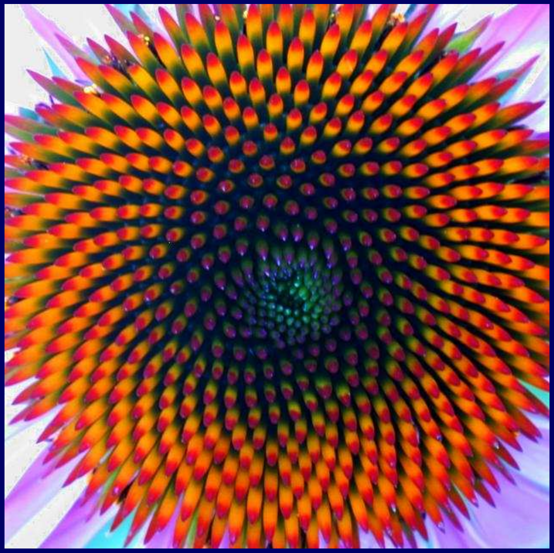
Great Theoretical Ideas In Computer Science

Steven Rudich CS 15-251 Spring 2005

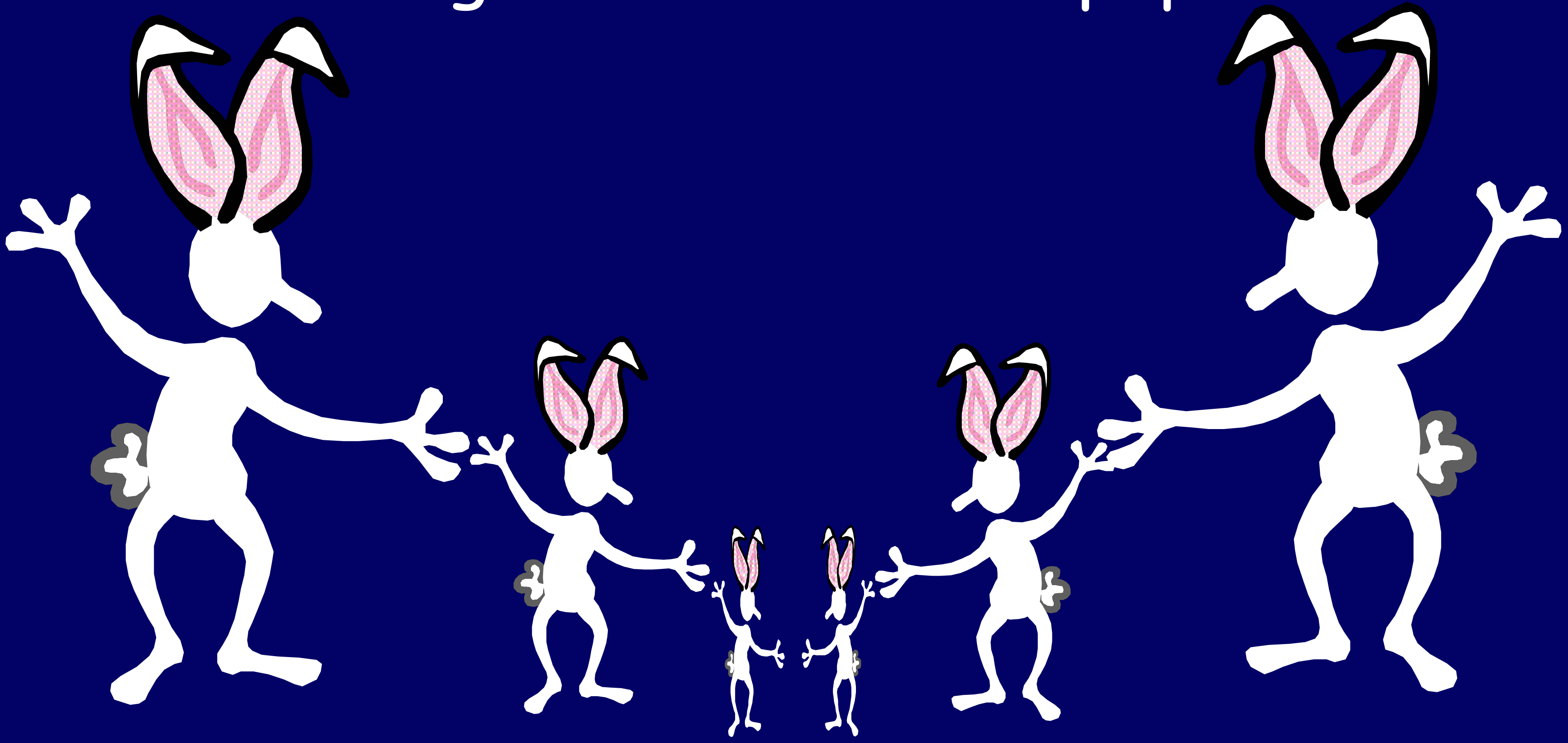
Lecture 13 Feb 22, 2005 Carnegie Mellon University

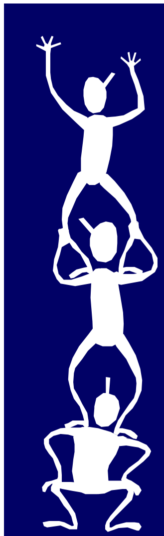
The Fibonacci Numbers And An Unexpected Calculation.



Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.





Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:

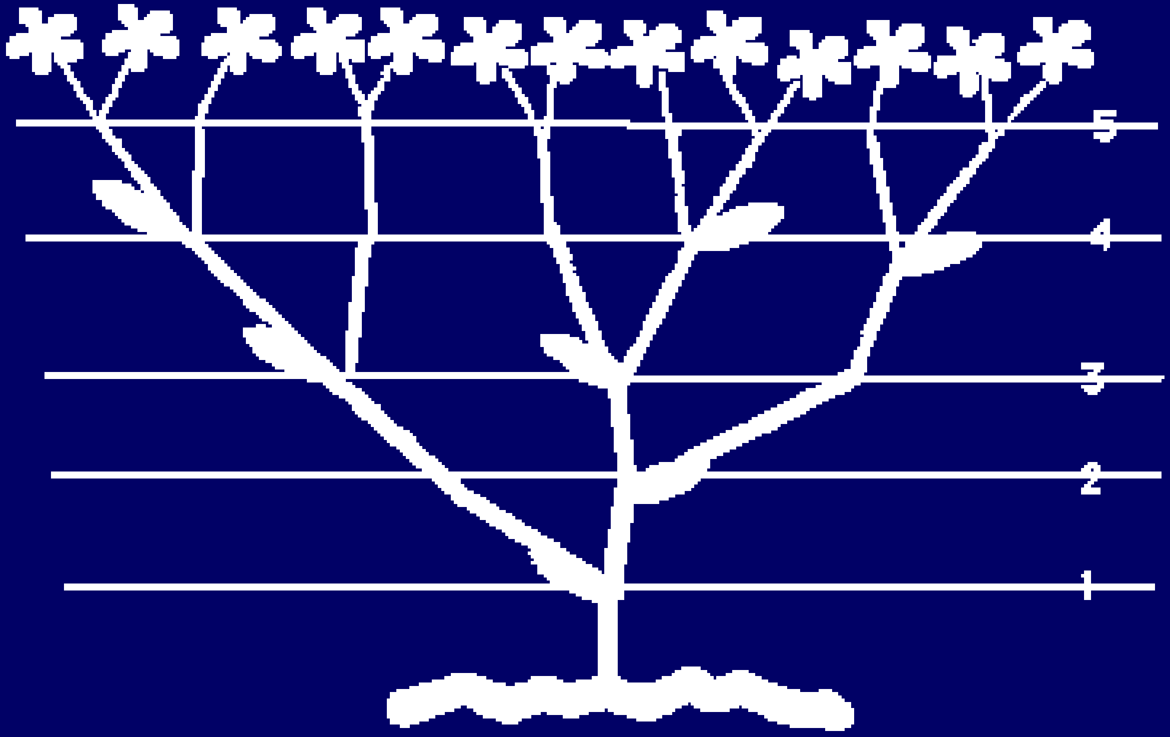
Fib(0) = 0; Fib (1) = 1

Inductive Rule

For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13

Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

Counting Petals

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)

8 petals: delphiniums

13 petals: ragwort, corn marigold, cineraria, some daisies

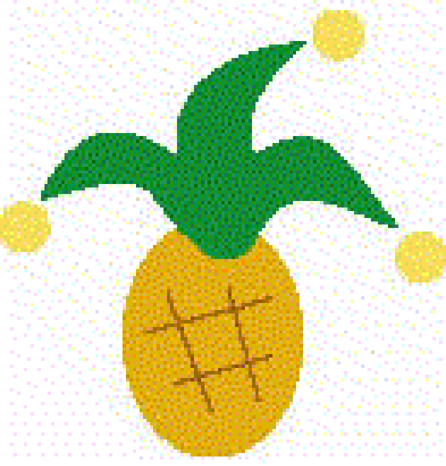
21 petals: aster, black-eyed susan, chicory

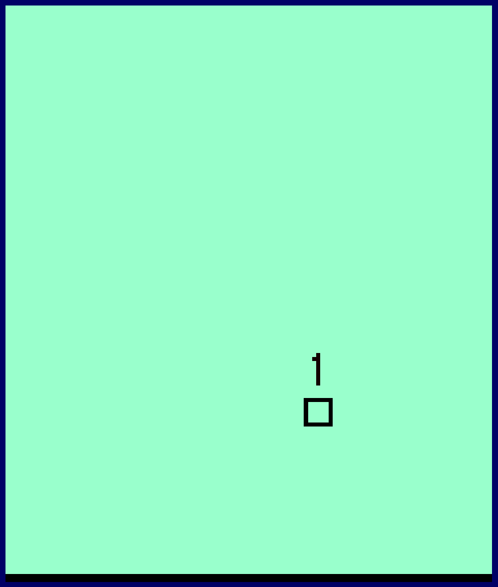
34 petals: plantain, pyrethrum

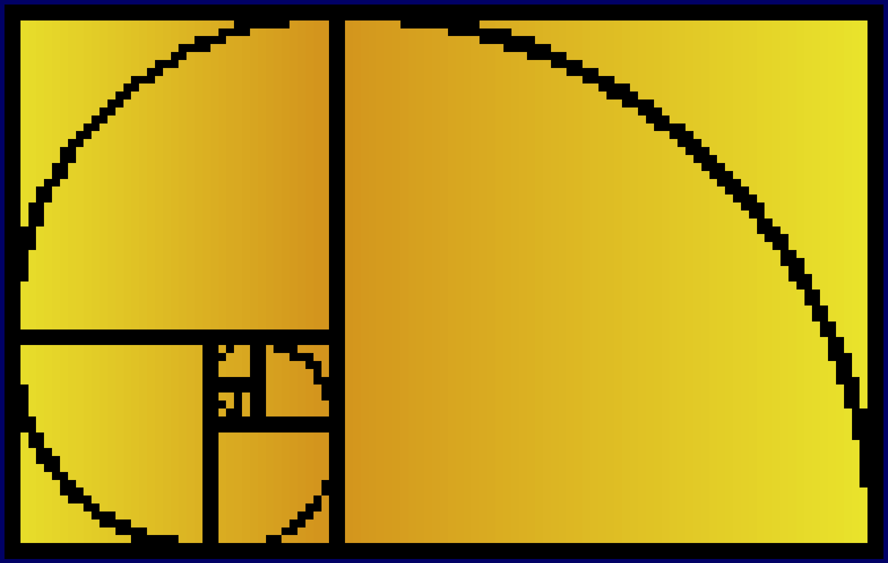
55, 89 petals: michaelmas daisies, the asteraceae family.

Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral. The ratio of consecutive ring lengths approaches the Golden Ratio.

