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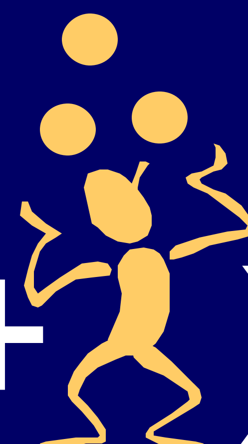
CS 15-251 Spring 2003

Lecture 11 Feb 14, 2004

Carnegie Mellon University

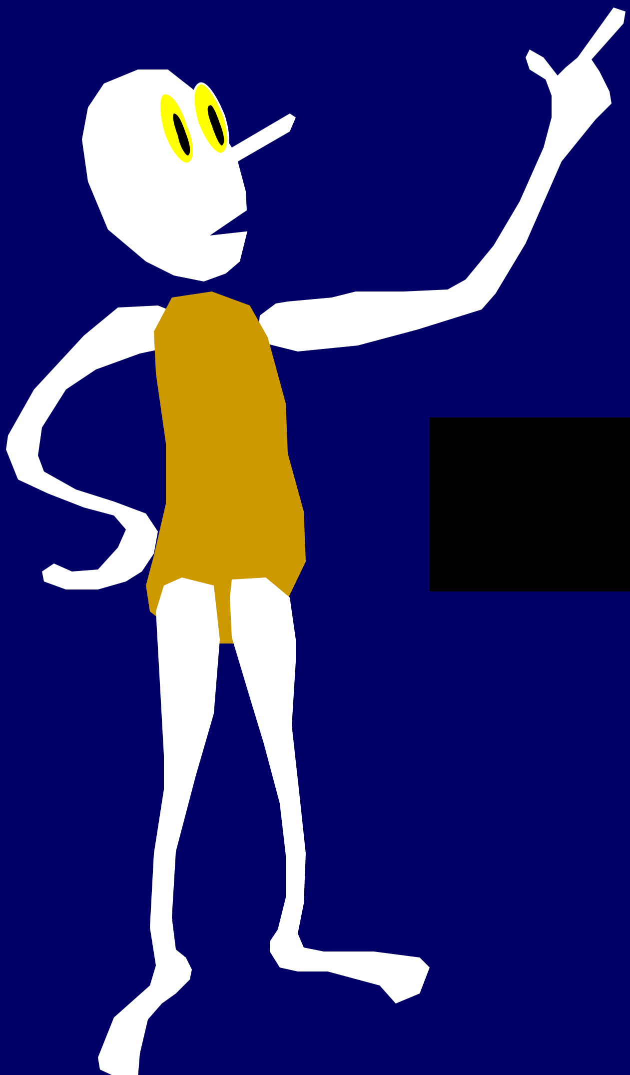
Counting III: Pascal's Triangle, Polynomials, and Vector Programs





