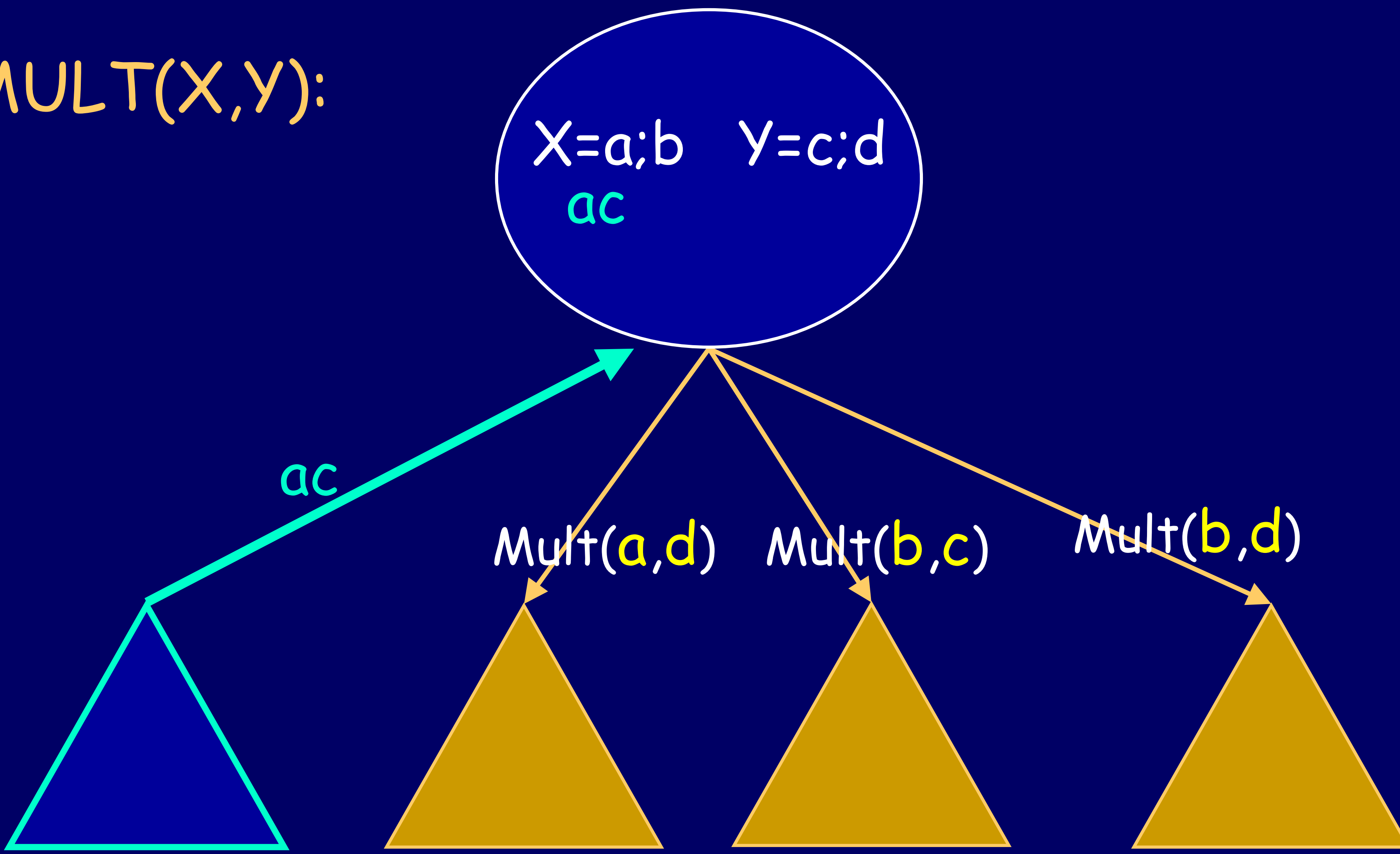
Divide, Conquer, and Glue

MULT(X,Y):



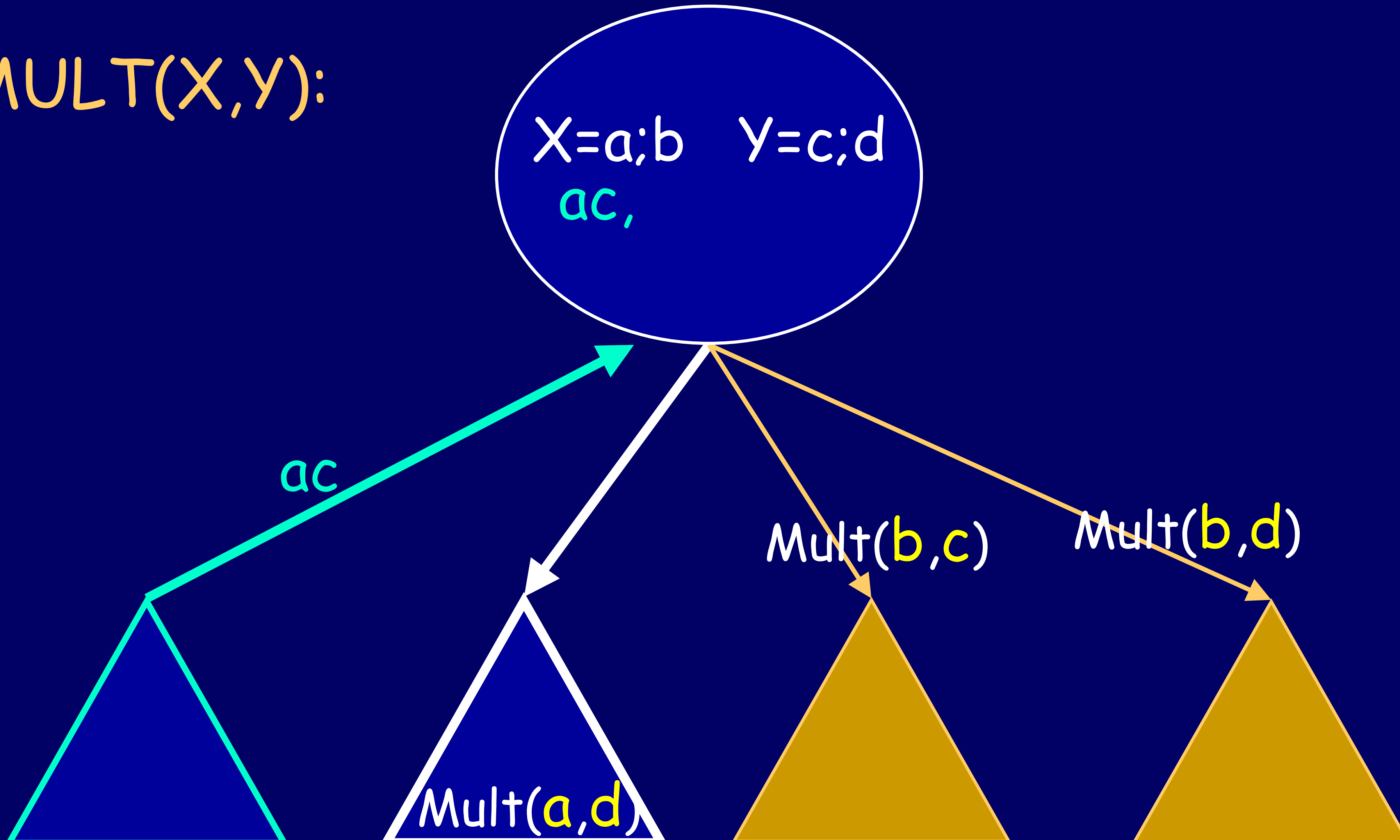
Divide, Conquer, and Glue

MULT(X,Y):



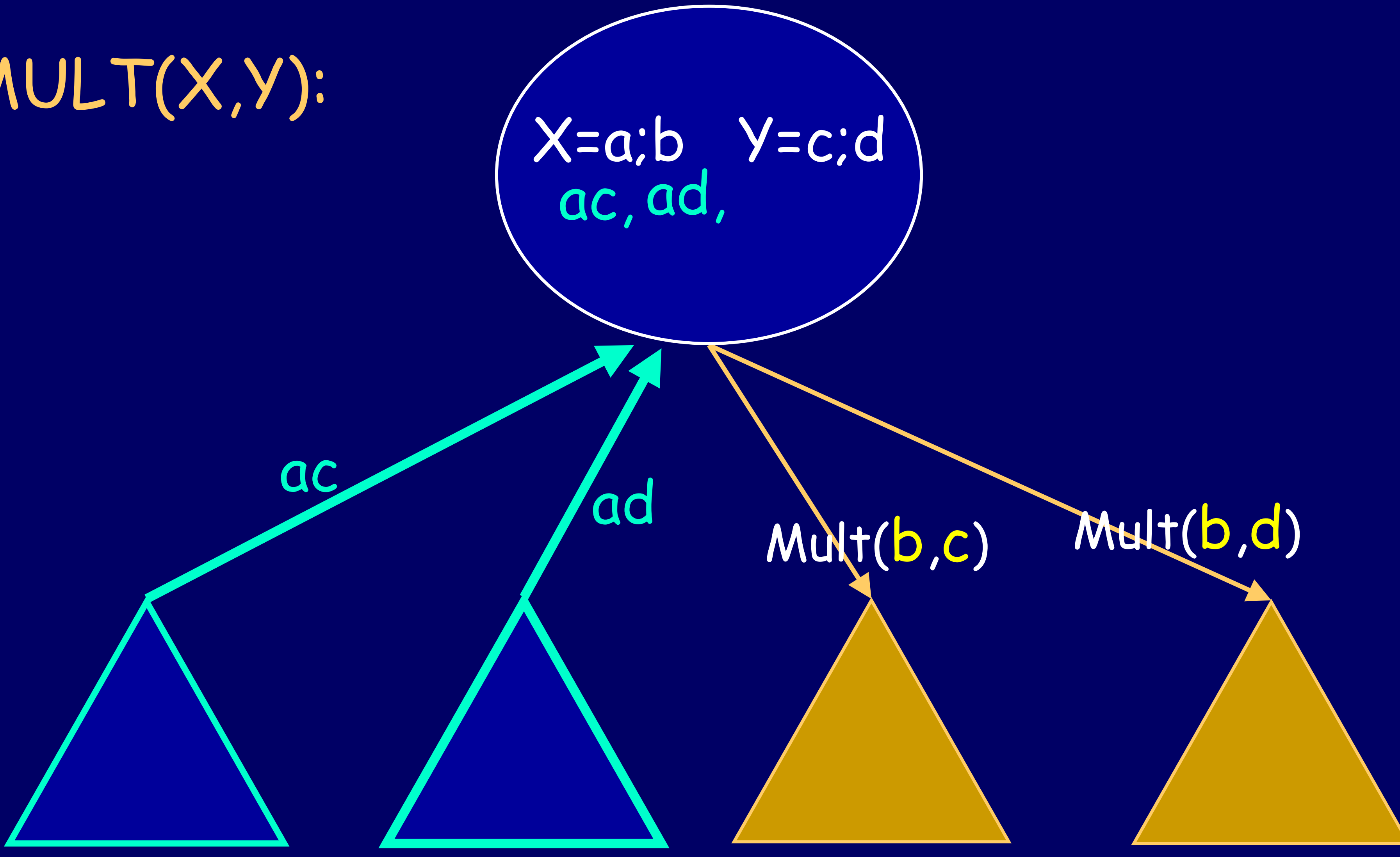
Divide, Conquer, and Glue

MULT(X,Y):