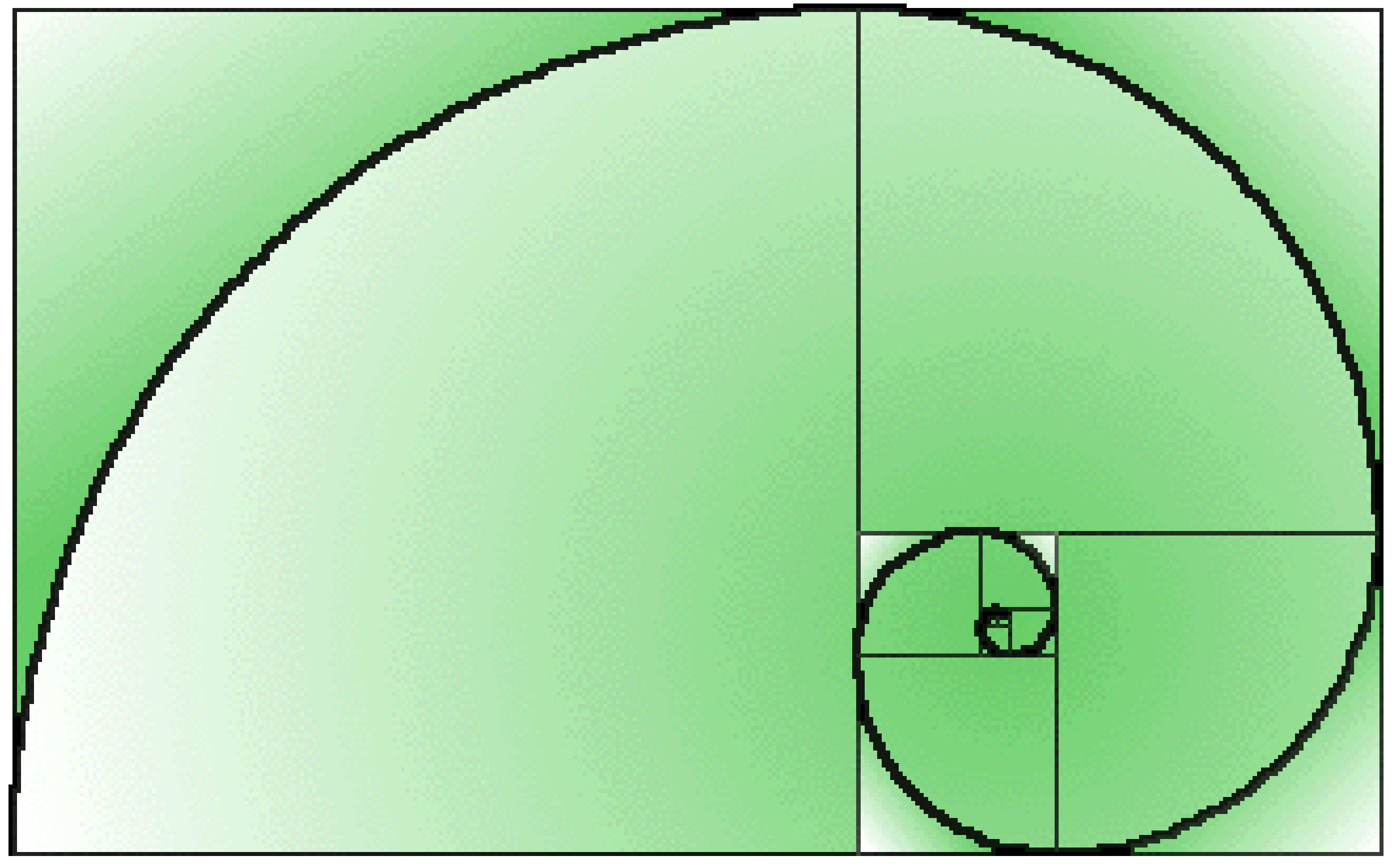



1



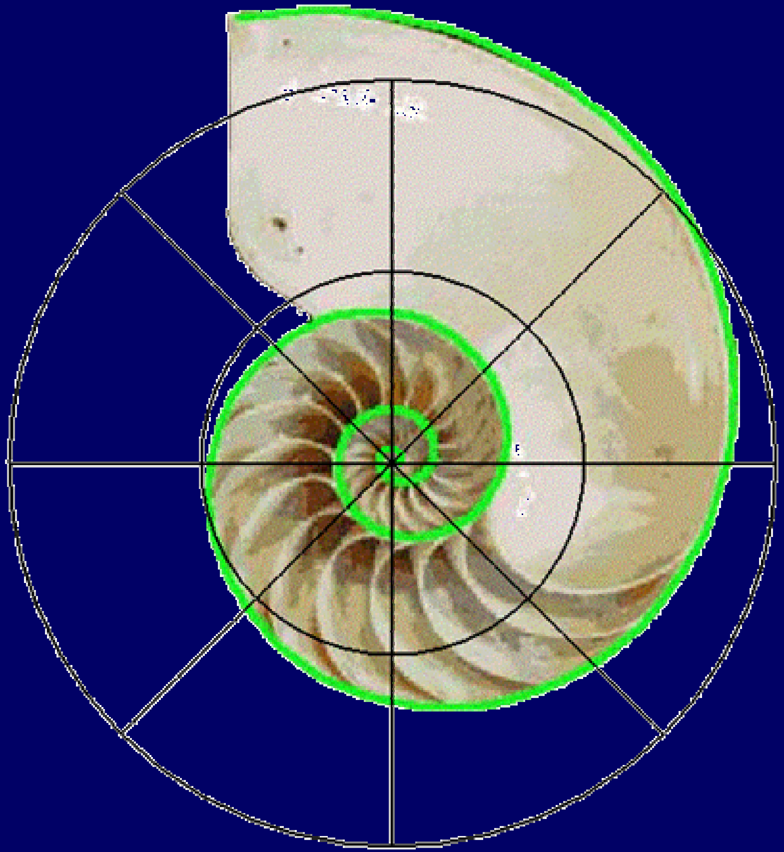
Bernoulli Spiral

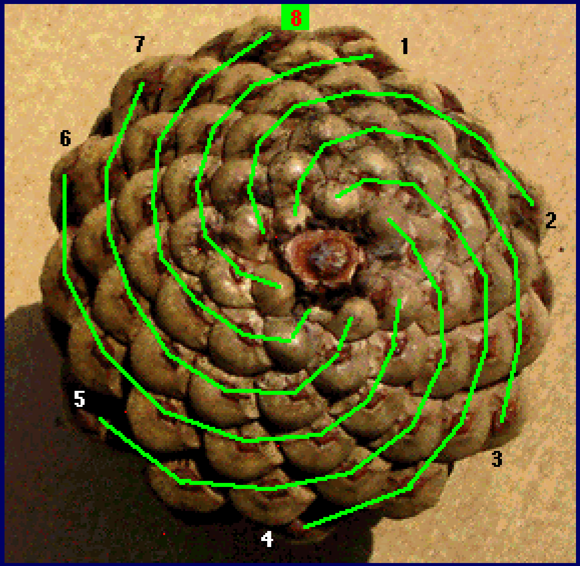
When the growth of the organism is proportional to its size

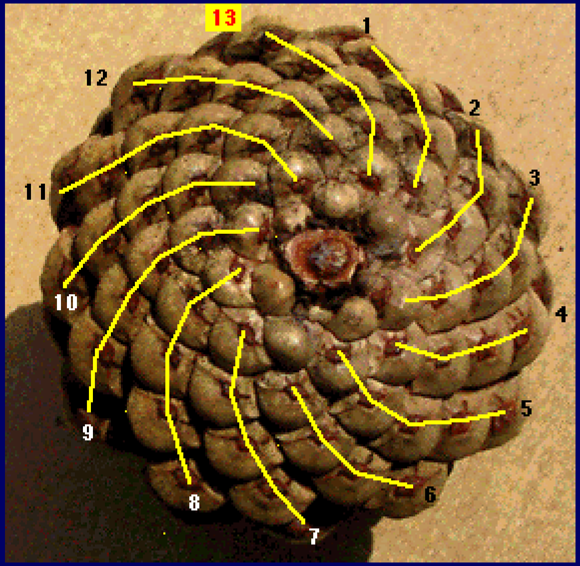


Bernoulli Spiral

When the growth of the organism is proportional to its size




8
7
1
6
2
5
3
4


13
1
12
2
3
11
4
10
5
9
6
8
7

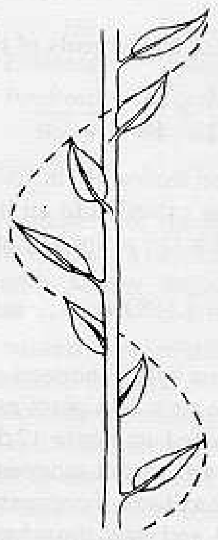
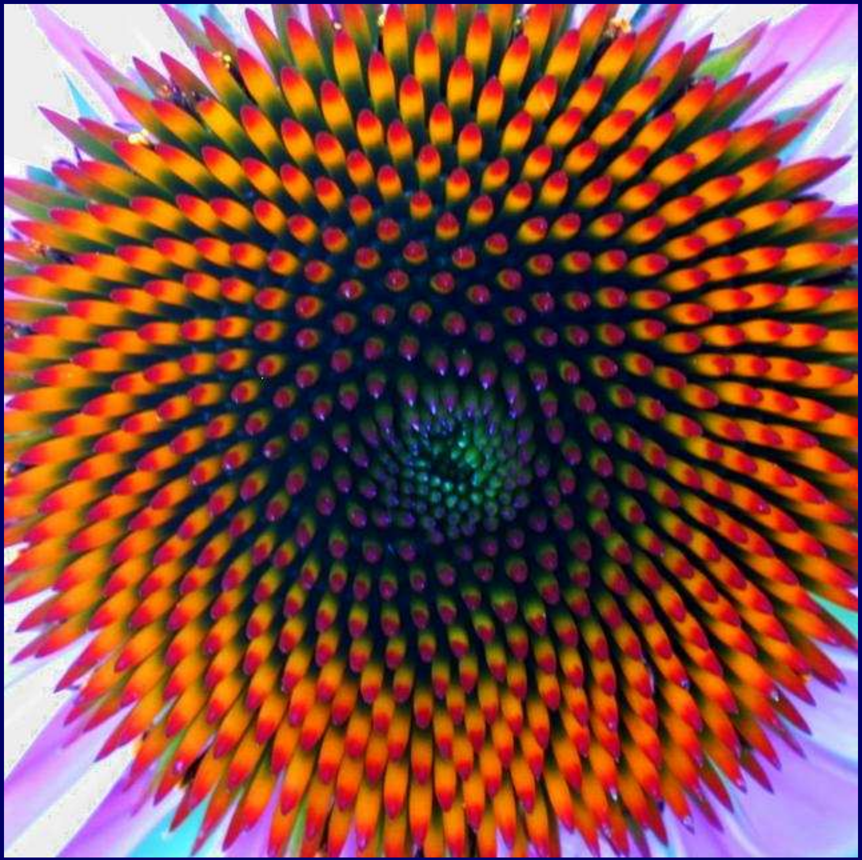


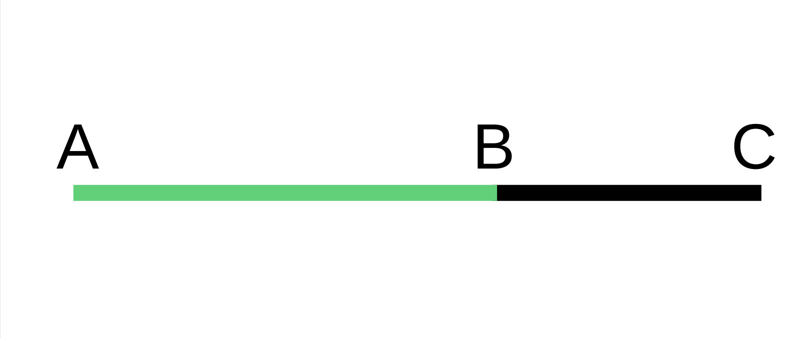
Fig. 12.4. Phyllotaxis



Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$



$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\boxed{\phi^2 - \phi - 1 = 0}$$

Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

The Divine Quadratic

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{\sqrt{5} + 1}{2}$$

$$\phi = 1 + 1/\phi$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\ldots.}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Best Rational Approximations to ϕ

We already saw the convergents of this CF

[1,1,1,1,1,1,1,1,1,1,1,...]

are of the form

Fib(n+1)/Fib(n)

Hence: $$\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$$

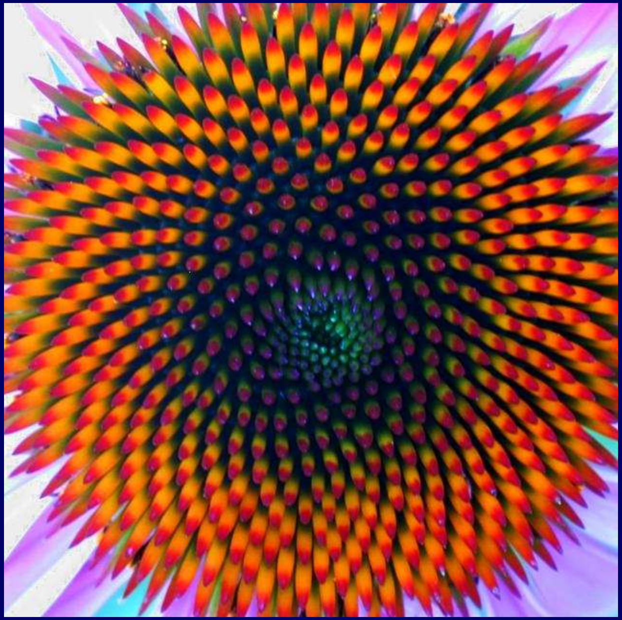
1,1,2,3,5,8,13,21,34,55,....

2/1 = 2
3/2 = 1.5
5/3 = 1.666...
8/5 = 1.6
13/8 = 1.625
21/13 = 1.6153846...
34/21 = 1.61904...

ϕ = 1.6180339887498948482045

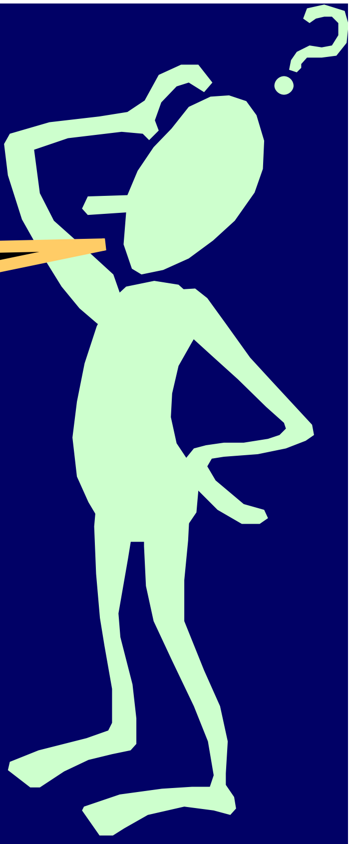
Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$



1.6180339887498948482045.....

Is there
life after
π and e?

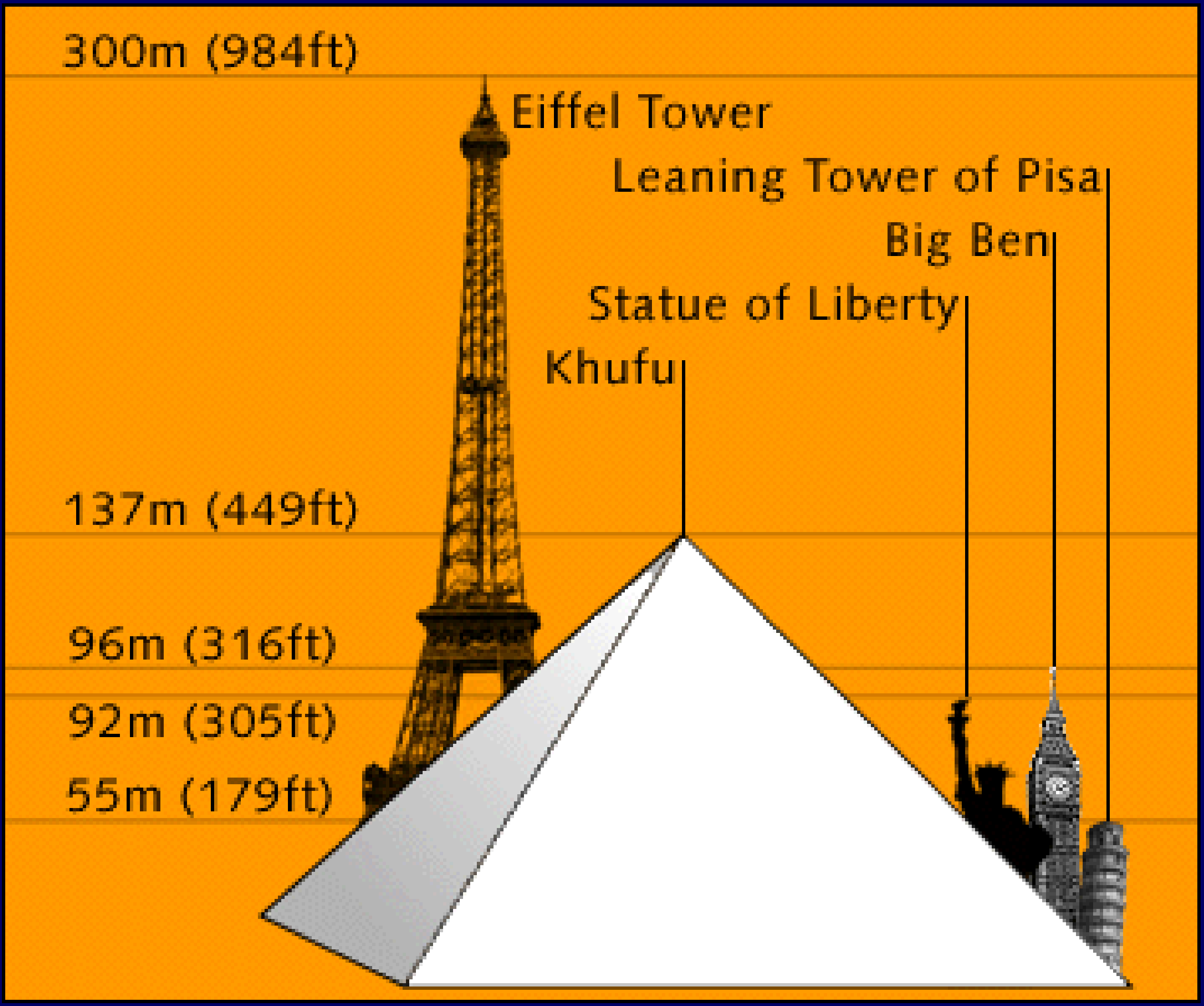


Khufu

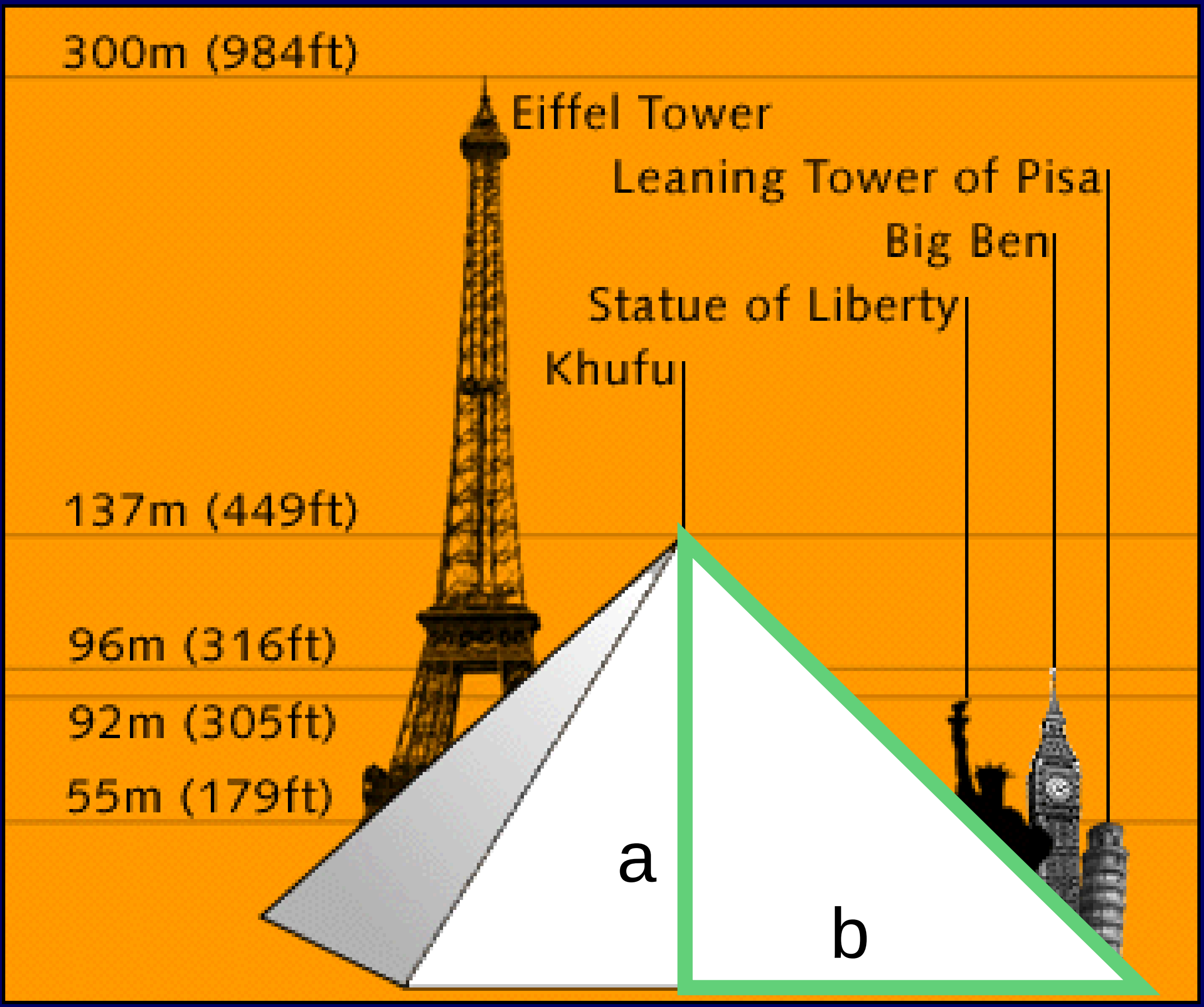
- 2589-2566 B.C.
- 2,300,000 blocks averaging 2.5 tons each