$$x^1 + x^2 + x^3$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

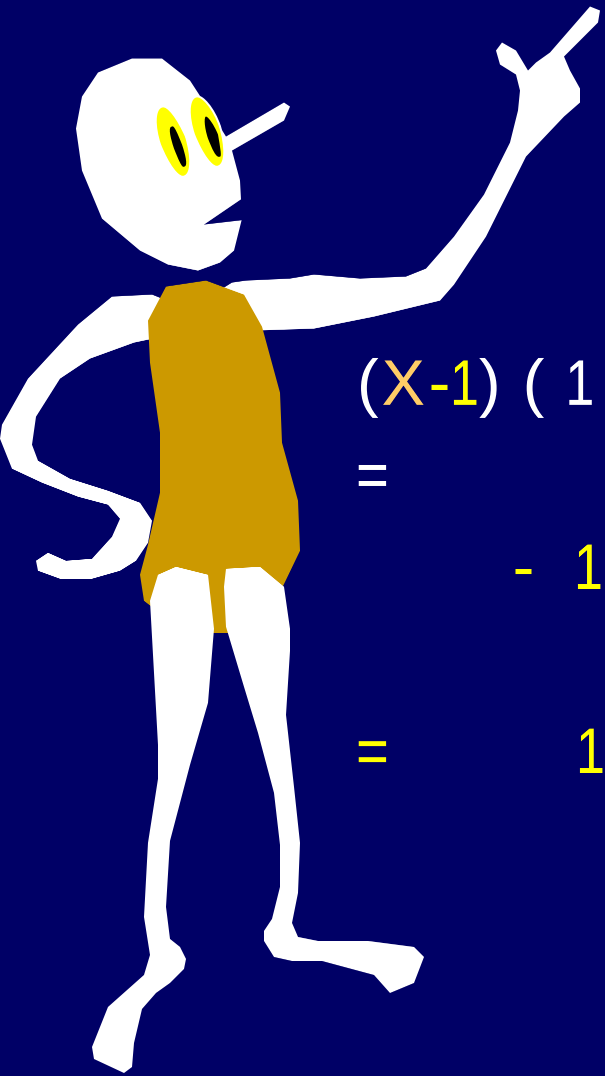


The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$



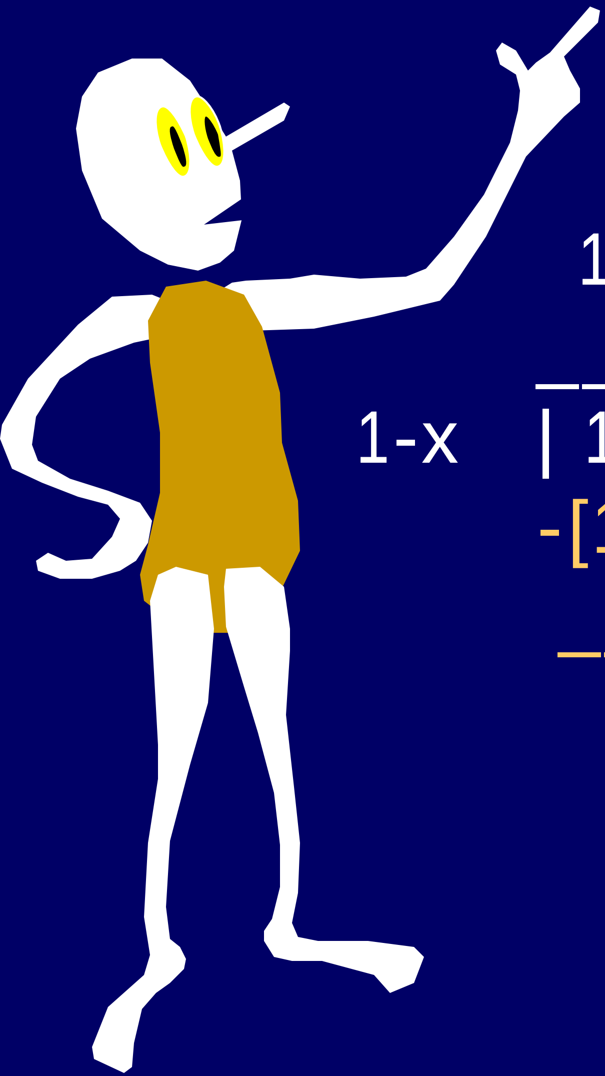
$$(X-1)\,(1 + X^1 + X^2 + X^3 + \dots + X^n + \dots)$$