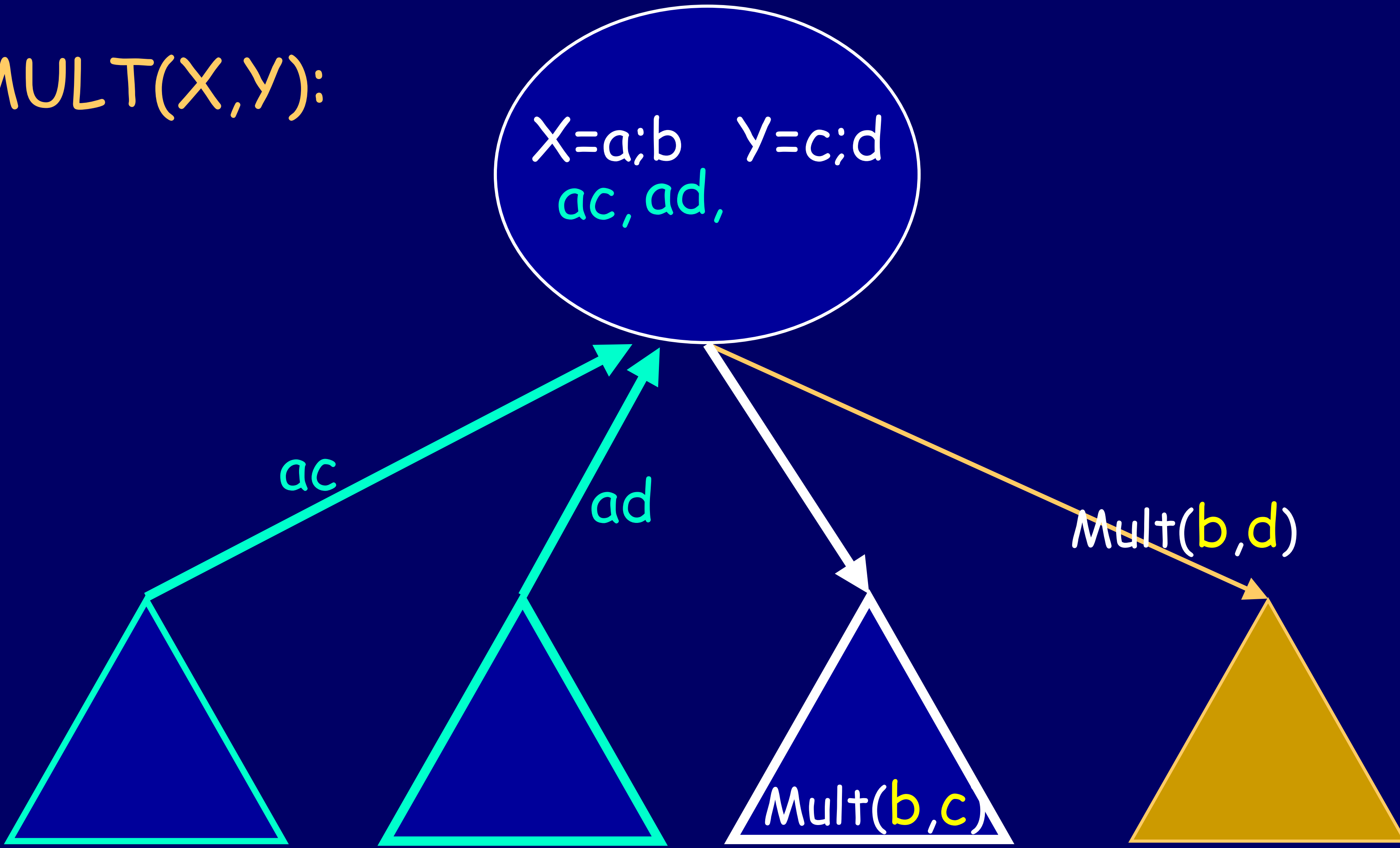
Divide, Conquer, and Glue

MULT(X,Y):



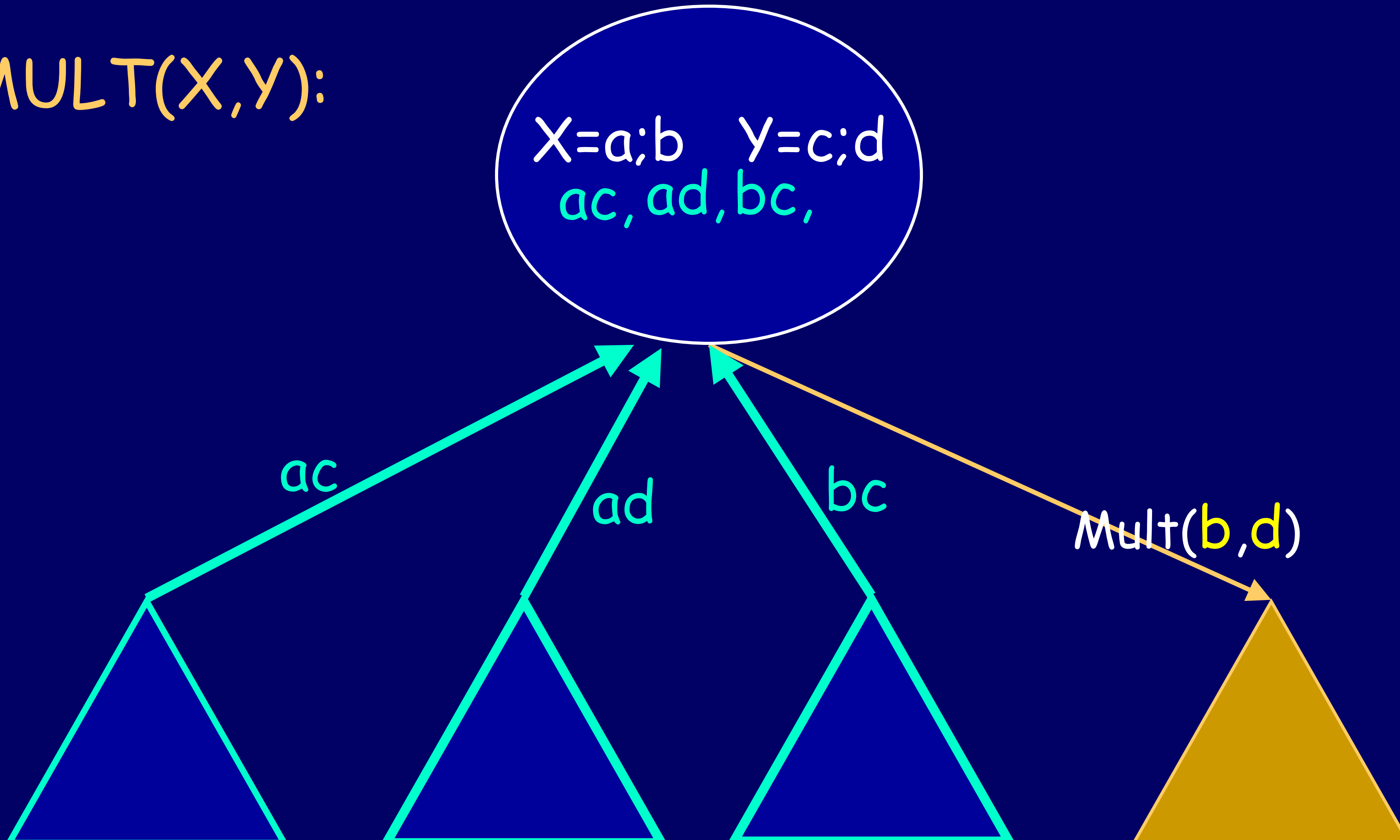
Divide, Conquer, and Glue

MULT(X,Y):



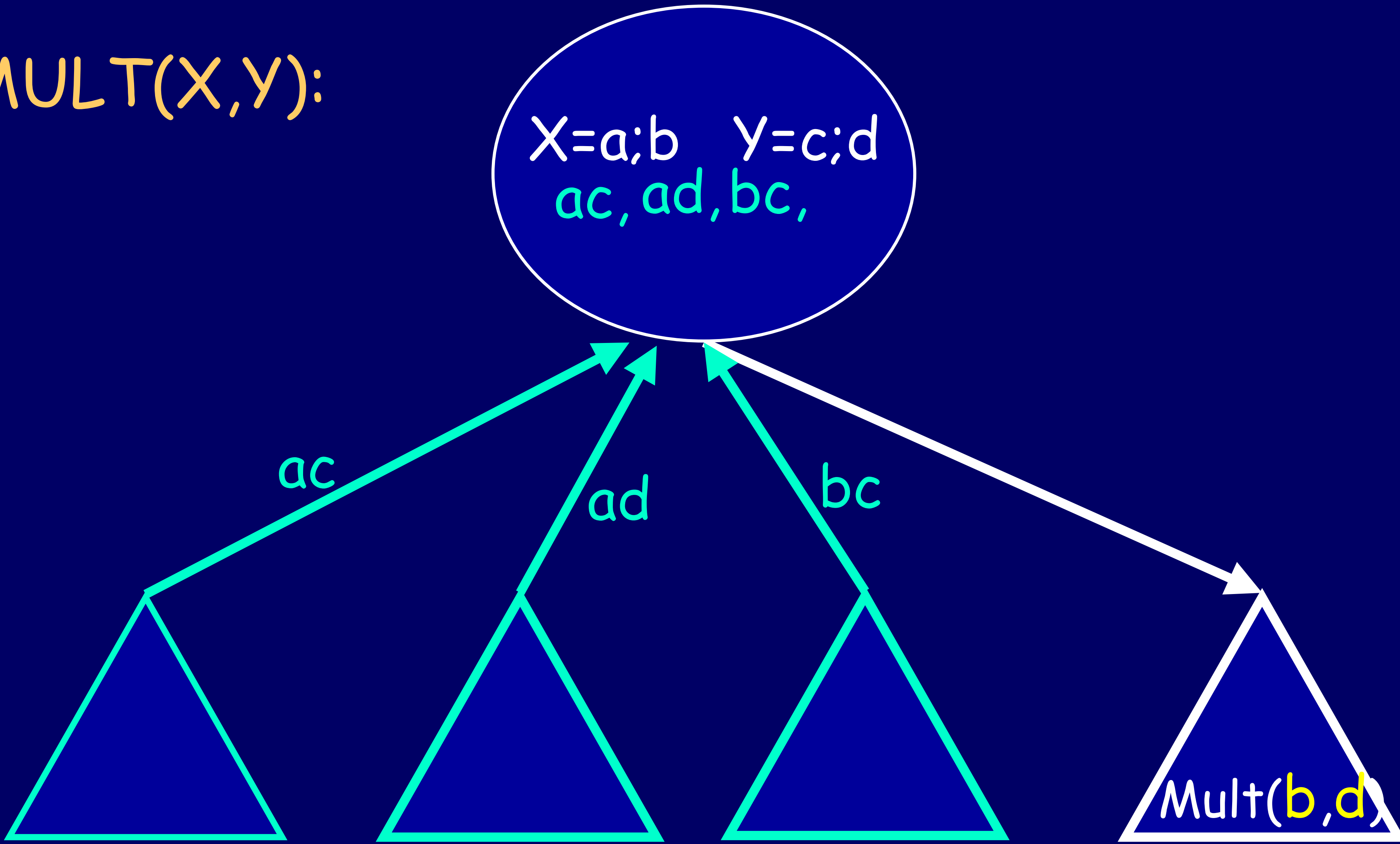
Divide, Conquer, and Glue

MULT(X,Y):



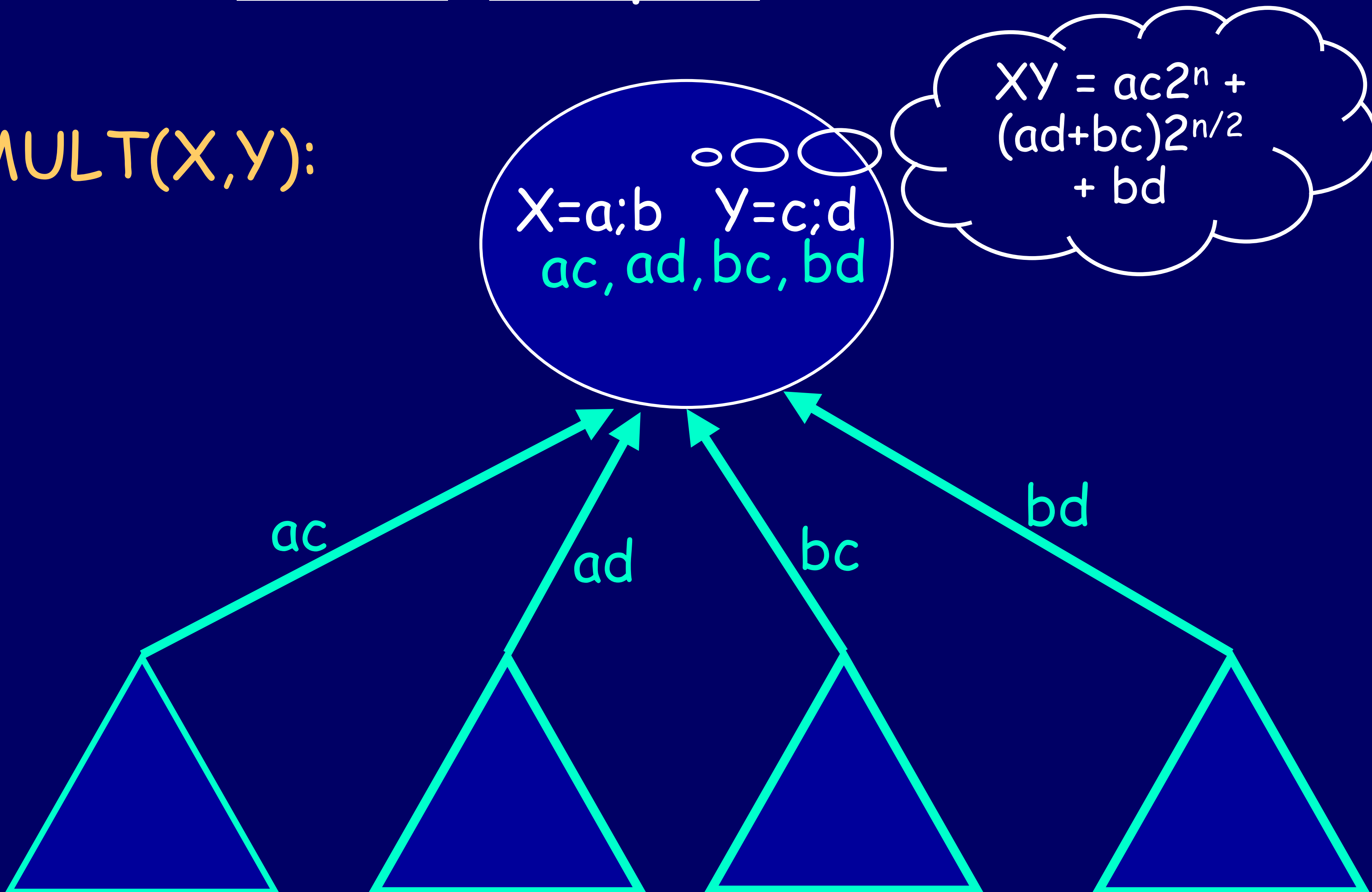
Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