Package

Great Pyramid at Gizeh



$$\frac{a}{b} = 1.618$$



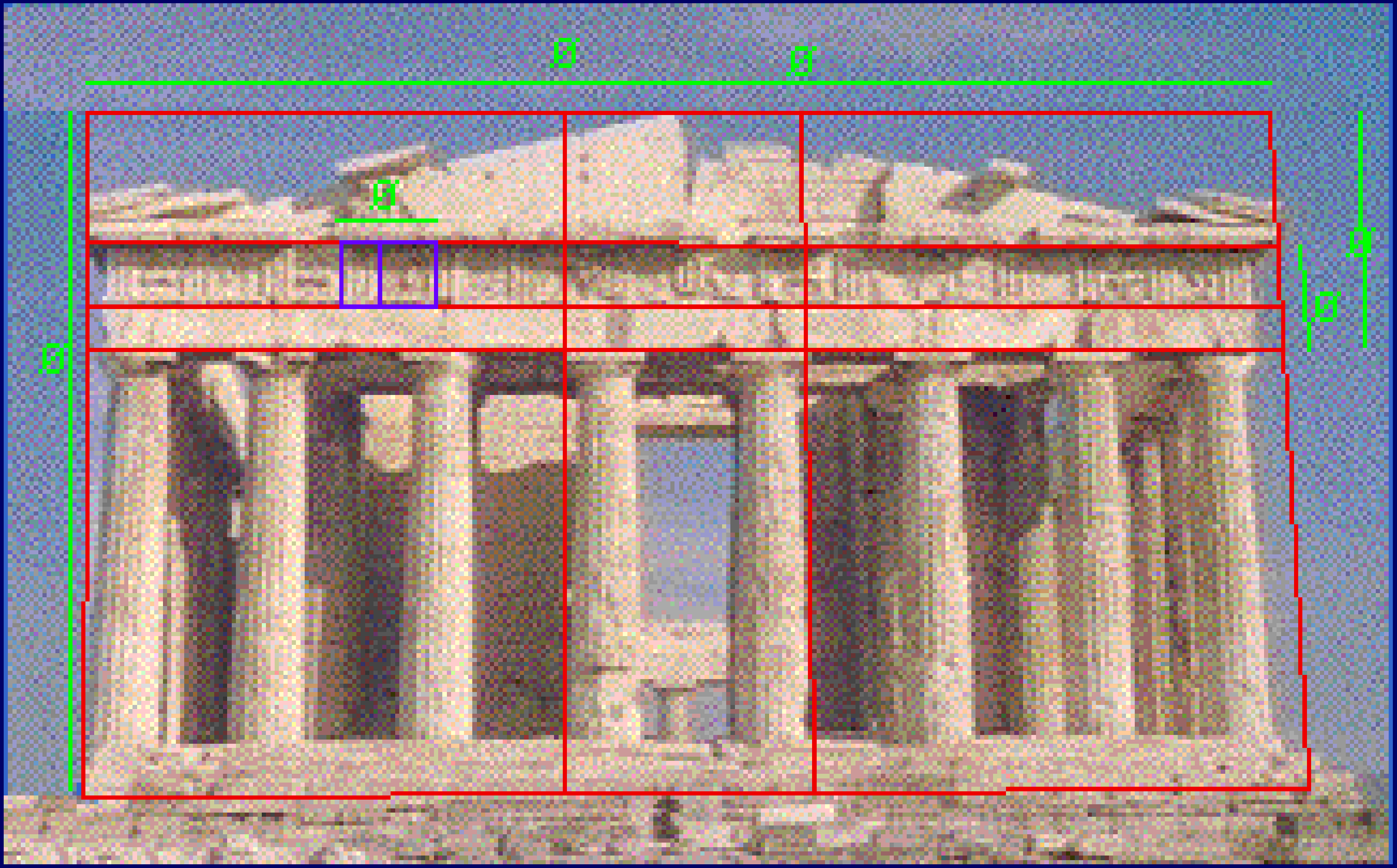
The ratio of the altitude of a face to half the base

Golden Ratio: the divine proportion

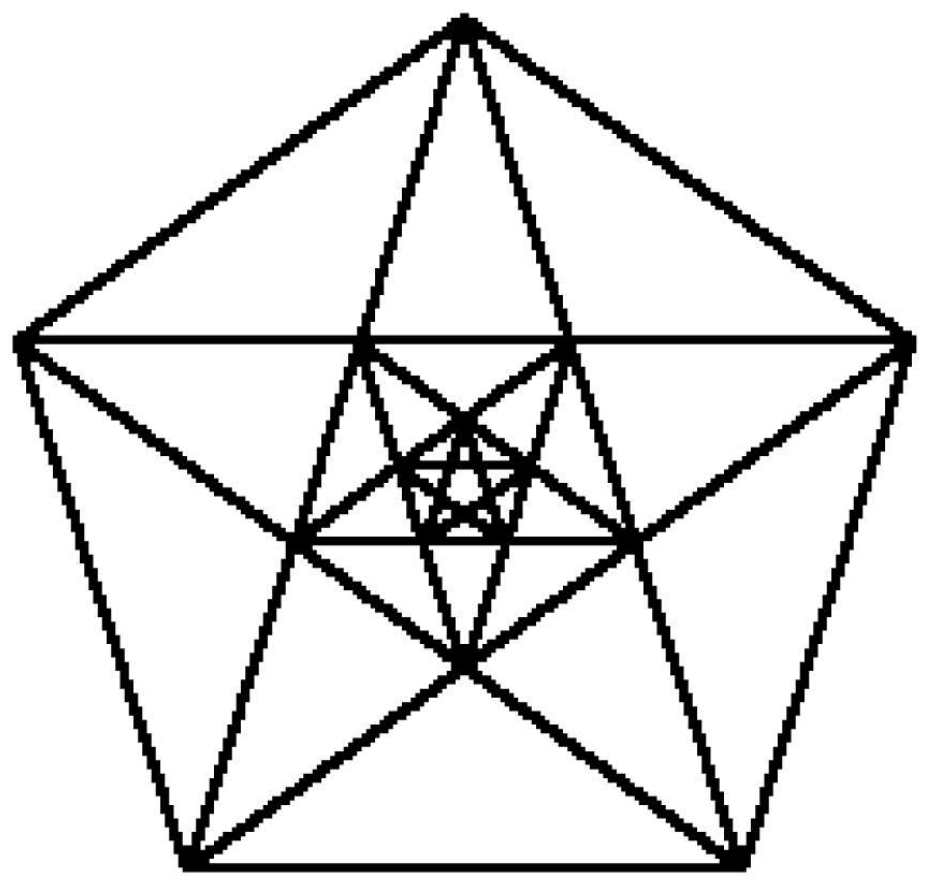
$\phi = 1.6180339887498948482045...$

"Phi" is named after the Greek sculptor Phidias

Parthenon, Athens (400 B.C.)



Pentagon

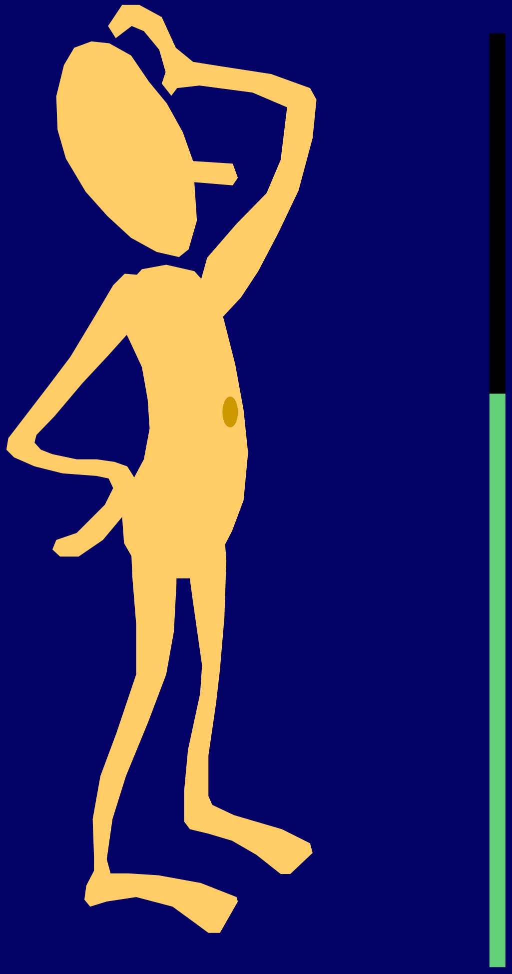


Golden Ratio
Divine Proportion

ϕ = 1.6180339887498948482045…

"Phi" is named after the Greek sculptor Phidias

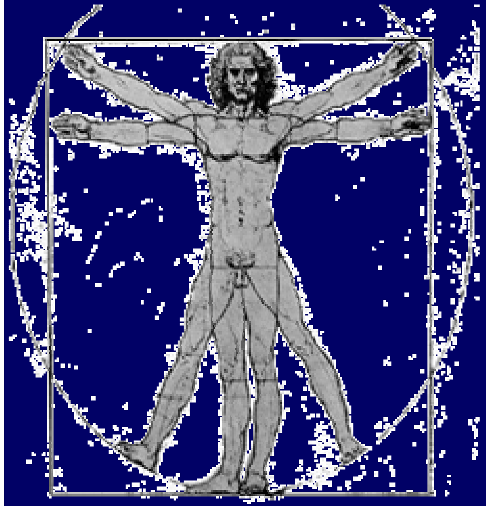
Ratio of height of the person to height of a person's navel



Divina Proportione
Luca Pacioli (1509)

Pacioli devoted an entire book to the marvelous properties of ϕ. The book was illustrated by a friend of his named:

Leonardo Da Vinci



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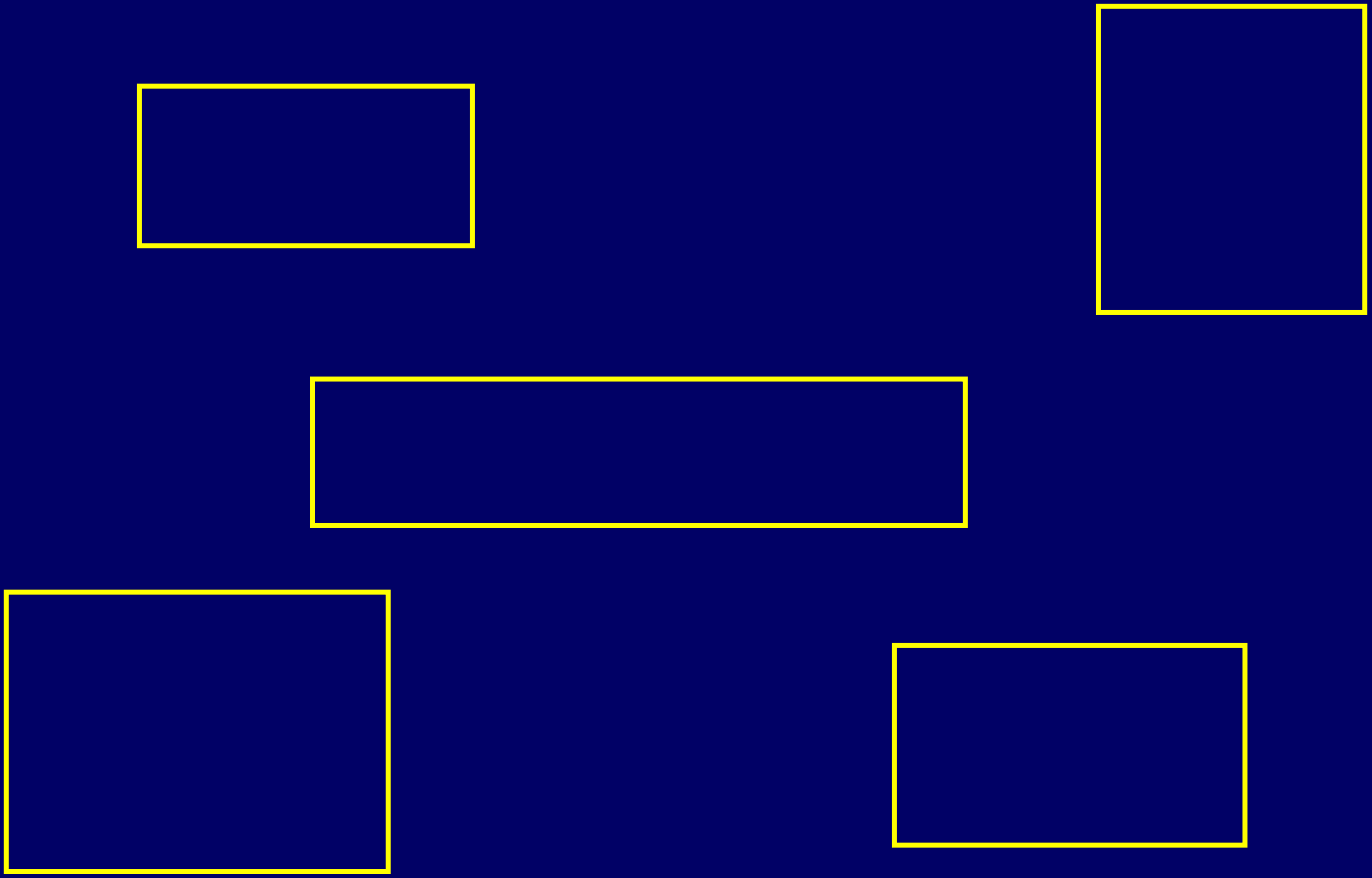
"For the sake of our salvation this list of effects must end."