$$= \quad X^1 + X^2 + X^3 + \dots \quad + X^n + X^{n+1} + \dots$$

$$- \;1 - X^1 - X^2 - X^3 - \dots - X^{n-1} - X^n - X^{n+1} - \dots$$

$$= \quad 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



$1 + x + \ldots\ldots$

$1\text{-}x \; | \; 1$

$-[1 - x]$

x

$-[x - x^2]$

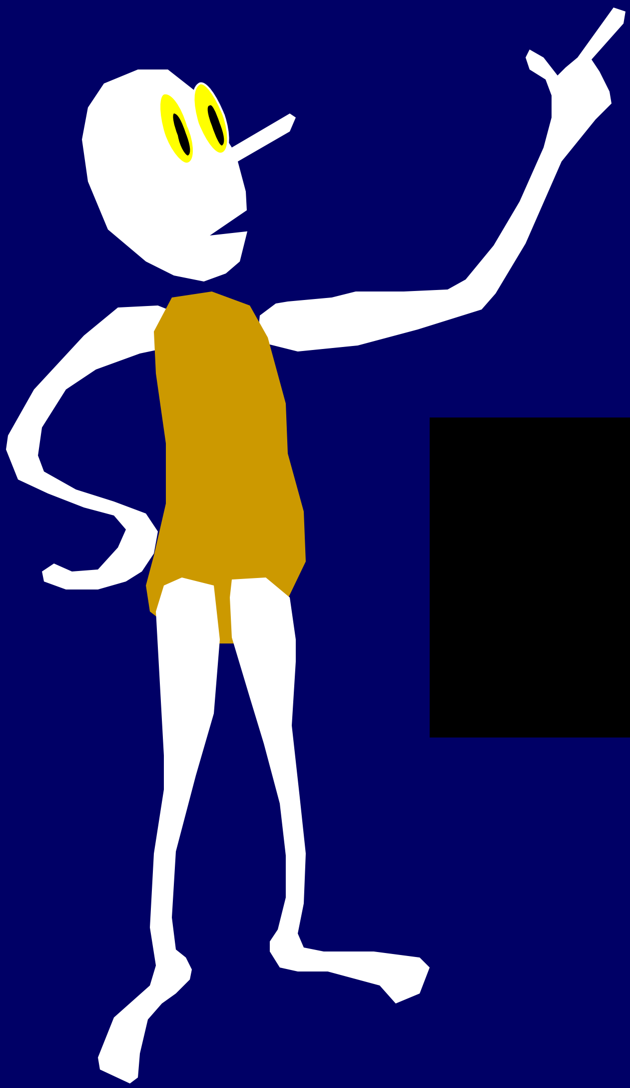
x^2

$-\ldots\ldots$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots\ldots = \frac{1}{1 - aX}$$

Geometric Series (Linear Form)

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots\right) = \frac{1}{(1 - aX)(1-bX)}$$



Geometric Series (Quadratic Form)

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots\right) =$$

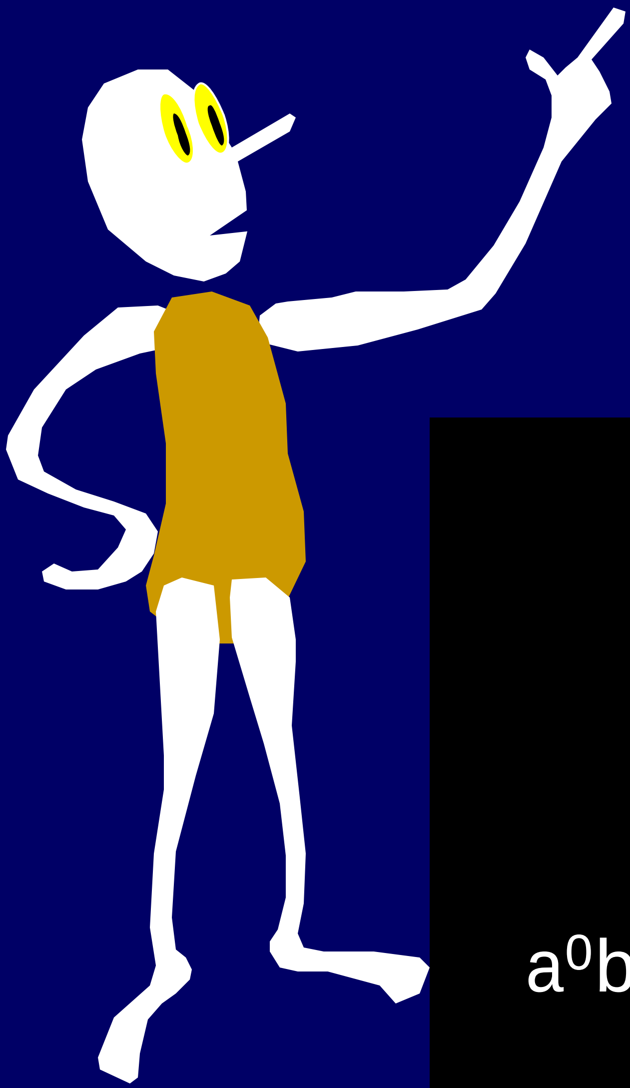
$$1 + c_1X^1 + \ldots + c_k X^k + \ldots$$

Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\ (1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + .. + c_k X^k + \dots$$