MULT(X,Y):

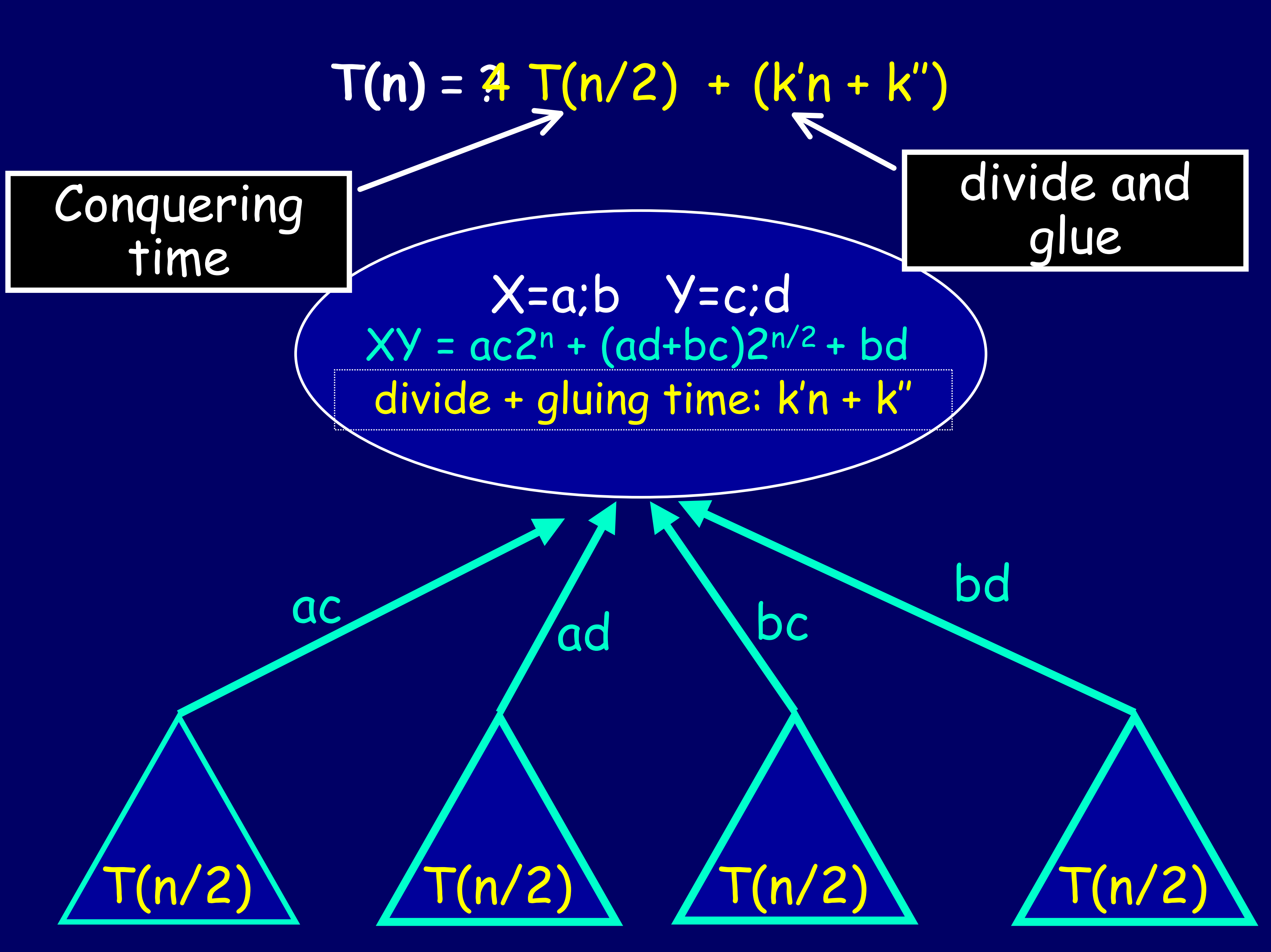


Time required by MULT

$T(n)$ = time taken by MULT on two n-bit numbers

What is $T(n)$? What is its growth rate?

Big Question: Is it $\Theta(n^2)$?


T(n) = ?4 T(n/2) + (k'n + k'')
Conquering time
divide and glue
X=a;b Y=c;d
XY = ac2^n + (ad+bc)2^{n/2} + bd
divide + gluing time: k'n + k''
ac
ad
bc
bd
T(n/2)
T(n/2)
T(n/2)
T(n/2)

Recurrence Relation

$T(1) = k$ for some constant k

$T(n) = 4\,T(n/2) + k'n + k''$ for constants k' and k''

<u>MULT(X,Y):</u>

If $|X| = |Y| = 1$ then return XY

break X into a;b and Y into c;d

return

MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) $2^{n/2}$ + MULT(b,d)

Let's be concrete and keep it simple

$T(1) =$ ~~k~~ 1 for some constant k

$T(n) = 4\,T(n/2) +$ ~~k'~~$n +$ ~~k''~~ for constants k' and k''

<u>MULT(X,Y):</u>

If $|X| = |Y| = 1$ then return XY

break X into a;b and Y into c;d

return

MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) $2^{n/2}$ + MULT(b,d)

Let's be concrete and keep it simple

$T(1) = 1$

$T(n) = 4\ T(n/2) + \ n$

(Notice that T(n) is inductively defined only for positive powers of 2.)

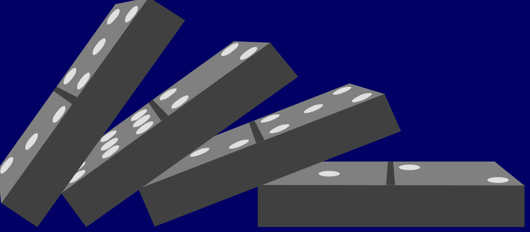
What is the growth rate of T(n)?

Technique 1: Guess and Verify

Guess: $G(n) = 2n^2 - n$

Verify: $G(1) = 1$ and $G(n) = 4\, G(n/2) + n$

$$
\begin{aligned}
& 4\,[2(n/2)^2 - n/2] + n \\
= \quad & 2n^2 - 2n + n \\
= \quad & 2n^2 - n \\
= \quad & G(n)
\end{aligned}
$$



Technique 1: Guess and Verify

Guess: $G(n) = 2n^2 - n$

Verify: $G(1) = 1$ and $G(n) = 4\,G(n/2) + n$

Similarly: $T(1) = 1$ and $T(n) = 4\,T(n/2) + n$

So $T(n) = G(n) = \Theta(n^2)$

Technique 2:
Guess Form and Calculate Coefficients

Guess: $T(n) = an^2 + bn + c$ for some a,b,c

Calculate: $T(1) = 1 \Rightarrow a + b + c = 1$

$$T(n) = 4\,T(n/2) + n$$

$$\Rightarrow \quad an^2 + bn + c = 4\,[a(n/2)^2 + b(n/2) + c] + n$$

$$= an^2 + 2bn + 4c + n$$

$$\Rightarrow \quad (b+1)n + 3c = 0$$

Therefore: $b=-1 \qquad c=0 \qquad a=2$

Technique 3: Labeled Tree Representation

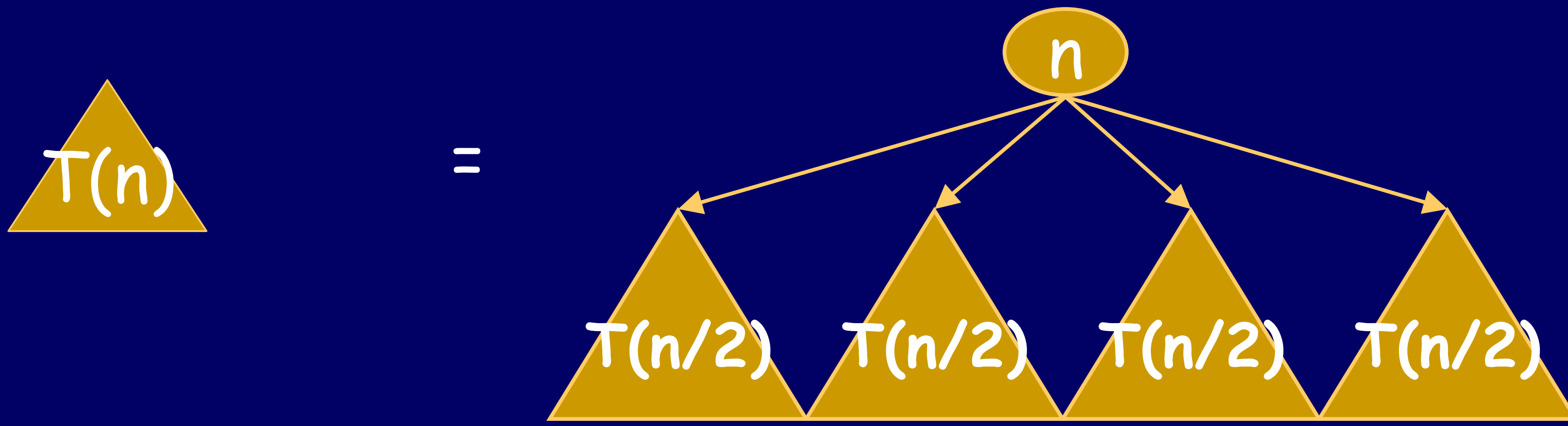
Definition:
Labeled Tree

A tree node-labeled by S is a tree T = <V,E> with an associated function Label: V to S