Aesthetics

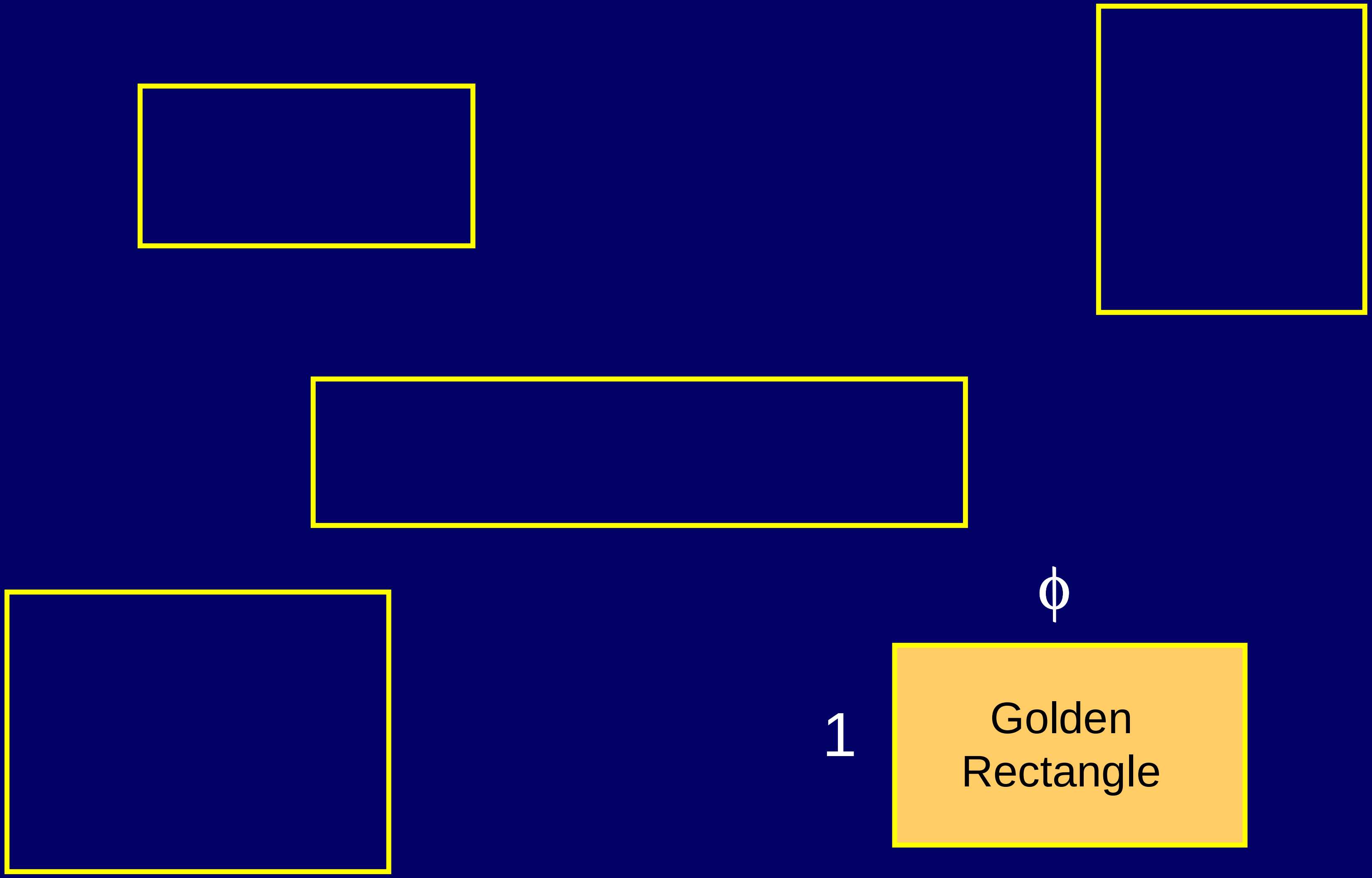
ϕ plays a central role in renaissance art and architecture.

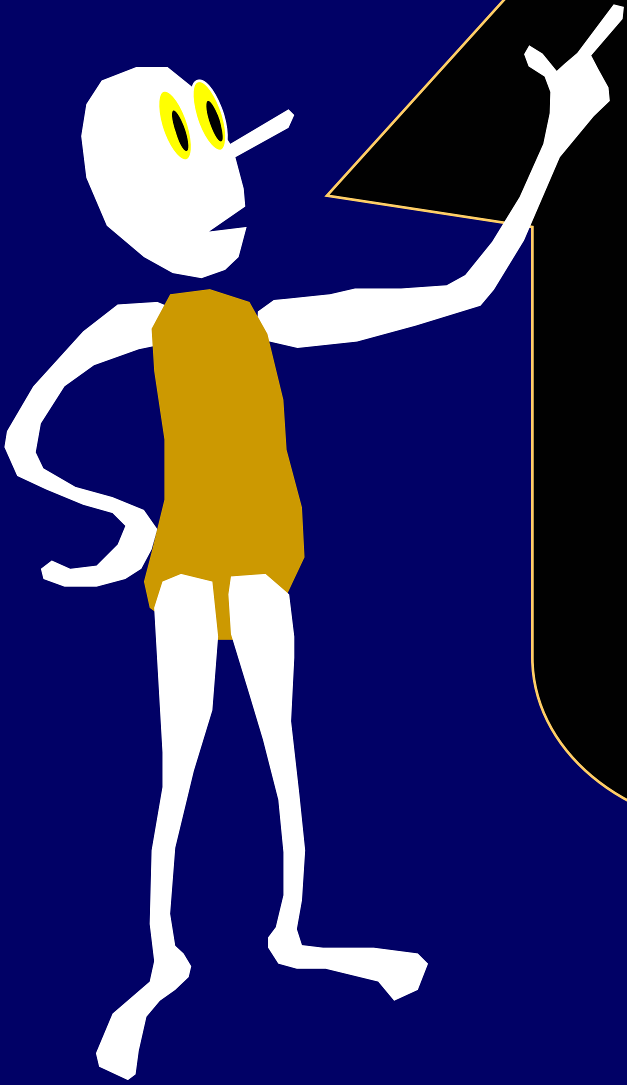
After measuring the dimensions of pictures, cards, books, snuff boxes, writing paper, windows, and such, psychologist Gustav Fechner claimed that the preferred rectangle had sides in the golden ratio (1871).

Which is the most attractive rectangle?



Which is the most attractive rectangle?





Let’s take a break from the Fibonacci Numbers in order to remark on polynomial division.

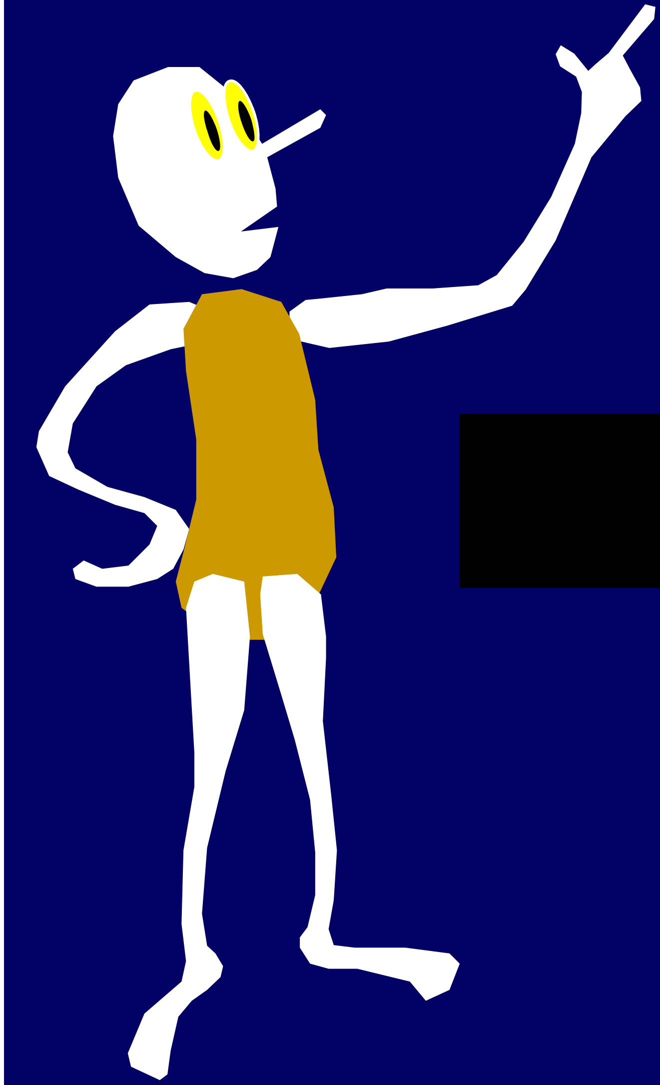
How to divide polynomials?

$$\frac{1}{1 - X}\ ?$$

$$\begin{array}{r|l}
 & 1 + X + X^2 \\ \hline
1 - X & 1 \\
 & \underline{-(1 - X)} \\
 & \qquad X \\
 & \qquad \underline{-(X - X^2)} \\
 & \qquad\qquad X^2 \\
 & \qquad\qquad \underline{-(X^2 - X^3)} \\
 & \qquad\qquad\qquad X^3 \quad \ldots
\end{array}$$

$$= \quad 1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7 + \ldots$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

The limit as n goes to infinity of

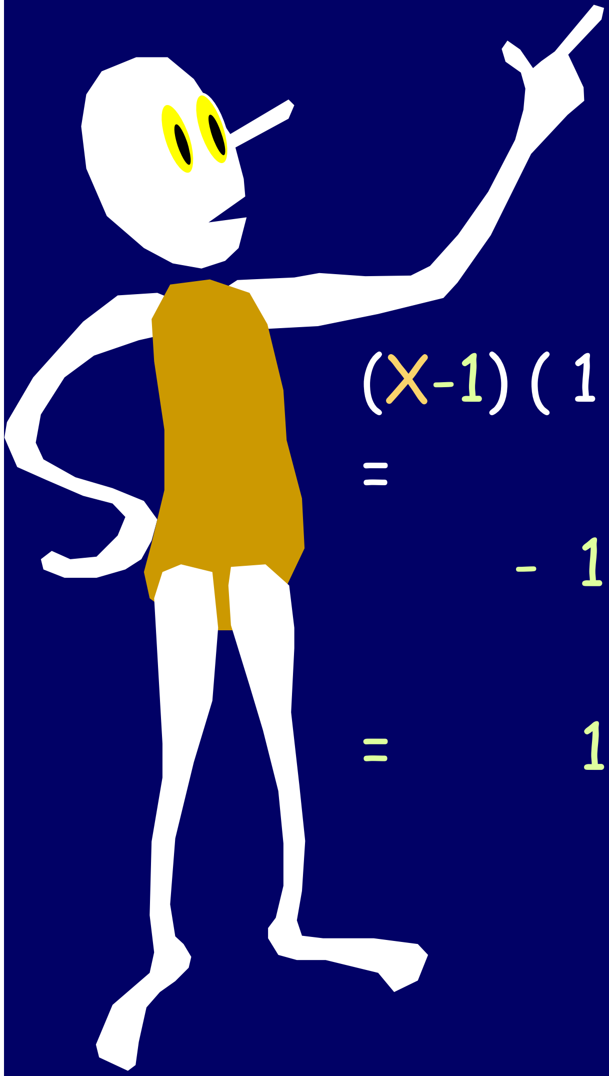
$$\frac{X^{n+1} - 1}{X - 1} = \frac{- 1}{X - 1}$$

$$= \frac{1}{1 - X}$$

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + ... = \frac{1}{1 - X}$$



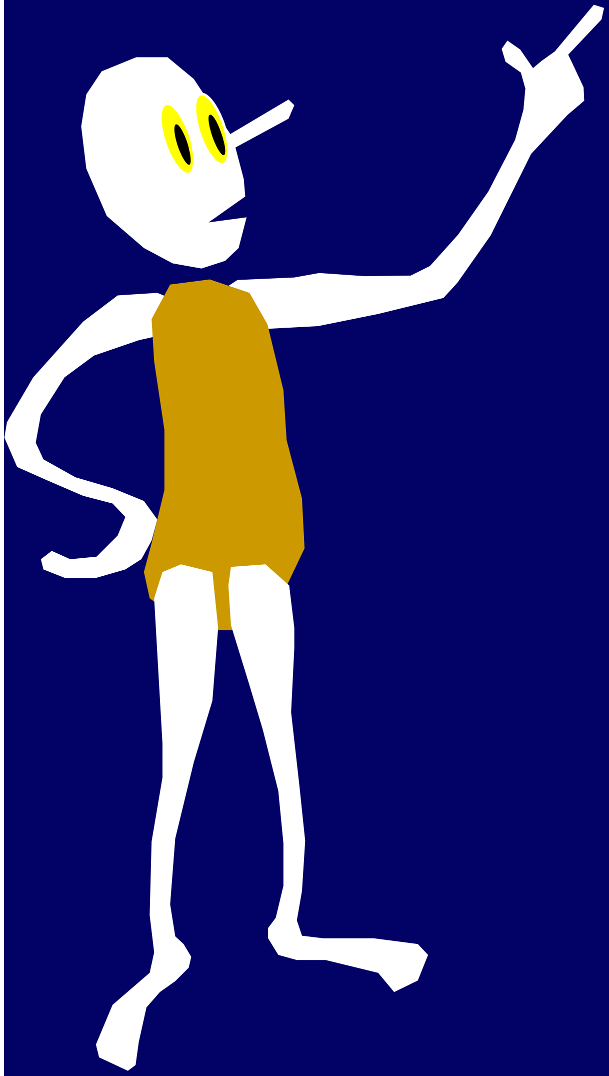
$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$$

$$= \quad X^1 + X^2 + X^3 + \ldots \quad + X^n + X^{n+1} + ...$$

$$- 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$$

$$= \quad 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



$$\begin{array}{r|l}
 & 1 + X + X^2 + \ldots \\ \hline
1 - X & 1 \\
 & \underline{-(1 - X)} \\
 & \quad X \\
 & \quad \underline{-(X - X^2)} \\
 & \qquad X^2 \\
 & \qquad \underline{-(X^2 - X^3)} \\
 & \qquad\quad X^3 \\
 & \qquad\qquad \ldots
\end{array}$$

Something a bit more complicated

$$X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6$$

$$1 - X - X^2 \; \overline{\big) \; X}$$

$$-(X - X^2 - X^3)$$

$$X^2 + X^3$$

$$-(X^2 - X^3 - X^4)$$

$$2X^3 + X^4$$

$$-(2X^3 - 2X^4 - 2X^5)$$

$$3X^4 + 2X^5$$

$$-(3X^4 - 3X^5 - 3X^6)$$

$$5X^5 + 3X^6$$

$$-(5X^5 - 5X^6 - 5X^7)$$

$$8X^6 + 5X^7$$

$$-(8X^6 - 8X^7 - 8X^8)$$

$$\boxed{\frac{X}{1 - X - X^2}}$$

Hence

$$\frac{X}{1 - X - X^2}$$

$$= 0\times1 + 1\,X^1 + 1\,X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \ldots$$

$$= F_0\,1 + F_1\,X^1 + F_2\,X^2 + F_3\,X^3 + F_4\,X^4 + F_5\,X^5 + F_6\,X^6 + \ldots$$

Going the Other Way

$(1 - X - X^2) \times$

$(F_0\, 1 + F_1\, X^1 + F_2\, X^2 + \ldots + F_{n-2}\, X^{n-2} + F_{n-1}\, X^{n-1} + F_n\, X^n + \ldots$

$= (F_0\, 1 + F_1\, X^1 + F_2\, X^2 + \ldots + F_{n-2}\, X^{n-2} + F_{n-1}\, X^{n-1} + F_n\, X^n + \ldots$

$- F_0\, X^1 - F_1\, X^2 - \ldots - F_{n-3}\, X^{n-2} - F_{n-2}\, X^{n-1} - F_{n-1}\, X^n - \ldots$

$- F_0\, X^2 - \ldots - F_{n-4}\, X^{n-2} - F_{n-3}\, X^{n-1} - F_{n-2}\, X^n - \ldots$

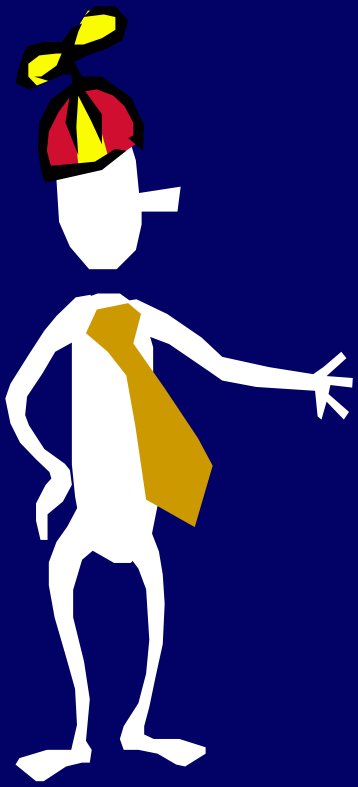
$= F_0\, 1 + (F_1 - F_0)\, X^1$

$= X$

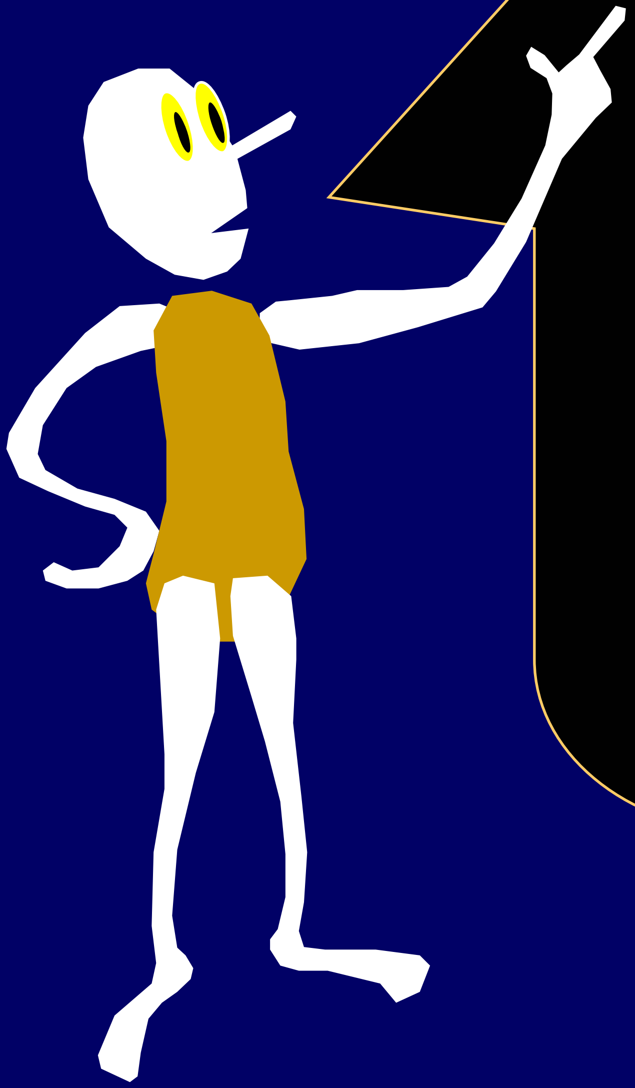
$\boxed{F_0 = 0, F_1 = 1}$

Thus

$$F_0\,1 + F_1\,X^1 + F_2\,X^2 + \ldots + F_{n-1}\,X^{n-1} + F_n\,X^n + \ldots$$



$$= \frac{X}{1 - X - X^2}$$



I was trying to get away from them!

Vector Recurrence Relations

Let P be a vector program that takes input.

A vector relation is any statement of the form:

$$V^{\rightarrow} = P(V^{\rightarrow})$$

If there is a unique $V^{\rightarrow}$ satisfying the relation, then $V^{\rightarrow}$ is said to be defined by the relation $V^{\rightarrow} = P(V^{\rightarrow})$.

Fibonacci Numbers

Recurrence Relation Definition:

$$F_0 = 0, \quad F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2}, n > 1$$

Vector Recurrence Relation Definition:

$F^{\rightarrow}$ = RIGHT($F^{\rightarrow}$+ <1>) + RIGHT(RIGHT($F^{\rightarrow}$))

$$\boxed{F^{\rightarrow} = \mathrm{RIGHT}(F^{\rightarrow} + \langle 1 \rangle) + \mathrm{RIGHT}(\mathrm{RIGHT}(F^{\rightarrow}))}$$

$$F^{\rightarrow} = a_0,\ a_1 \qquad ,\ a_2,\ a_3,\ a_4,\ .\ .\ .$$

$$\mathrm{RIGHT}(F^{\rightarrow} + \langle 1 \rangle) = 0,\ a_0 + 1,\ a_1,\ a_2,\ a_3,$$

$$\mathrm{RIGHT}(\mathrm{RIGHT}(F^{\rightarrow}))$$

$$= 0,\ 0 \qquad ,\ a_0,\ a_1,\ a_2,\ a_3,\ .$$

$$\boxed{F^{\rightarrow} = \text{RIGHT}(F^{\rightarrow} + 1) + \text{RIGHT}(\text{RIGHT}(F^{\rightarrow}))}$$

$$F = a_0 + a_1 X + a_2 X^2 + a_3 X^3 +$$

$$\text{RIGHT}(F + 1) = (F+1) X$$

$$\text{RIGHT}(\text{RIGHT}(F))$$

$$= F X^2$$

$$\boxed{F = (F + 1)\,X + F\,X^2}$$

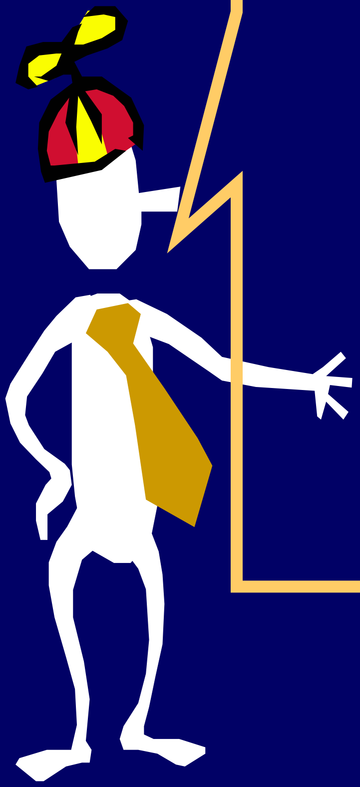
$$F = \quad a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \;.\;\;.\;\;.$$

$$\mathrm{RIGHT}(F + 1) = \;(F{+}1)\,X$$

$$\mathrm{RIGHT}(\,\mathrm{RIGHT}(\,F\,)\,)$$

$$= \qquad F\,X^2$$

$$F = FX + X + FX^2$$

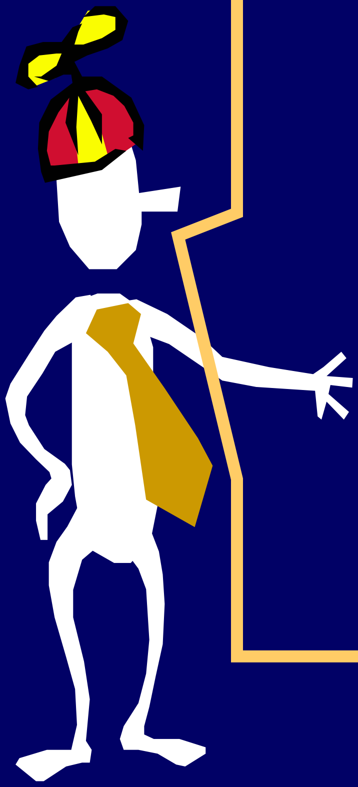


Solve for F.

$F - FX - FX^2 = X$

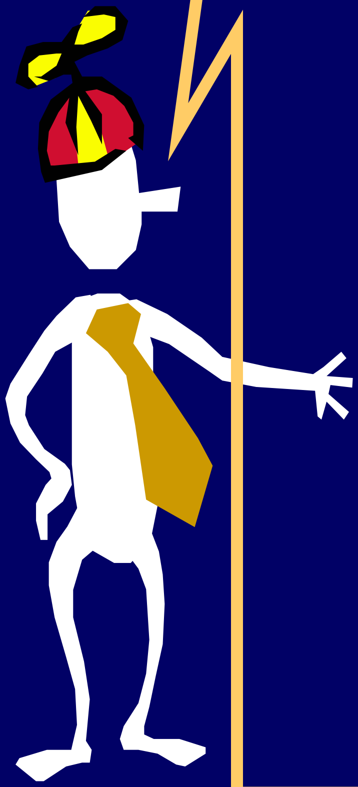
$F(1-X-X^2) = X$

$F = X/(1-X-X^2)$



What is the Power Series Expansion of $x / (1-x-x^2)$?

What does this look like when we expand it as an infinite sum



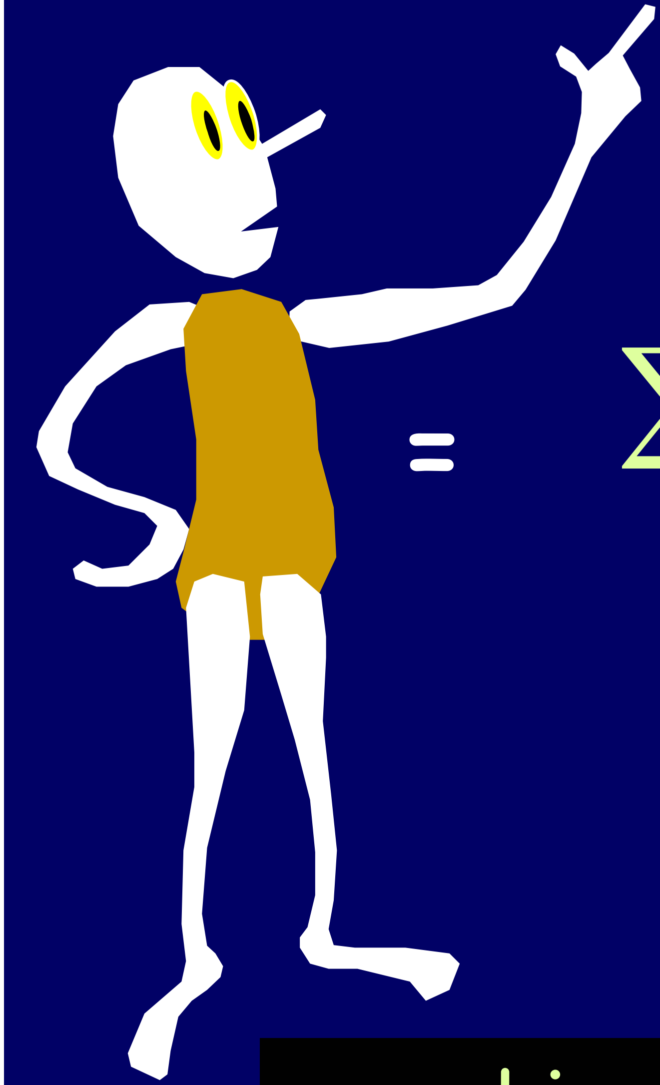
Since the bottom is quadratic we can factor it.

$$X / (1-X-X^2) =$$

$$X/(1- \phi X)(1 - (-\phi)^{-1}X)$$

$$\text{where } \phi = \frac{1+\sqrt{5}}{2}$$

"The Golden Ratio"

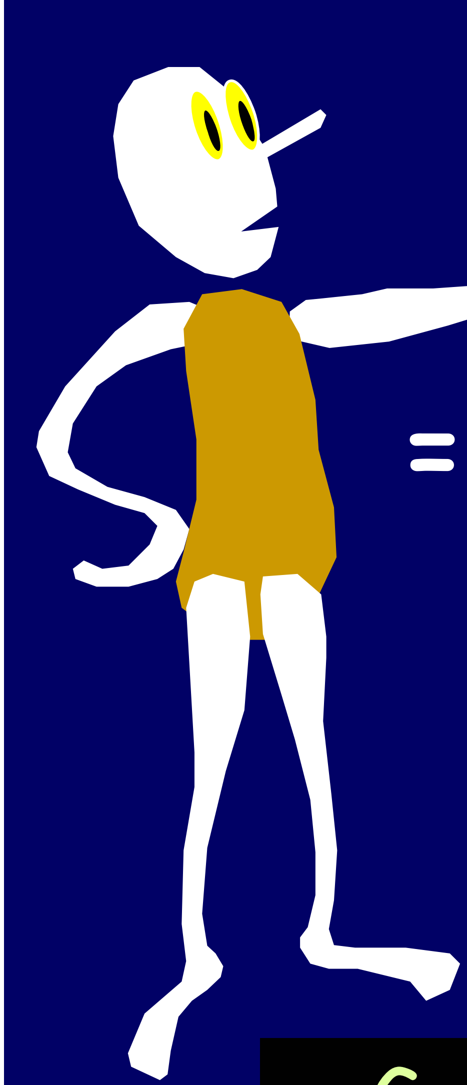


$$\frac{X}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \quad \sum_{n=0..\infty} \quad ? \quad X^n$$

Linear factors on the bottom

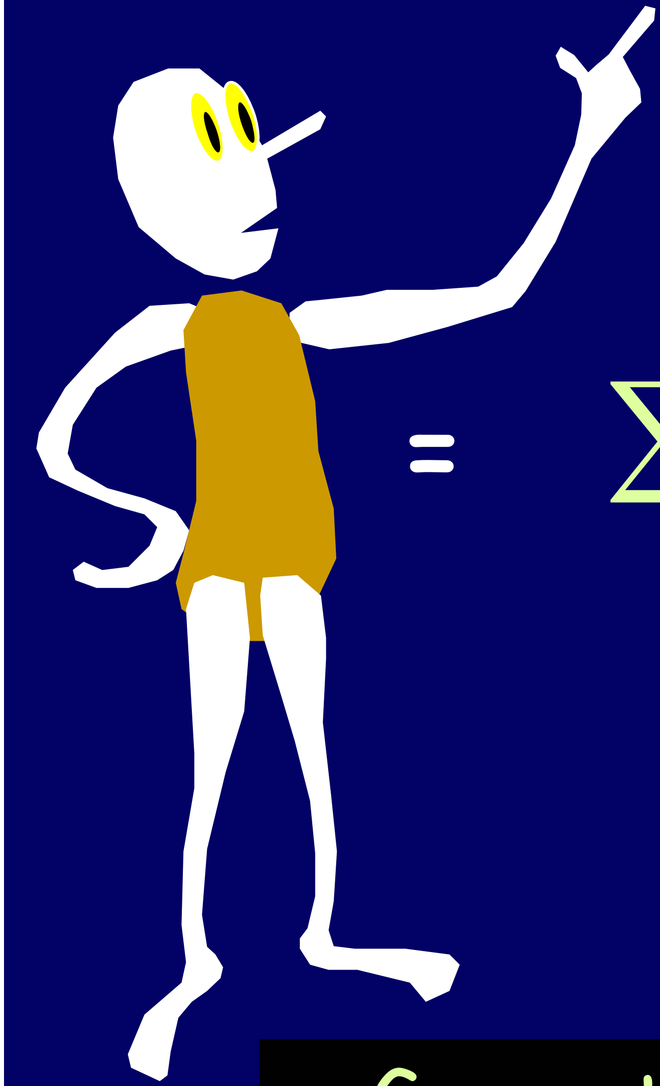
$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$



$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

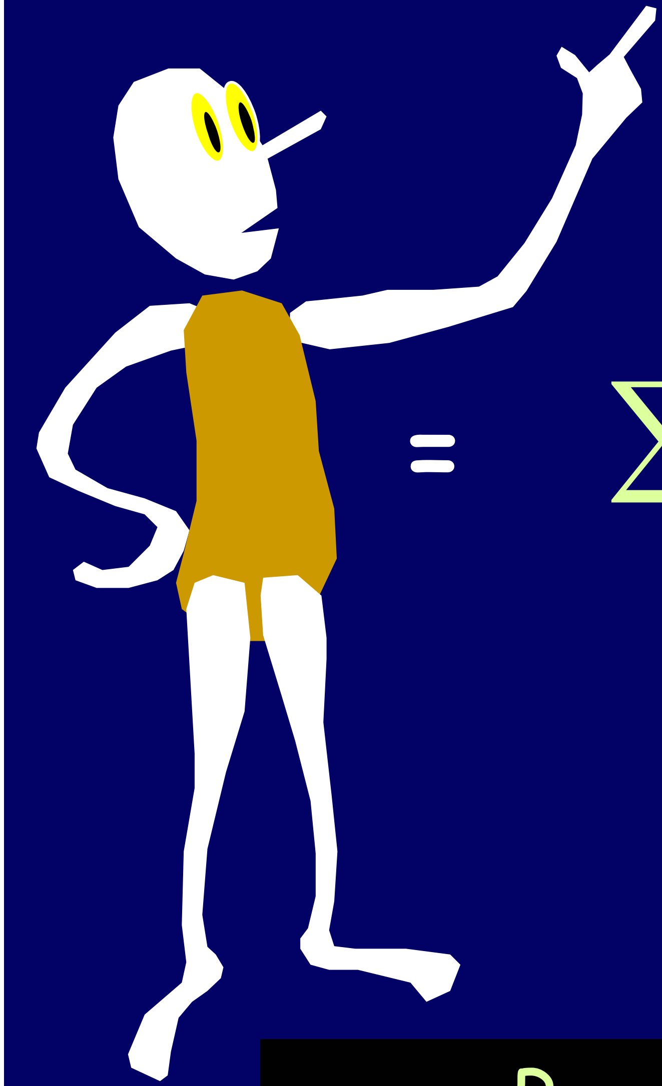
Geometric Series (Quadratic Form)



$$\frac{1}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \quad \sum_{n=0..\ \infty} \quad \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^n$$