From Lecture #2

Technique 3: Labeled Tree Representation

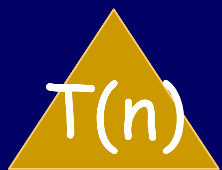
$T(n) = n + 4\,T(n/2)$



$T(1) = 1$

Node labels: time <u>not</u> spent conquering

$$T(n) = n + 4\,T(n/2)$$



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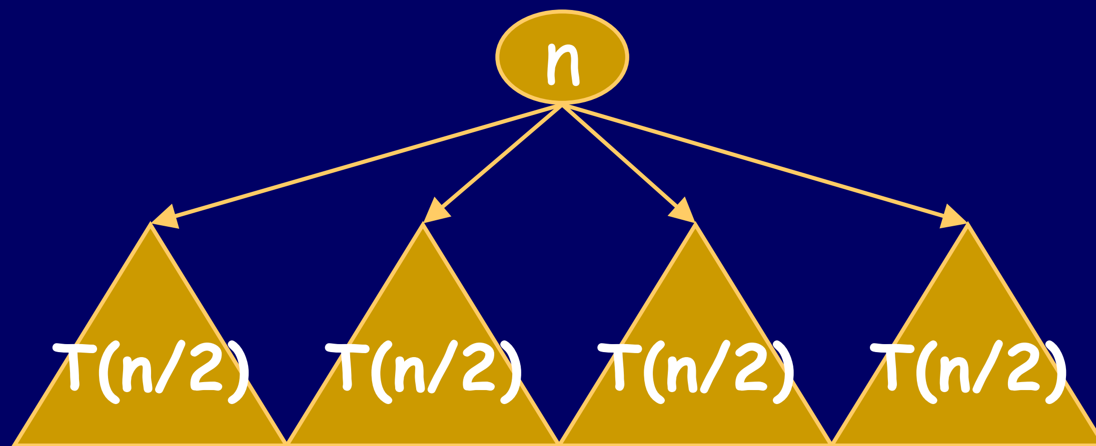
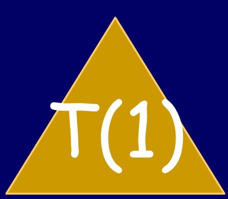
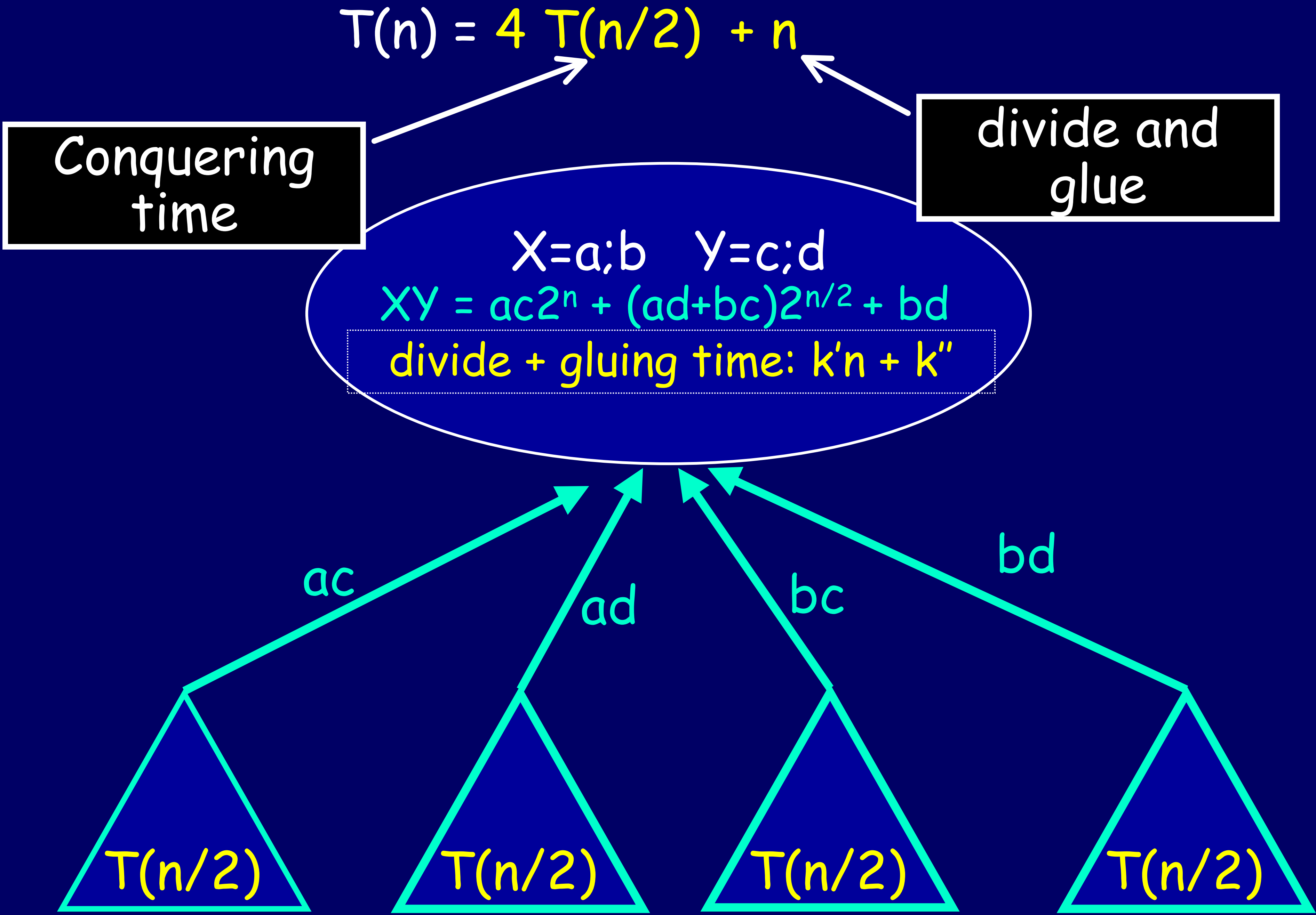