Geometric Series (Quadratic Form)

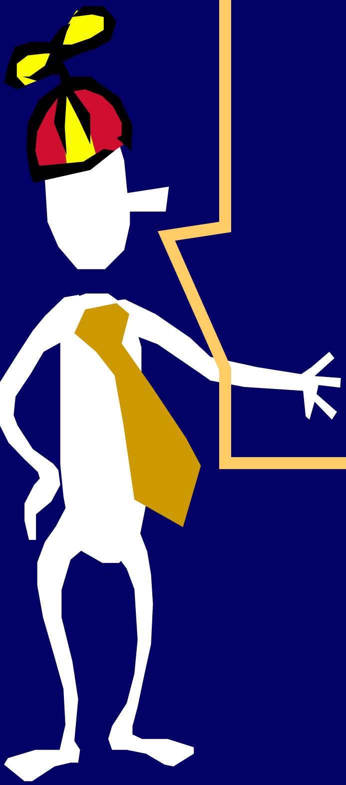


$$\frac{X}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \sum_{n=0.. \infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^{n+1}$$

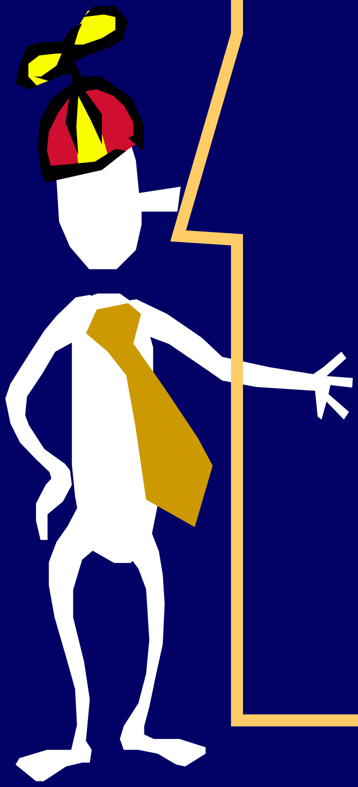
Power Series Expansion of F

$$\frac{x}{1-x-x^2} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$



$$\frac{x}{1-x-x^2} = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}} \left(\phi^i - \left(-\frac{1}{\phi} \right)^i \right) x^i$$

Leonhard Euler (1765)
J. P. M. Binet (1843)
A de Moivre (1730)



The ith Fibonacci number is:

$$\frac{1}{\sqrt{5}}\left(\phi^{i}-\left(-\frac{1}{\phi}\right)^{i}\right)$$

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

Example: $f_5 = 5$

4 = 2 + 2
2 + 1 + 1
1 + 2 + 1
1 + 1 + 2
1 + 1 + 1 + 1

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$f_2 = 1$

1 = 1

$f_1 = 1$

0 = the empty sum

$f_3 = 2$

2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

# of sequences beginning with a 1

# of sequences beginning with a 2

Fibonacci Numbers Again

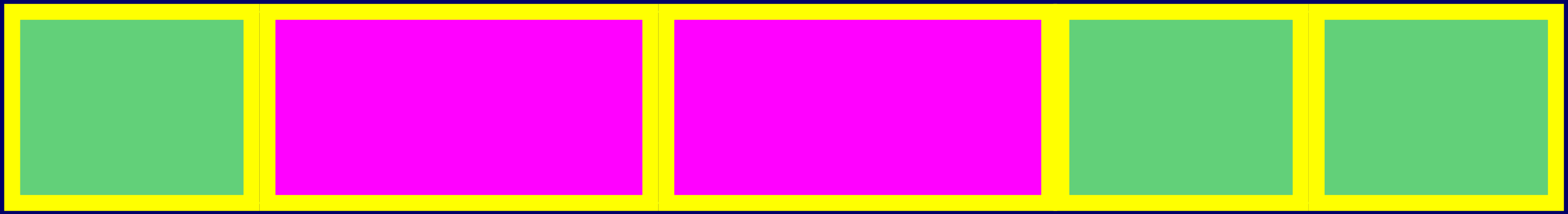
Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

$$f_1 = 1 \qquad f_2 = 1$$

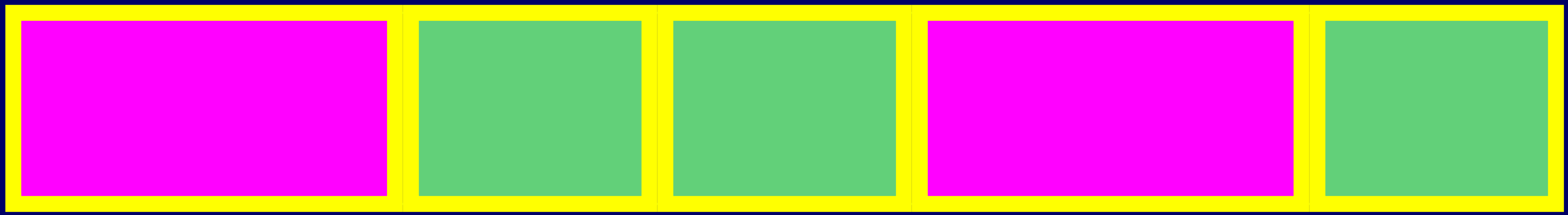
Visual Representation: Tiling

Let f_{n+1} be the number of different ways to tile a 1 X n strip with squares and dominoes.



Visual Representation: Tiling

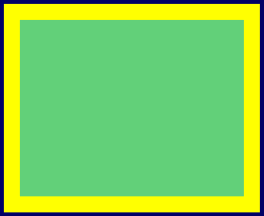
Let f_{n+1} be the number of different ways to tile a 1 X n strip with squares and dominoes.



Visual Representation: Tiling

1 way to tile a strip of length 0

1 way to tile a strip of length 1:



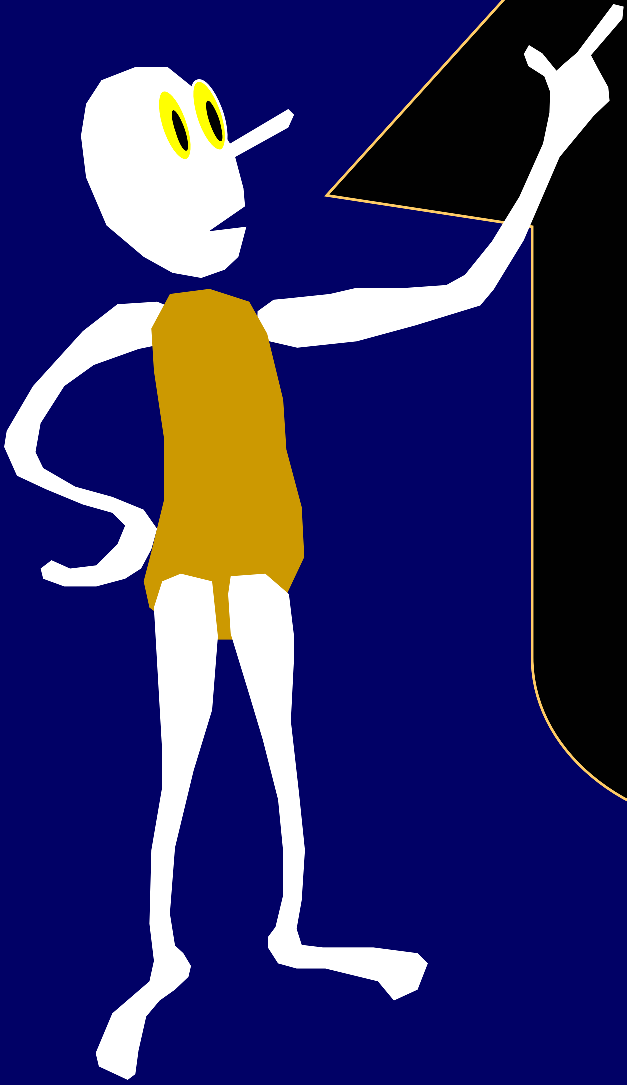
2 ways to tile a strip of length 2:

$f_{n+1} = f_n + f_{n-1}$

f_{n+1} is number of ways to title length n.

f_n tilings that start with a square.

f_{n-1} tilings that start with a domino.



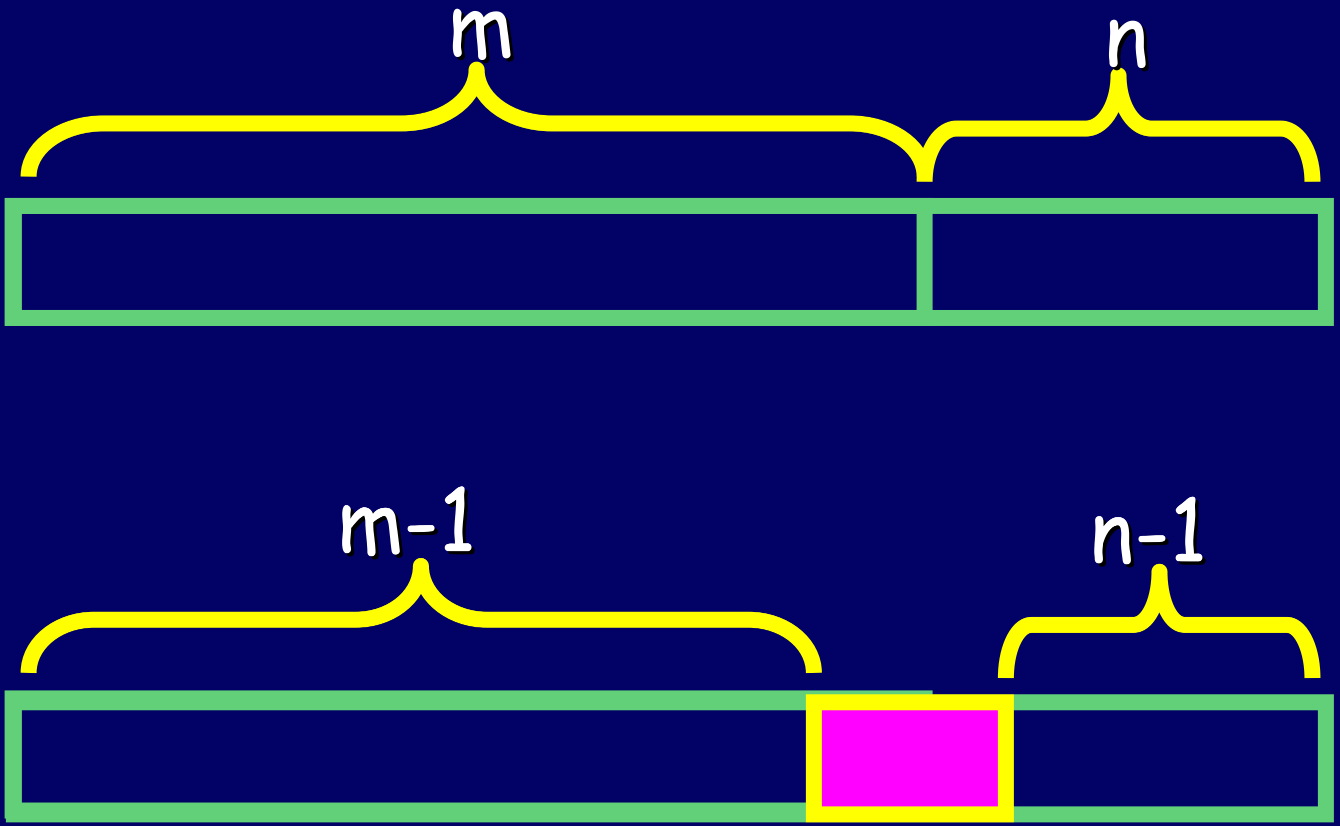
Let's use this visual representation to prove a couple of Fibonacci identities.

Fibonacci Identities

The Fibonacci numbers have many unusual properties. The many properties that can be stated as equations are called Fibonacci identities.