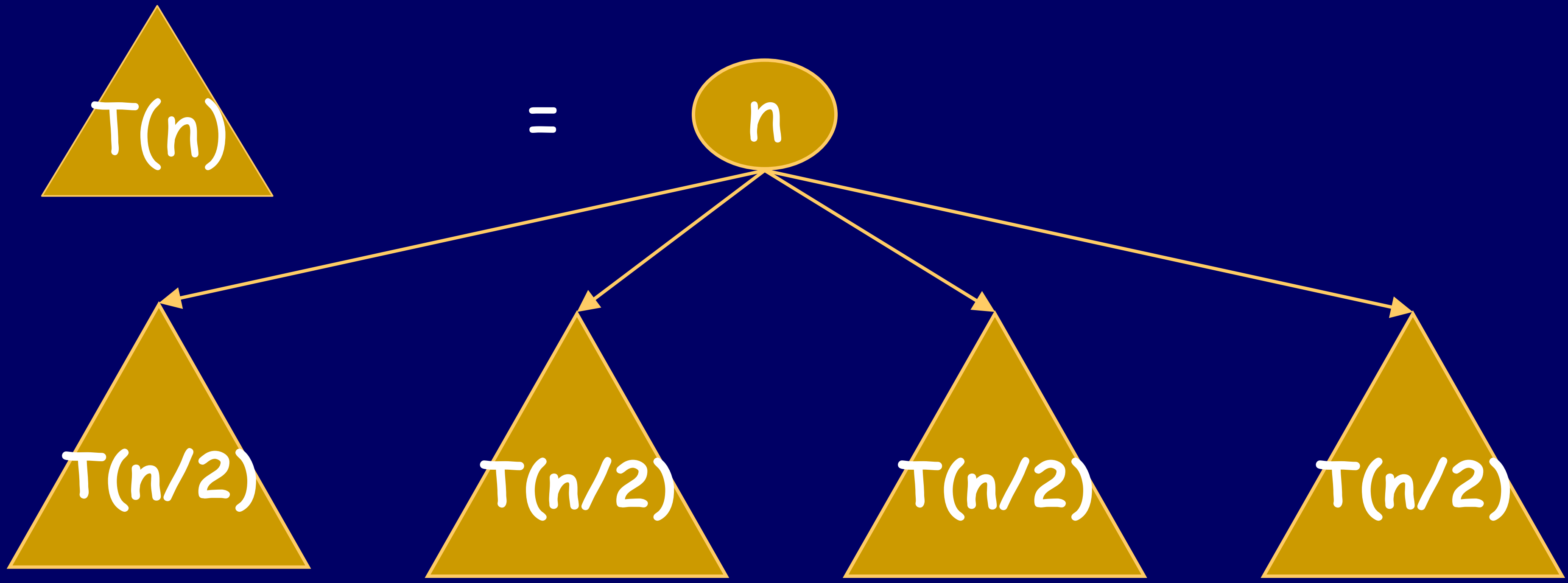
$$T(1) = 1$$

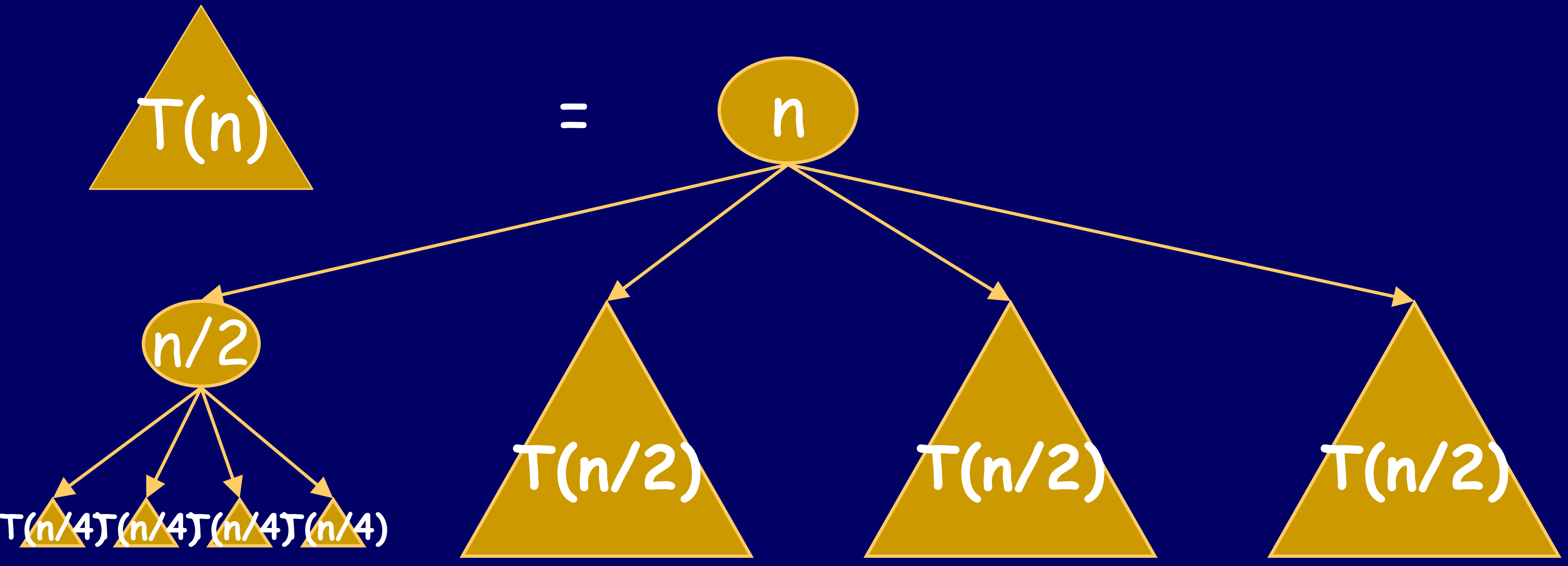


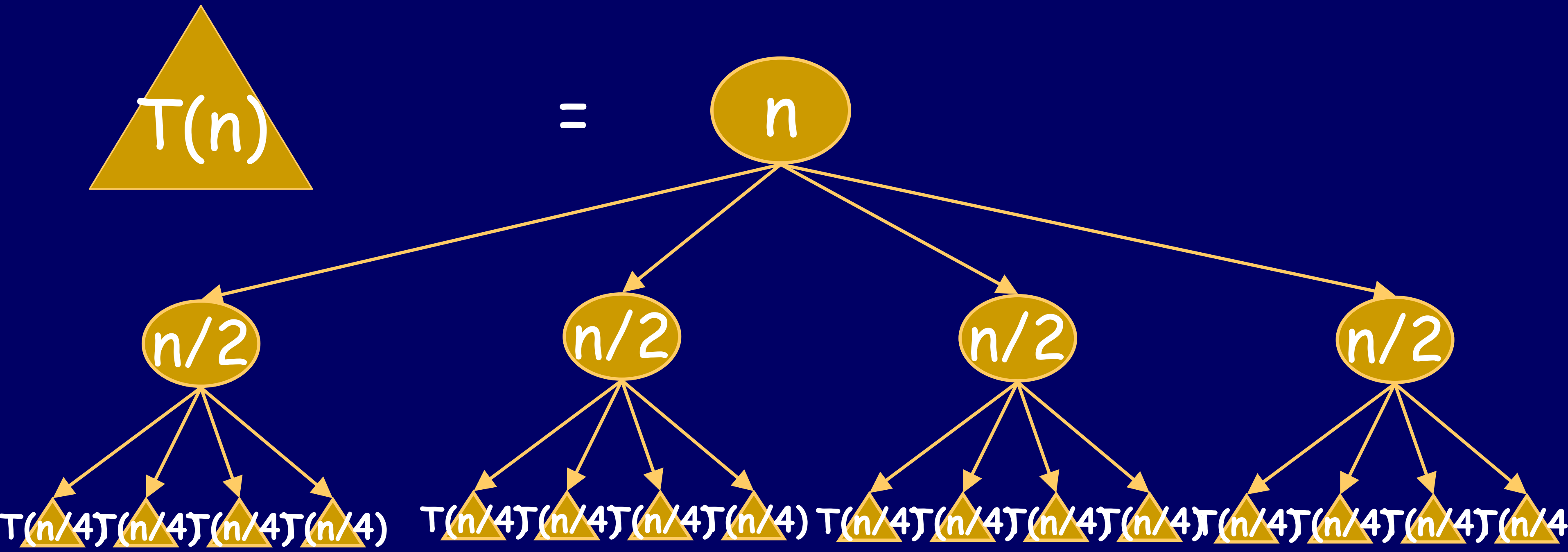
=

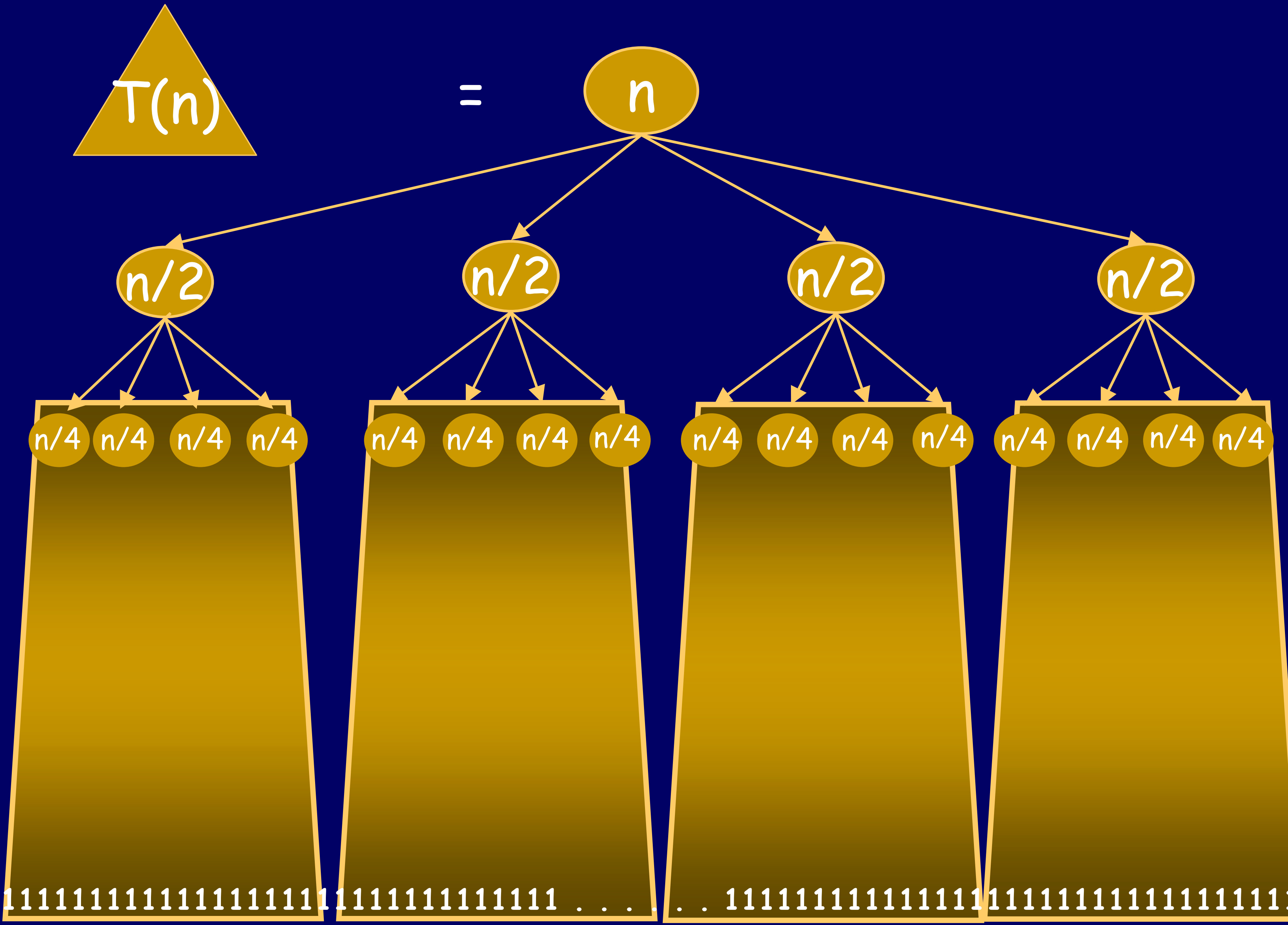


T(n) = 4 T(n/2) + n
Conquering time
divide and glue
X=a;b Y=c;d
XY = ac2^n + (ad+bc)2^{n/2} + bd
divide + gluing time: k'n + k"
ac
ad
bc
bd
T(n/2)
T(n/2)
T(n/2)
T(n/2)


T(n)
=
n
T(n/2)
T(n/2)
T(n/2)
T(n/2)


T(n)
=
n
n/2
T(n/2)
T(n/2)
T(n/2)
T(n/4)
T(n/4)
T(n/4)
T(n/4)


T(n)
=
n
n/2
n/2
n/2
n/2
T(n/4)T(n/4)T(n/4)T(n/4)
T(n/4)T(n/4)T(n/4)T(n/4)
T(n/4)T(n/4)T(n/4)T(n/4)
T(n/4)T(n/4)T(n/4)T(n/4



T(n)
=
n
n/2
n/2
n/2
n/2
n/4
n/4
n/4
n/4
n/4
n/4
n/4
n/4
n/4
n/4
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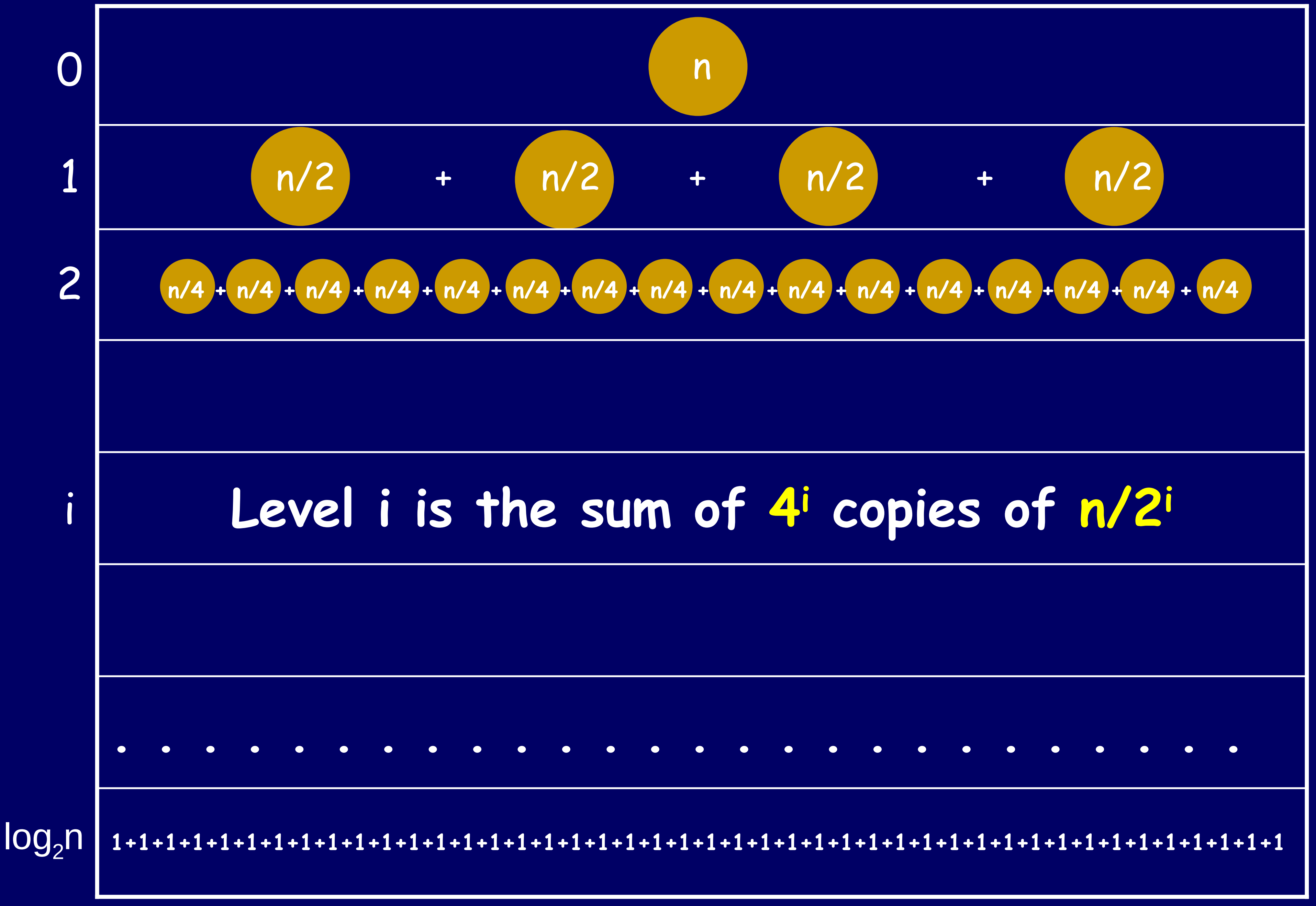
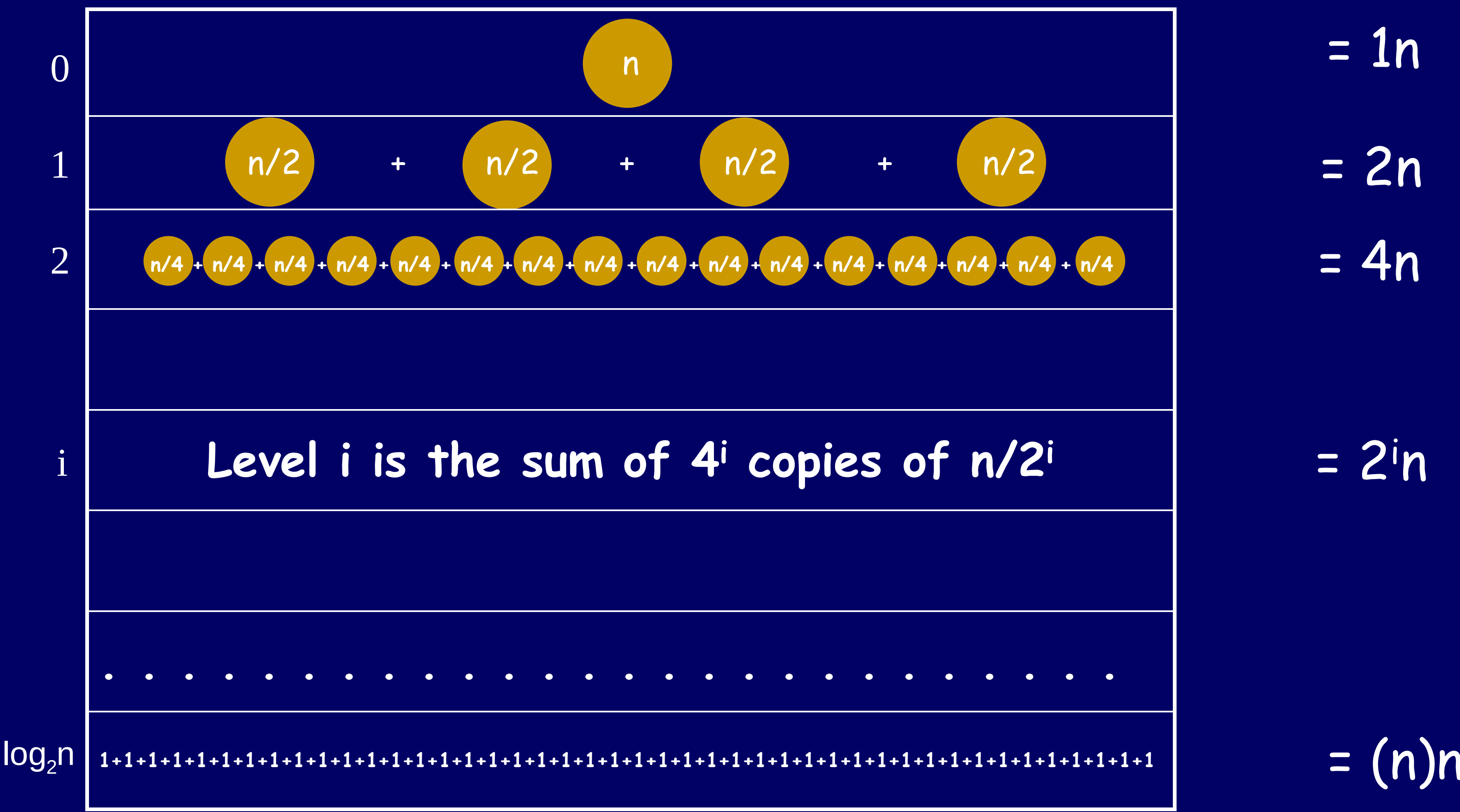
n/4
n/4
n/4
n/4


0
n
1
n/2 + n/2 + n/2 + n/2
2
n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4
i
Level i is the sum of 4^i copies of n/2^i
. . . .
log2n
1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1



$n(1+2+4+8+\ldots+n) = \quad n(2n-1) = \quad 2n^2-n$

Divide and Conquer MULT: $\Theta(n^2)$ time

Grade School Multiplication: $\Theta(n^2)$ time

Divide and Conquer MULT: $\Theta(n^2)$ time
Grade School Multiplication: $\Theta(n^2)$ time

In retrospect, it is obvious that the kissing number for Divide and Conquer MULT is n^2, since the leaves are in correspondence with the kisses.

MULT revisited

<u>MULT(X,Y):</u>

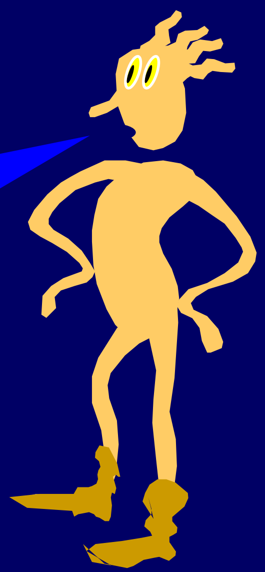
If |X| = |Y| = 1 then return XY

break X into a;b and Y into c;d

return

MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) $2^{n/2}$ + MULT(b,d)

MULT calls itself 4 times.
Can you see a way to reduce the number of calls?



Gauss' optimization

Input: a,b,c,d
Output: ac-bd, ad+bc

(a+bi)(c+di) =
[ac –bd] + [ad + bc] i

$X_1 = a + b$

$X_2 = c + d$

$X_3 = X_1 X_2 = ac + ad + bc + bd$

$X_4 = ac$

$X_5 = bd$

$X_6 = X_4 - X_5 = ac-bd$

$X_7 = X_3 - X_4 - X_5 = bc + ad$

Karatsuba, Anatolii Alexeevich (1937-)

Sometime in the late 1950's Karatsuba had formulated the first algorithm to break the kissing barrier!

Gaussified MULT
(Karatsuba 1962)

MULT(X,Y):

If |X| = |Y| = 1 then return XY

break X into a;b and Y into c;d

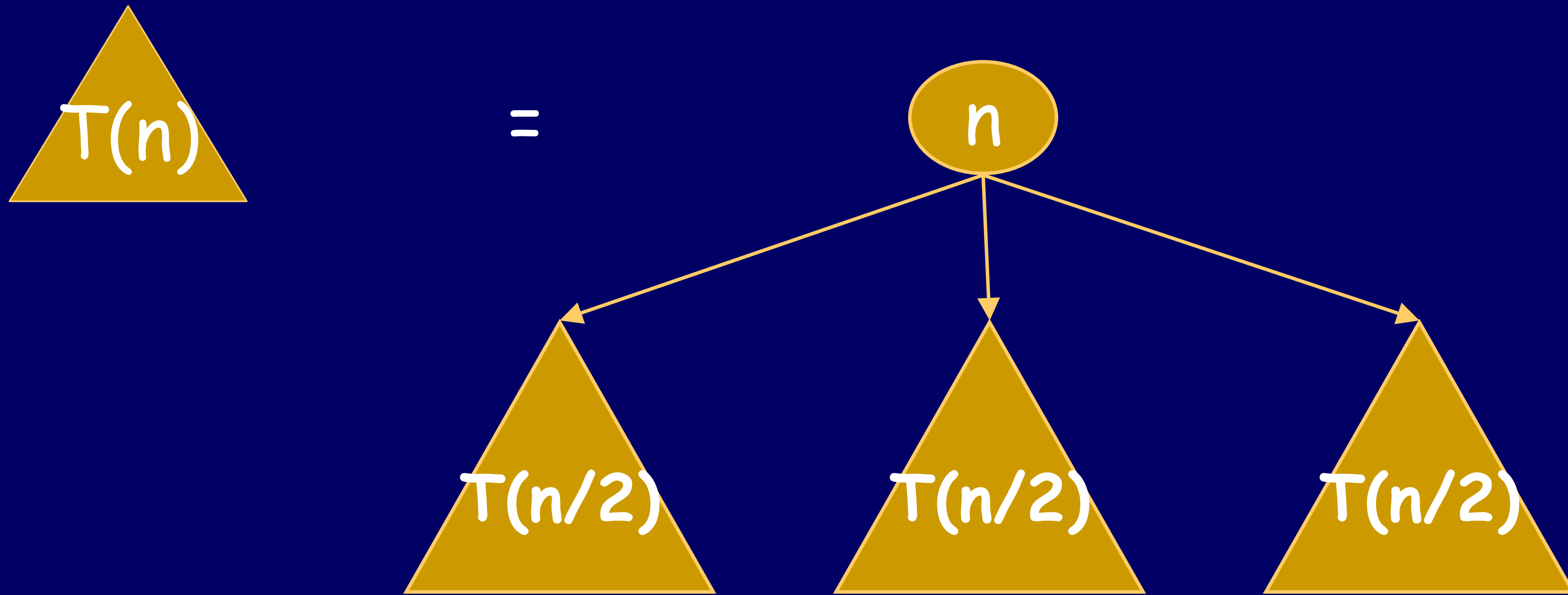
e = MULT(a,c) and f = MULT(b,d)

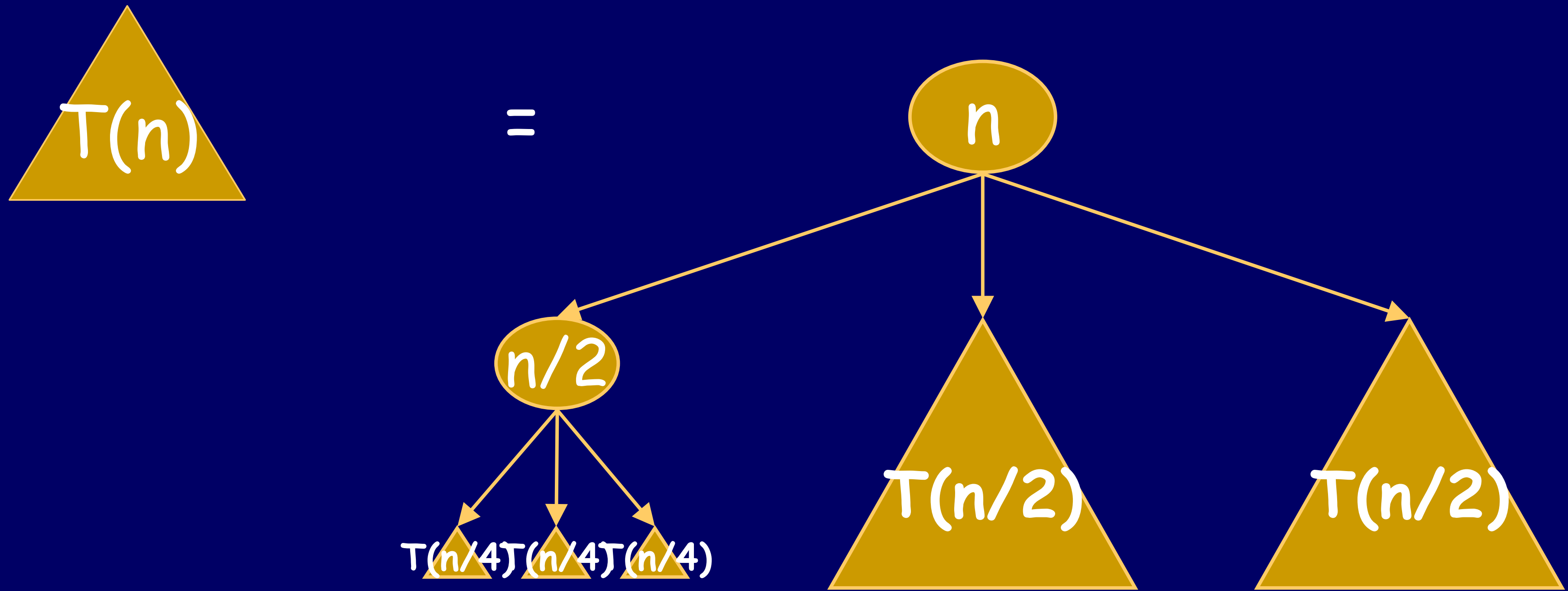
return

$e\ 2^n + (\text{MULT}(a+c,b+d) - e - f)\ 2^{n/2} + f$

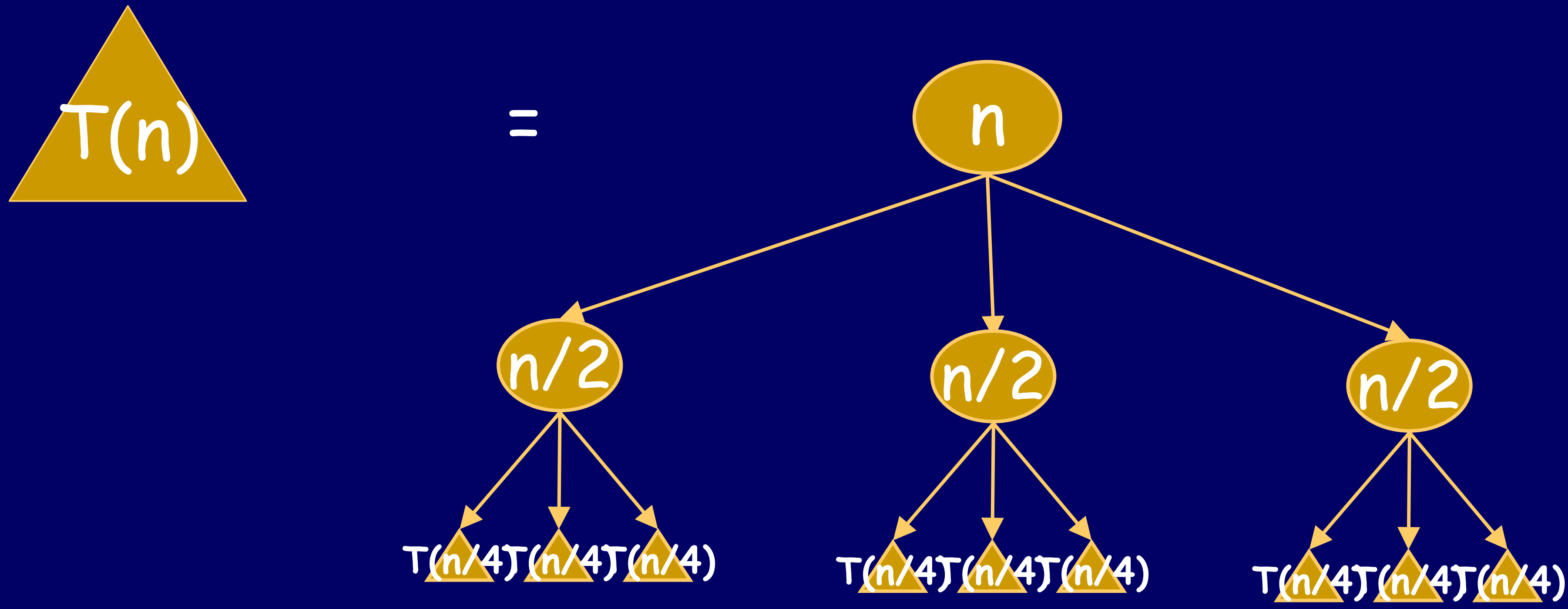
$$T(n) = 3\ T(n/2) + n$$

Actually: $T(n) = 2\ T(n/2) + T(n/2 + 1) + kn$

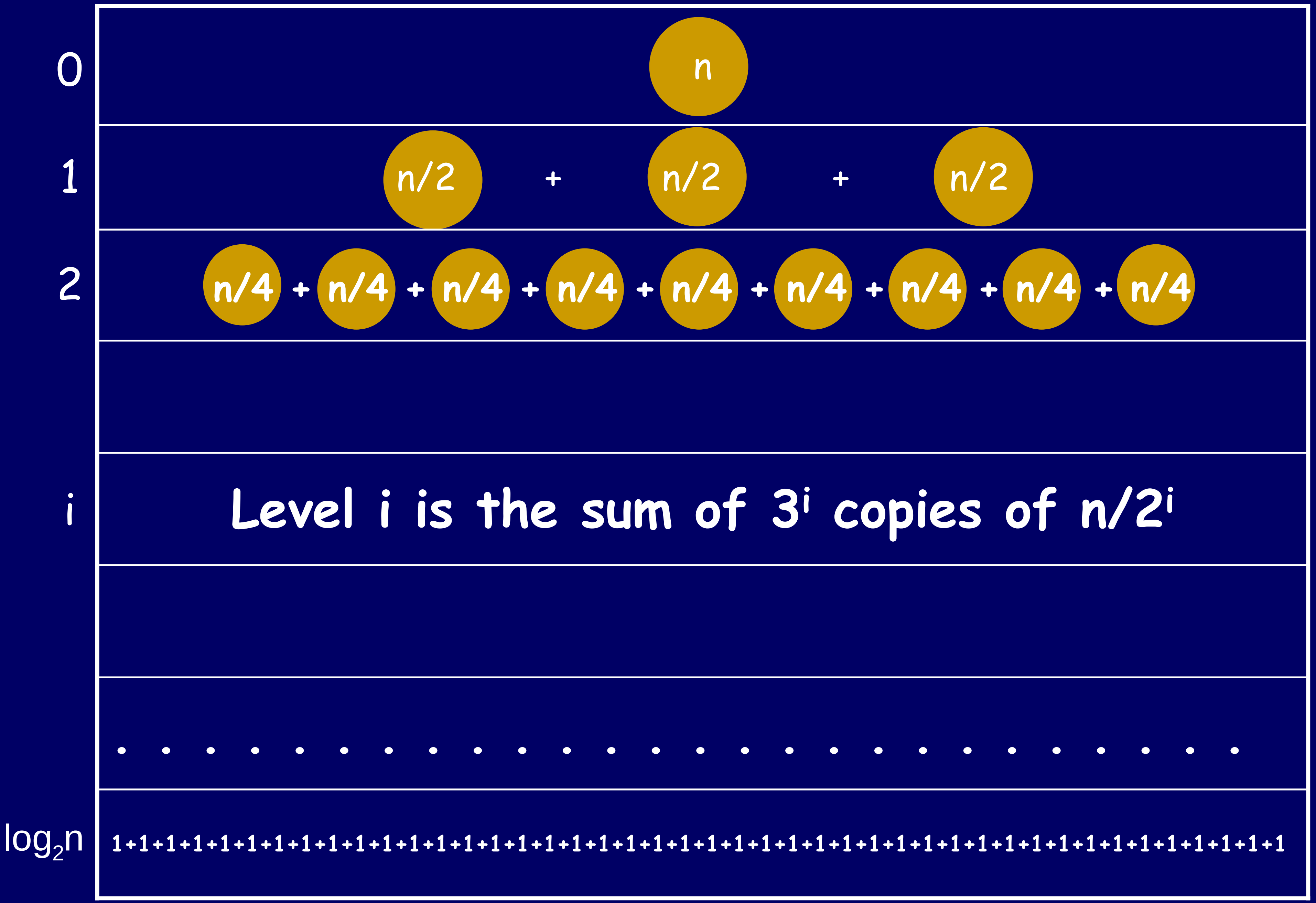

T(n)
=
n
T(n/2)
T(n/2)
T(n/2)