Ex: $F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$



$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

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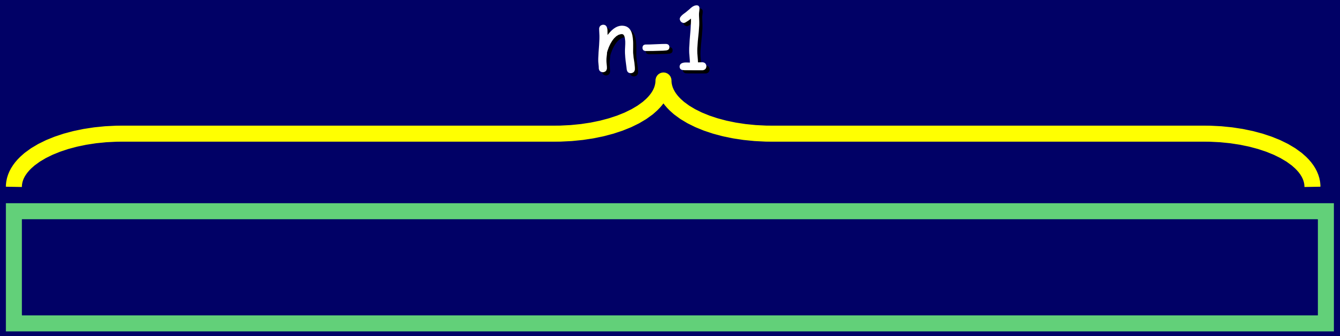
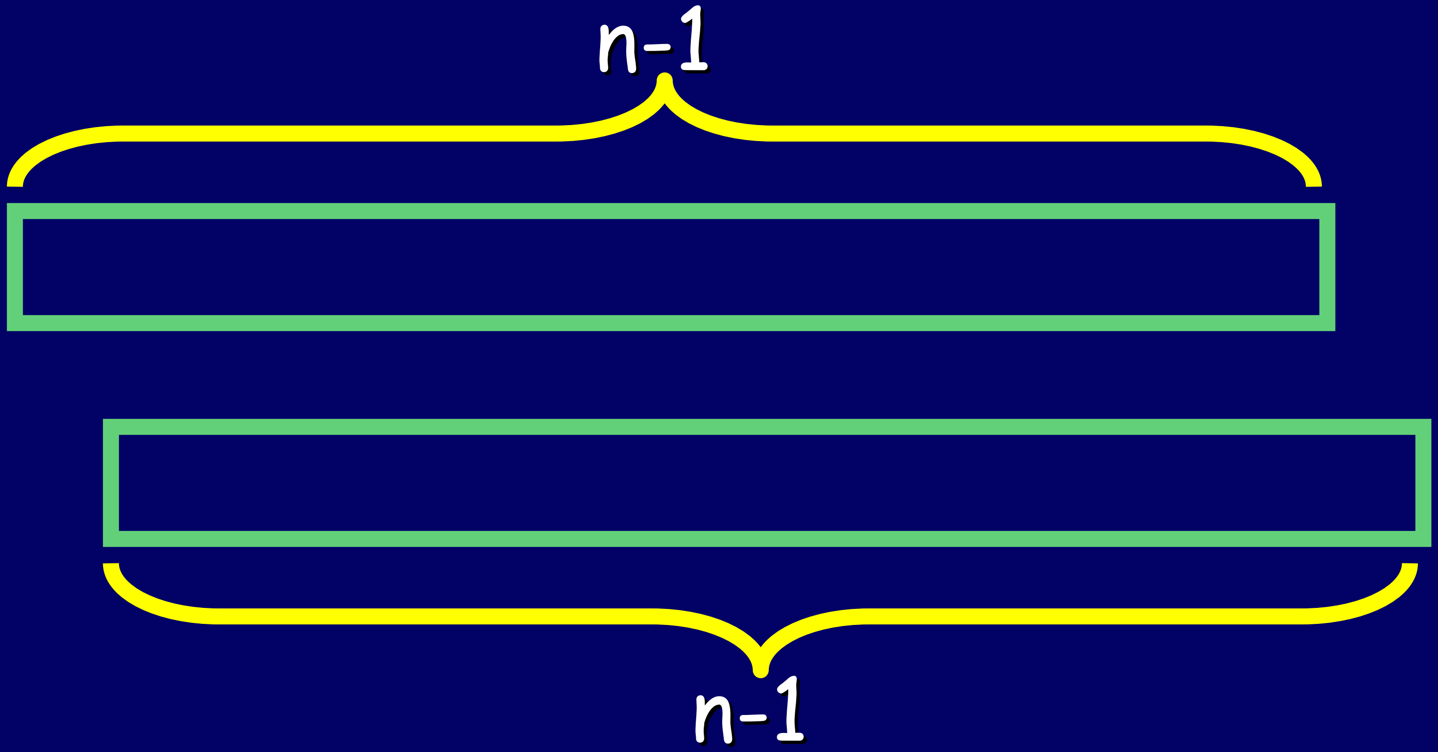
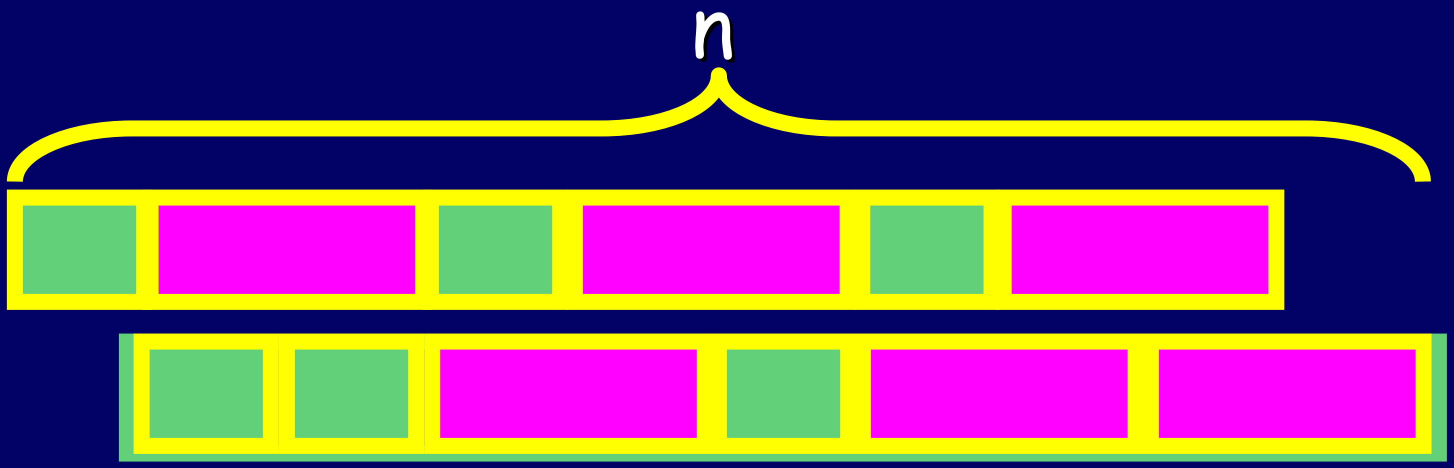


F_n tilings of a strip of length n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

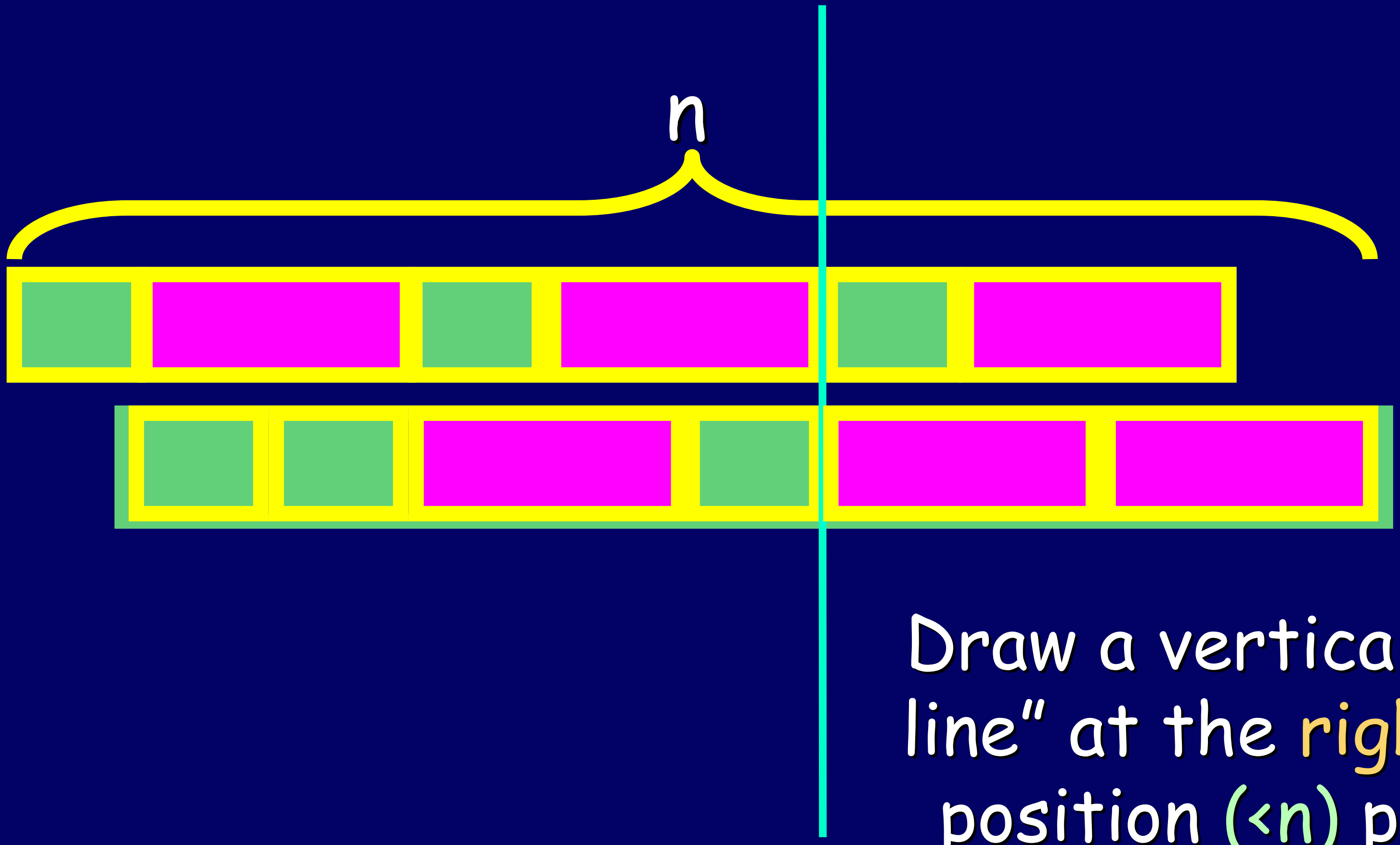


$$(F_n)^2 \quad = \quad F_{n-1}\, F_{n+1} \quad + \quad (-1)^n$$



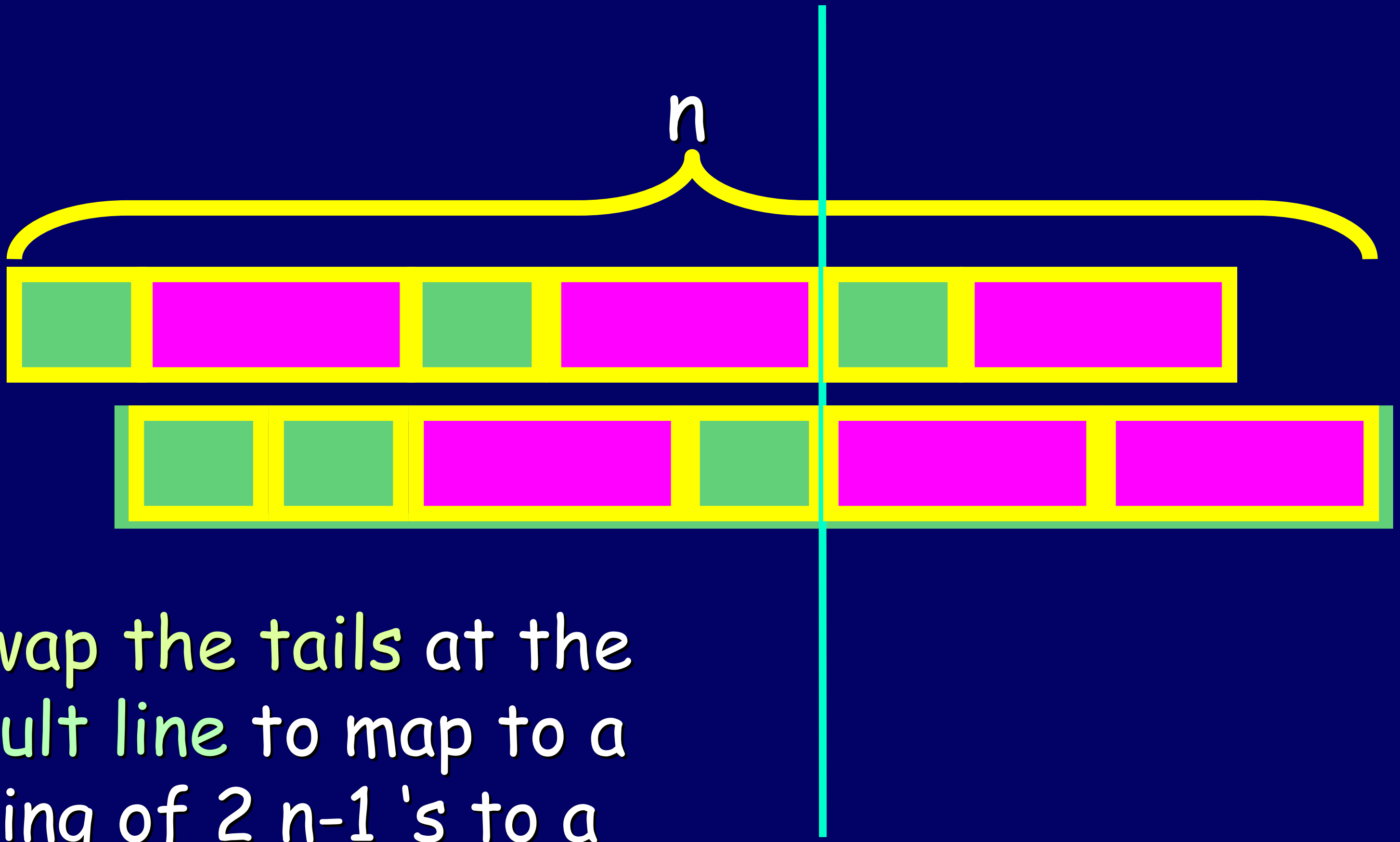
$(F_n)^2$ tilings of two strips of size n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



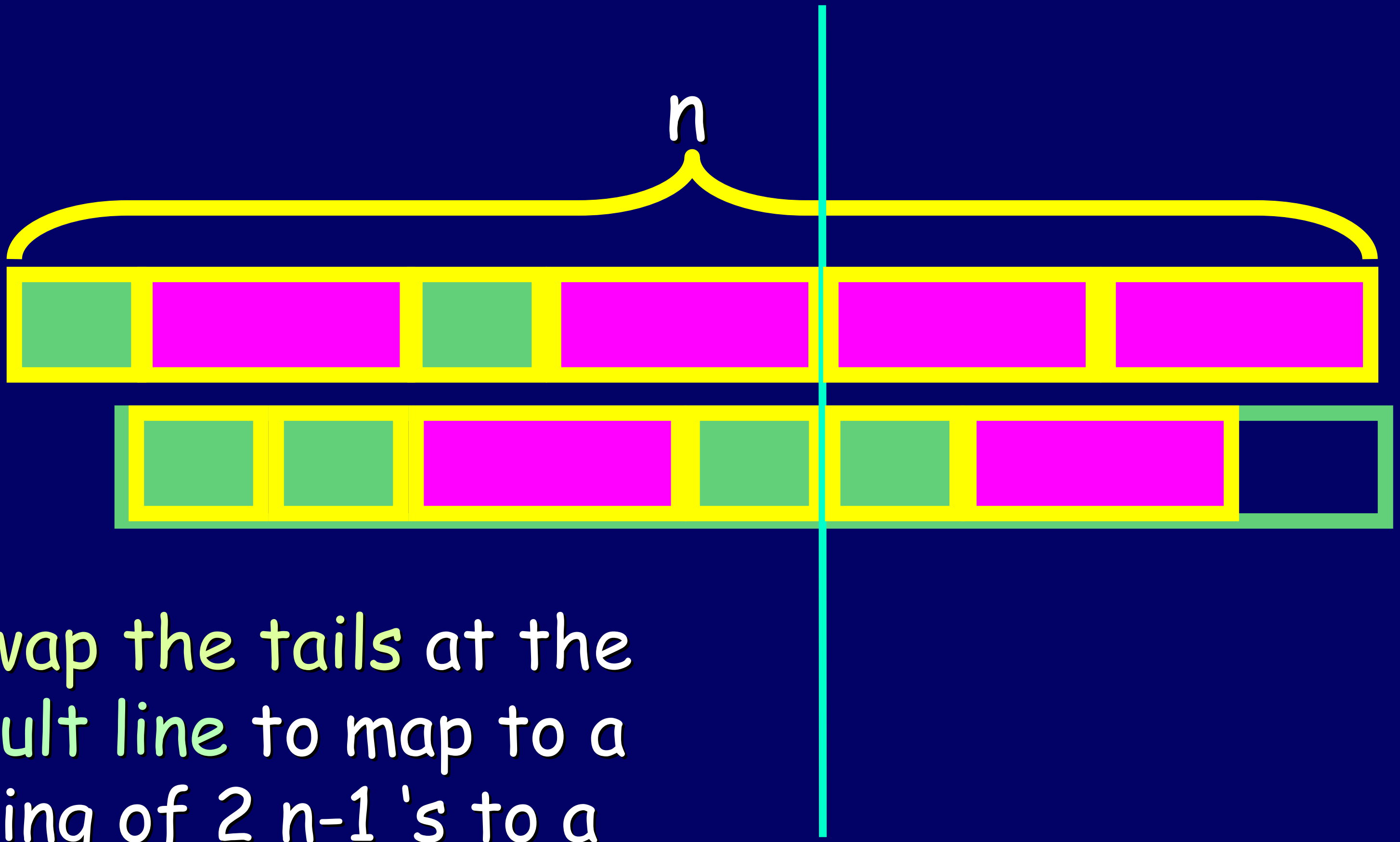
Draw a vertical "fault line" at the rightmost position (<n) possible without cutting any dominoes