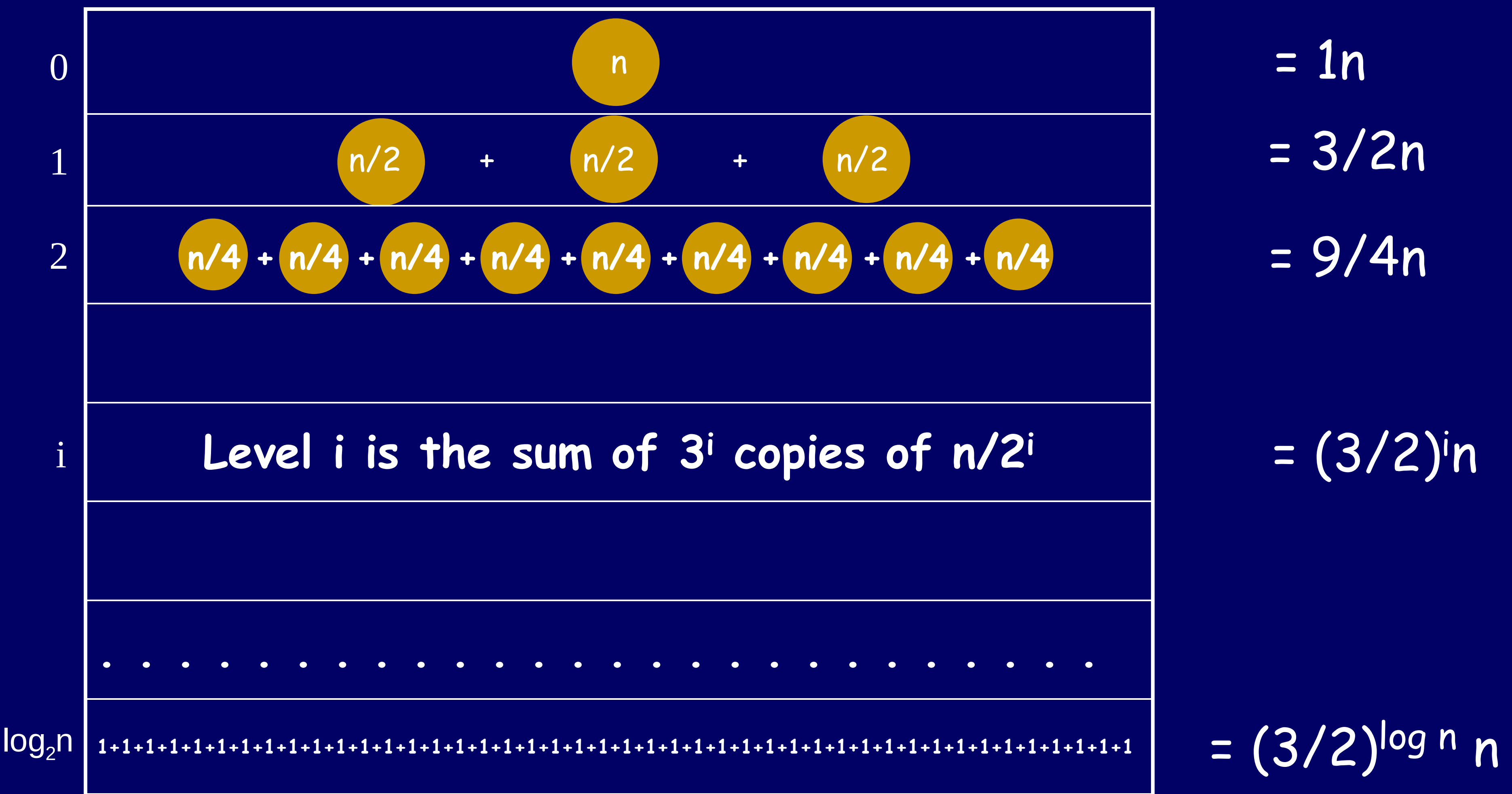
T(n)
=
n
n/2
T(n/2)
T(n/2)
T(n/4)
T(n/4)
T(n/4)



T(n)
=
n
n/2
n/2
n/2
T(n/4)
T(n/4)
T(n/4)
T(n/4)
T(n/4)
T(n/4)
T(n/4)
T(n/4)
T(n/4)

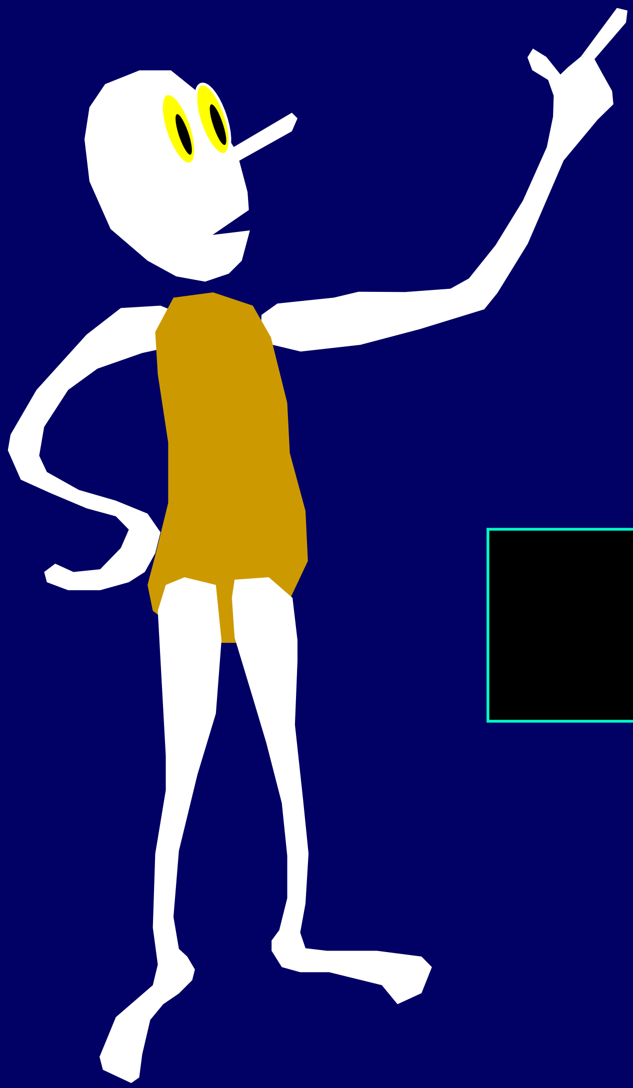


0
n
1
n/2 + n/2 + n/2
2
n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4
i
Level i is the sum of 3^i copies of $n/2^i$
. . . .
$\log_2 n$
1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1



$$n(1+3/2+(3/2)^2+ \ldots + (3/2)^{\log_2 n}) \quad = \quad 3n^{1.58\ldots} - 2n$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

Substituting into our formula....

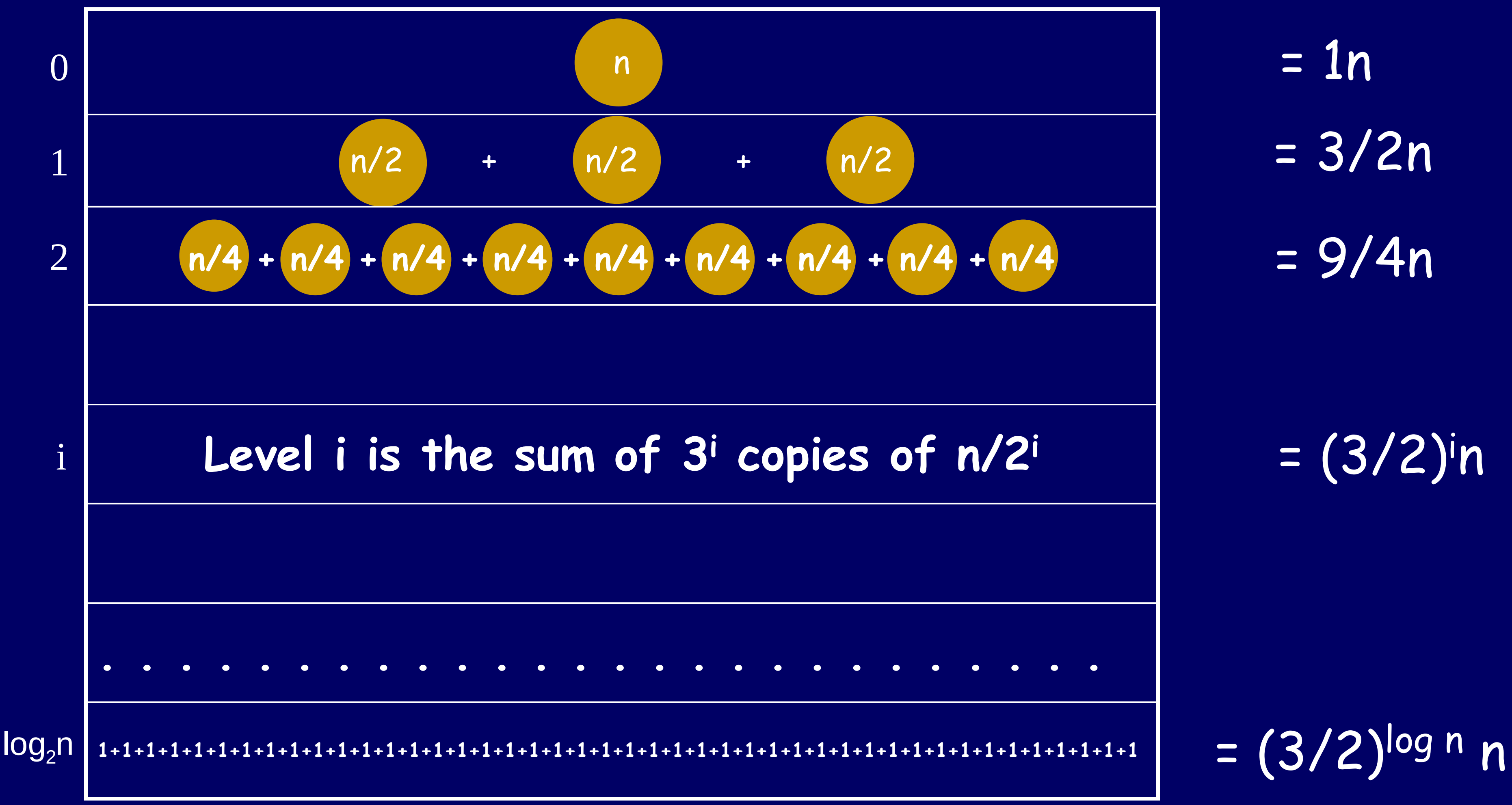
$$1 + X^1 + X^2 + X^3 + \ldots + X^{k-1} + X^k = \frac{X^{k+1} - 1}{X - 1}$$

We have: $X = 3/2$ $k = \log_2 n$

$$\frac{(3/2) \times (3/2)^{\log_2 n} - 1}{\frac{1}{2}} = 3 \times (3^{\log_2 n}/2^{\log_2 n}) - 2$$

$$= 3 \times (3^{\log_2 n}/n) - 2$$

$$= \frac{3\, n^{(\log_2 3)}}{n} - 2$$



$$n\left(\frac{3n^{\log_2 3}}{n} - 2\right) = 3n^{\log_2 3} - 2n$$

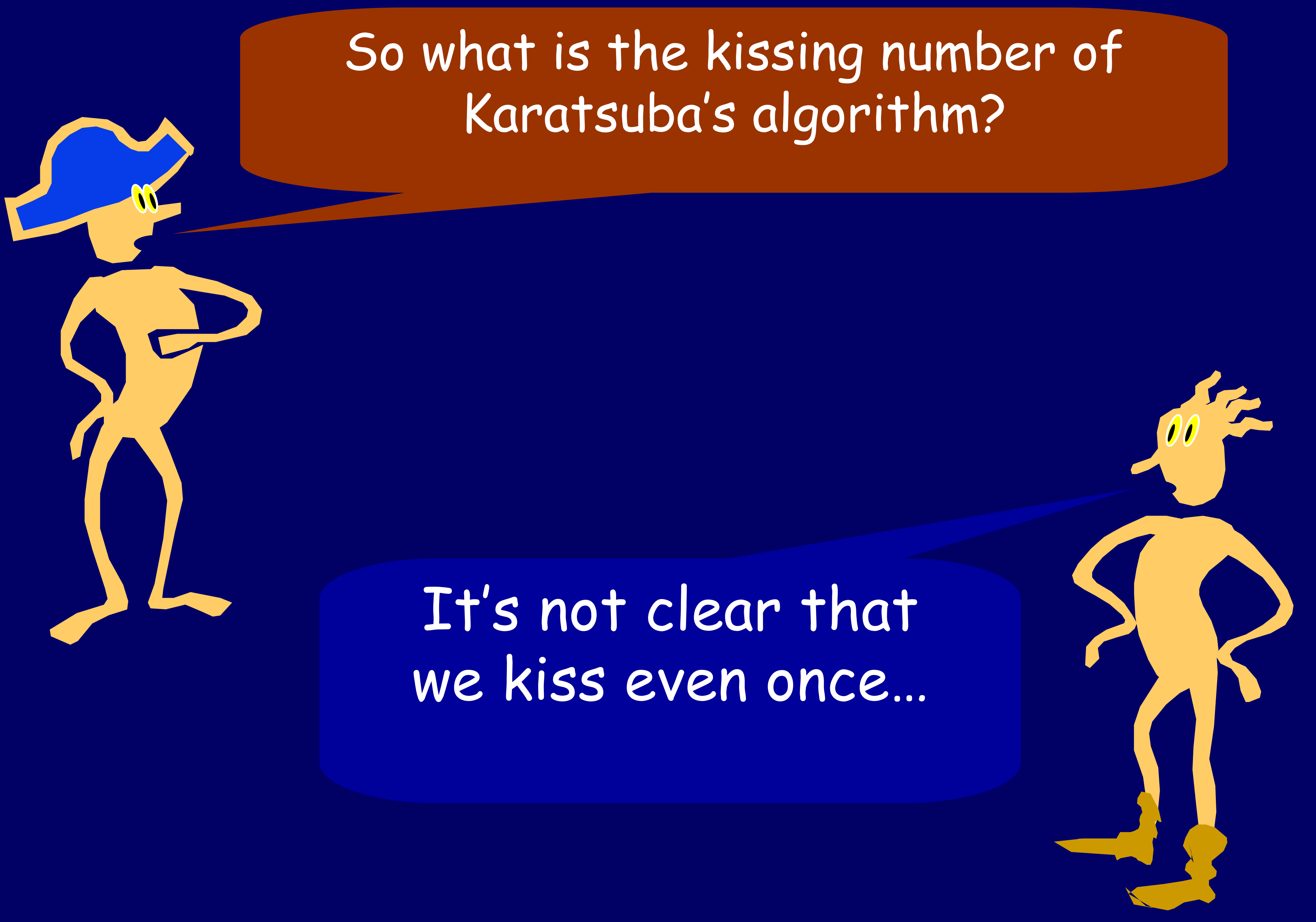
Dramatic improvement for large n

$$T(n) = 3n^{\log_2 3} - 2n$$
$$= \Theta(n^{\log_2 3})$$
$$= \Theta(n^{1.58\ldots})$$

A huge savings over $\Theta(n^2)$ when n gets large.

Strange! The Gauss optimization seems to only be worth a 25% speedup. It simply replaces every 4 multiplications with 3.
Where did the power come from?

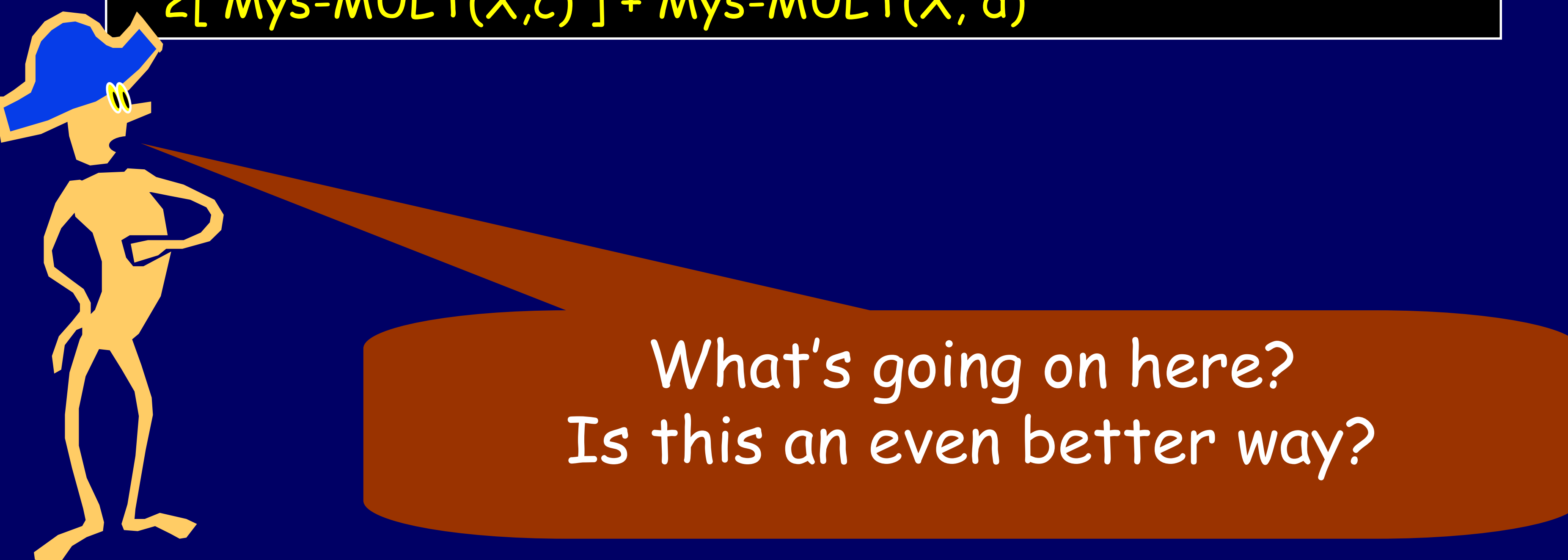
We applied the Gauss optimization RECURSIVELY!

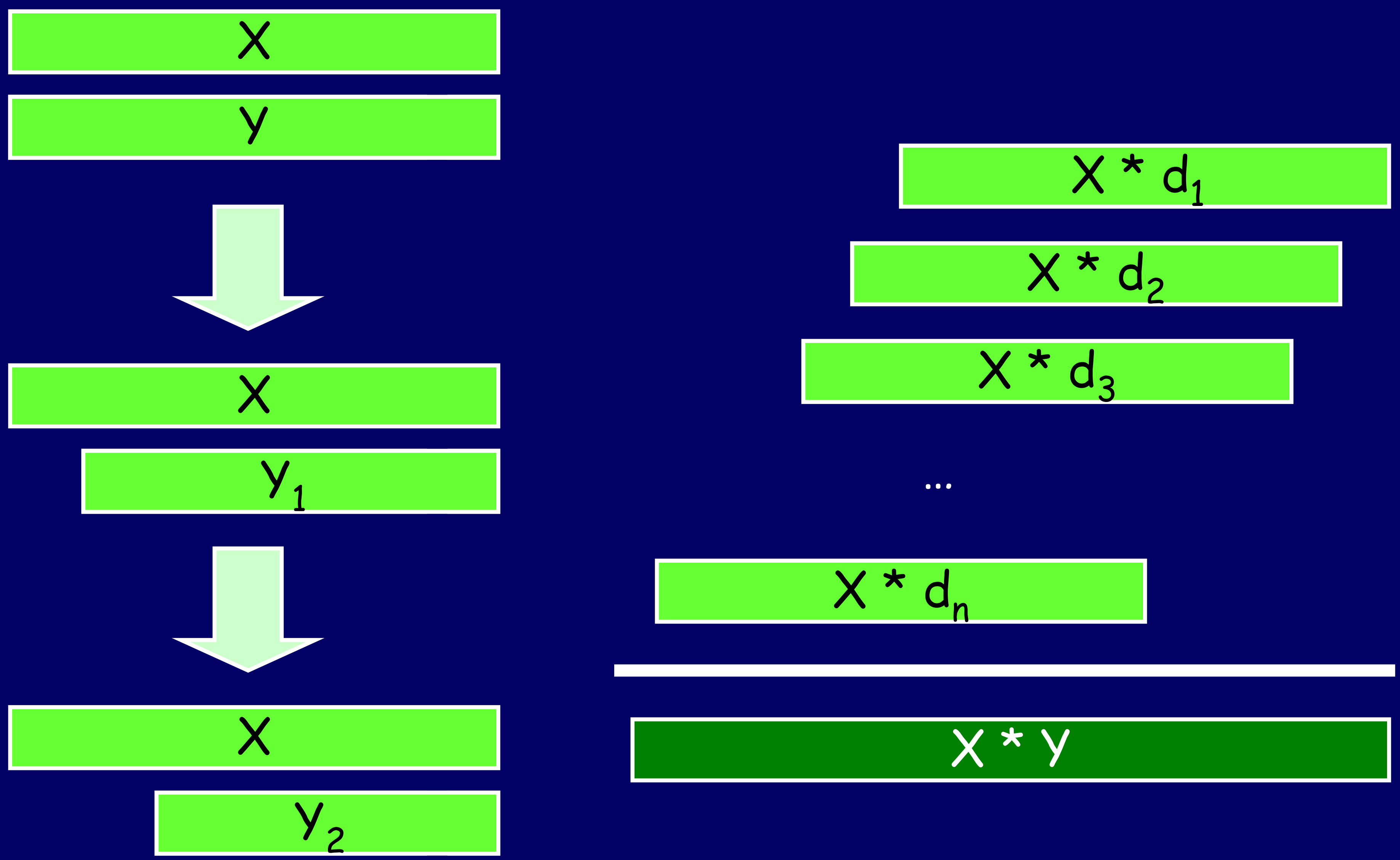

So what is the kissing number of Karatsuba's algorithm?
It's not clear that we kiss even once...

Mystery MULT

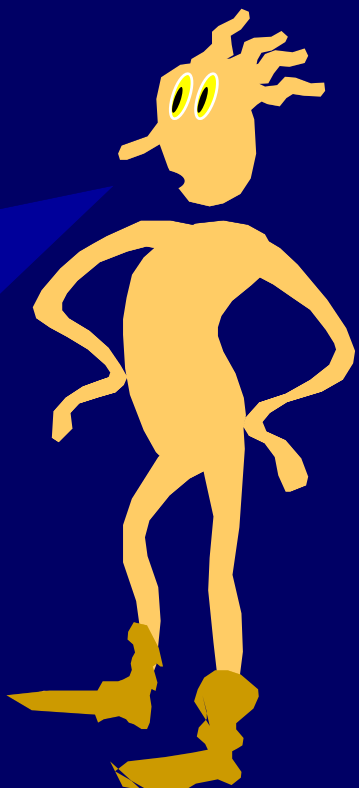
<u>Mys-MULT(X,Y):</u>

```
If |Y| = 1 then return X × Y
break Y into c;d where |d| = 1
return
  2[ Mys-MULT(X,c) ] + Mys-MULT(X, d)
```




X
y
X
y_1
X
y_2
X * d_1
X * d_2
X * d_3
...
X * d_n
X * y

Mys-MULT is the Egyptian method stated in recursive language.



Multiplication Algorithms

Kindergarten	$n2^n$
Grade School	n^2
Karatsuba	$n^{1.58...}$
Fastest Known	n logn loglogn

REFERENCES

Karatsuba, A., and Ofman, Y. *Multiplication of multidigit numbers on automata*. Sov. Phys. Dokl. 7 (1962), 595--596.