$$(F_n)^2 = F_{n-1}F_{n+1} + (-1)^n$$



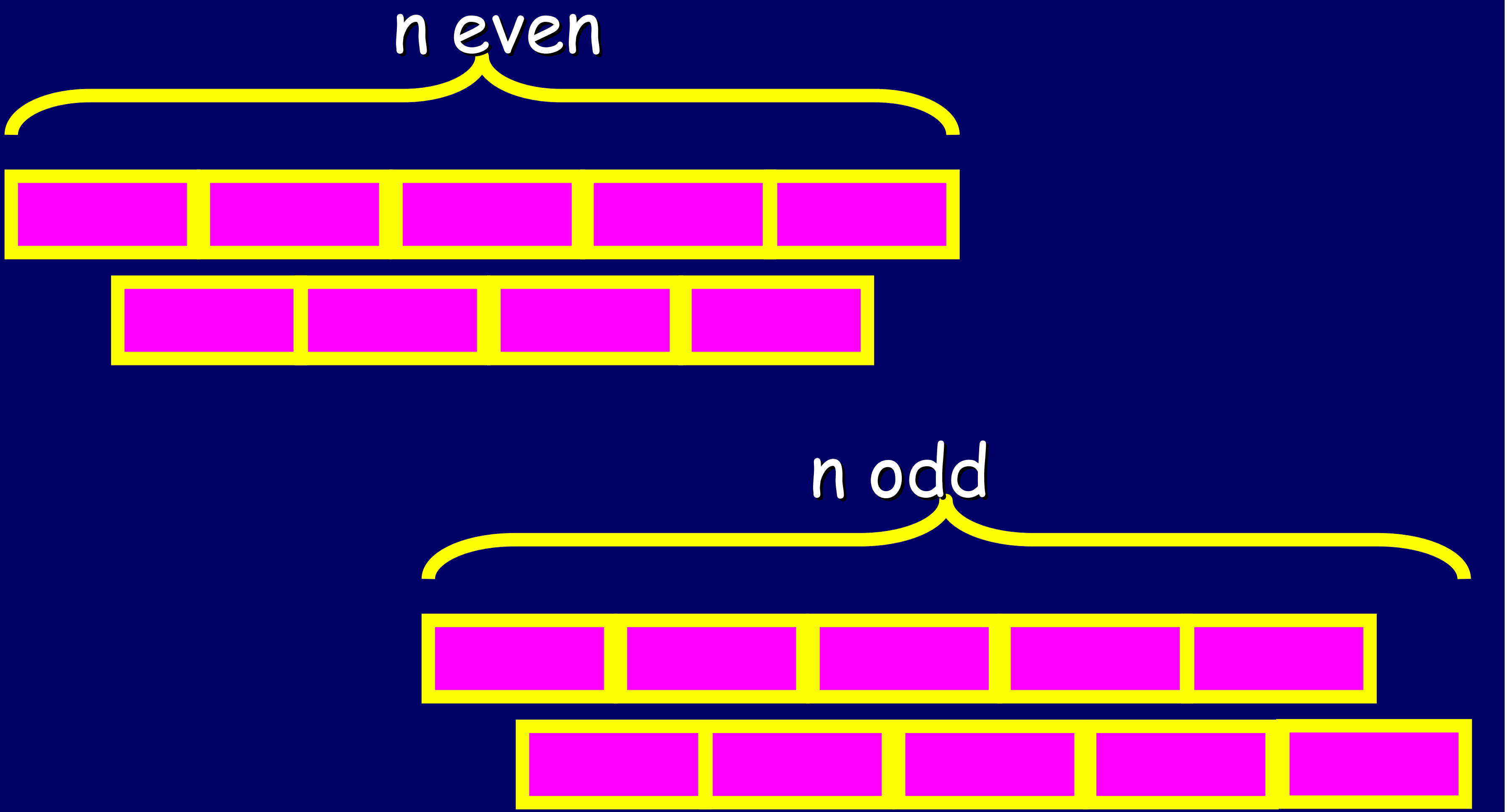
Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1}F_{n+1} + (-1)^n$$



Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n-1}$$



$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}} \boxed{-\frac{\left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}}$$

Less than .277

$$F_n = \text{closest integer to } \frac{\phi^n}{\sqrt{5}} = \left[\frac{\phi^n}{\sqrt{5}}\right]$$

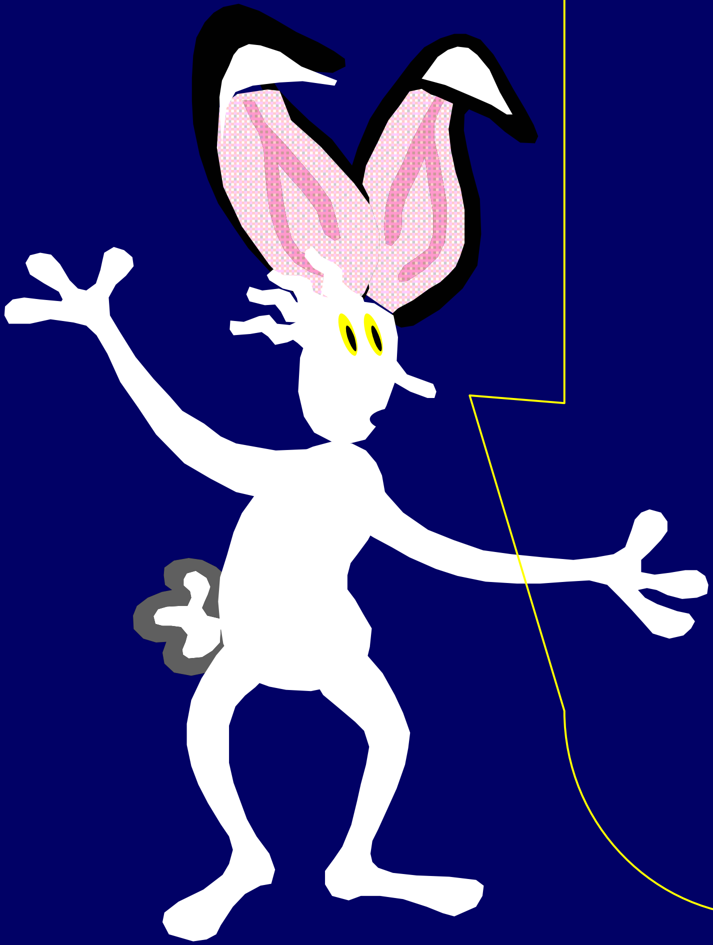
$$\frac{\mathbf{F_n}}{\mathbf{F_{n-1}}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} = \frac{\phi^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} + \frac{-\left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}}$$

$$\boxed{\lim_{n\to\infty} \frac{\mathbf{F_n}}{\mathbf{F_{n-1}}} = \phi}$$

1,1,2,3,5,8,13,21,34,55,....

2/1 = 2
3/2 = 1.5
5/3 = 1.666...
8/5 = 1.6
13/8 = 1.625
21/13= 1.6153846...
34/21= 1.61904...

ϕ = 1.6180339887498948482045



How could someone have figured that out?

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

POLYA:

When you want to find a solution to two simultaneous constraints, first characterize the solution space to one of them, and then find a solution to the second that is within the space of the first.

A technique to derive the formula for the Fibonacci numbers

F_n is defined by two conditions:

<u>Base condition</u>: $F_0=0$, $F_1=1$

<u>Inductive condition</u>: $F_n=F_{n-1}+F_{n-2}$

Forget the base condition and concentrate on satisfying the inductive condition

Inductive condition: $F_n=F_{n-1}+F_{n-2}$

Consider solutions of the form:

$F_n= c^n$ for some complex constant c

C must satisfy:

$$c^n - c^{n-1} - c^{n-2} = 0$$

$$c^n - c^{n-1} - c^{n-2} = 0$$

iff $c^{n-2}(c^2 - c^1 - 1) = 0$

iff $c=0$ or $c^2 - c^1 - 1 = 0$

Iff $c = 0$, $c = \phi$, or $c = -(1/\phi)$

$$c = 0,\ c = \phi,\ \text{or}\ c = -(1/\phi)$$

So for all these values of c the inductive condition is satisfied:

$$c^n - c^{n-1} - c^{n-2} = 0$$

Do any of them happen to satisfy the base condition as well? $c^0=0$ and $c^1=1$?

ROTTEN LUCK

Insight: if 2 functions $g(n)$ and $h(n)$ satisfy the inductive condition then so does $a\,g(n) + b\,h(n)$ for all complex a and b

$g(n)-g(n-1)-g(n-2)=0$ $ag(n)-ag(n-1)-ag(n-2)=0$	$h(n)-h(n-1)-h(n-2)=0$ $bh(n)-bh(n-1)-bh(n-2)=0$

$$(a\,g(n) + b\,h(n)) + (a\,g(n-1) + b\,h(n-1)) + (a\,g(n-2) + b\,h(n-2)) = 0$$

$\forall a,b \quad a\,\phi^n + b\,(-1/\phi)^n$ satisfies the inductive condition

Set a and b to fit the base conditions.

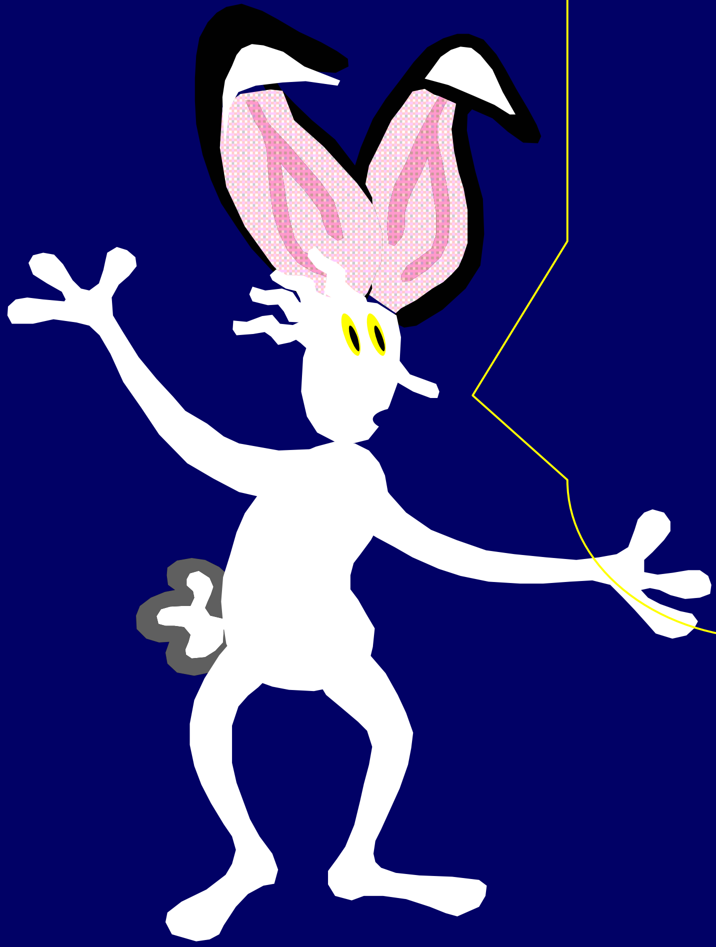
n=0 : $a + b = 0$

n=1 : $a\,\phi^1 + b\,(-1/\phi)^1 = 1$

Two equalities in two unknowns (a and b).

Now solve for a and b:

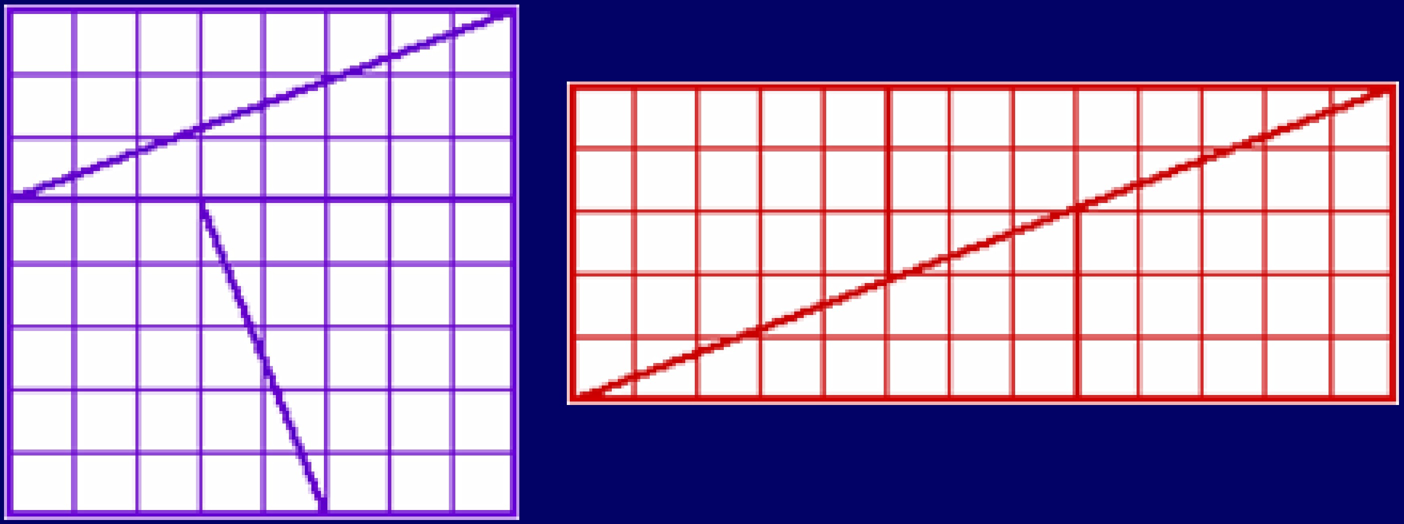
this gives $a = 1/\sqrt{5} \quad b = -1/\sqrt{5}$



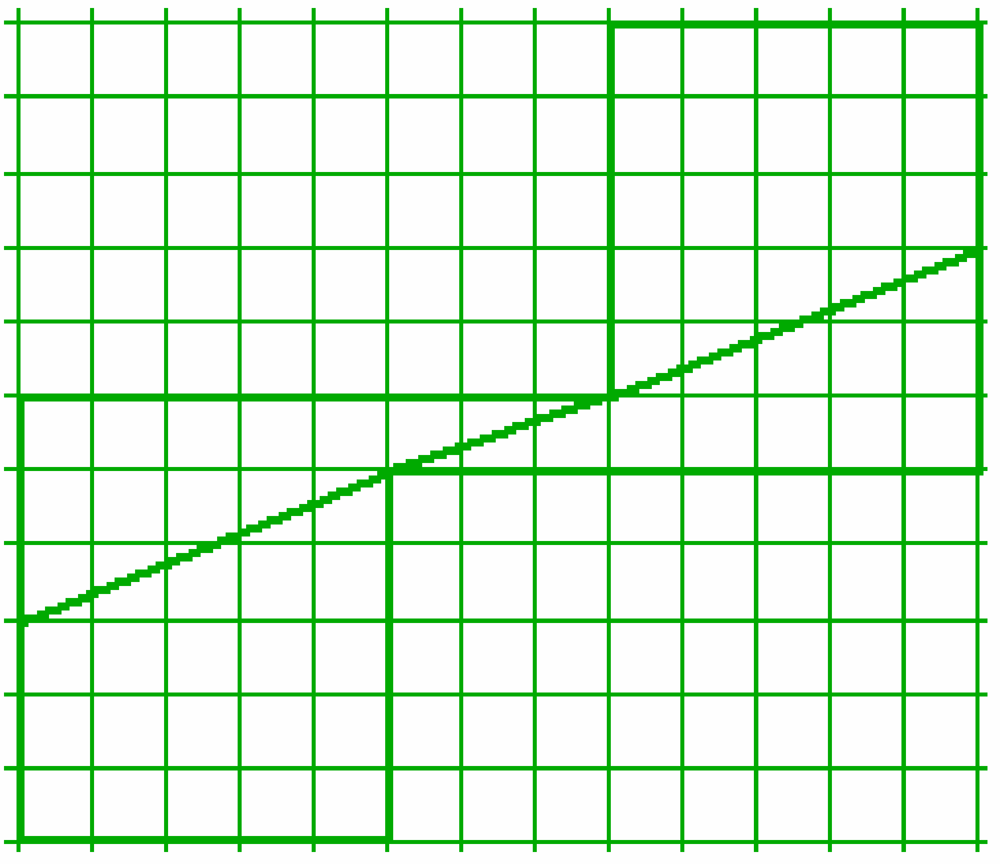
We have just proved Euler's result:

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

Fibonacci Magic Trick



Another Trick!



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"Recounting Fibonacci and Lucas Identities" by Arthur T. Benjamin and Jennifer J. Quinn, The CollegeMathematics Journal, Vol. 30, No. 5, 1999, pp. 359--366.