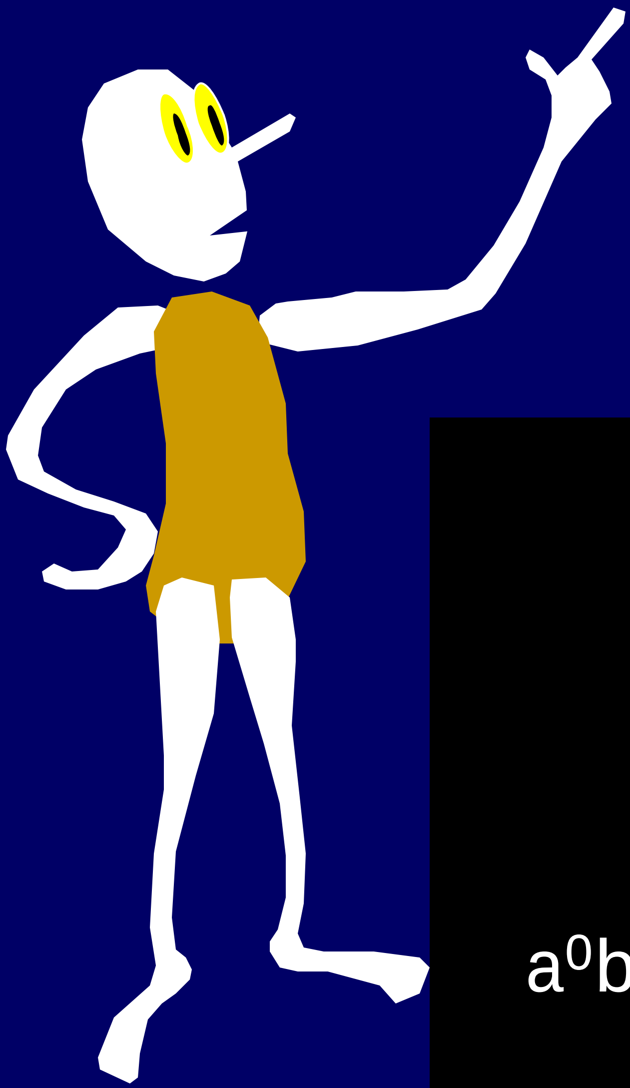


$$c_n =$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots\right) =$$

$$1 + c_1X^1 + \ldots + c_k X^k + \ldots$$

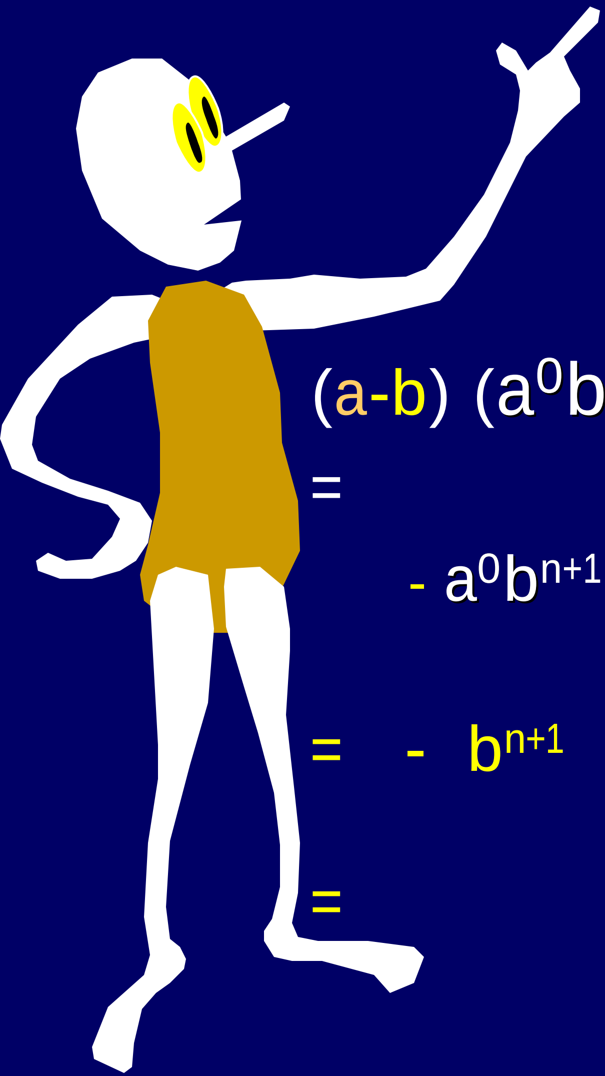


If a = b then

$$c_n = (n+1)(a^n)$$

$$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$$

$$a^0b^n + a^1b^{n-1} + \dots\ a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$$



$$(a-b)\ (a^0b^n + a^1b^{n-1} + \dots\ a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0)$$

$$= \quad a^1b^n + \dots\ a^{i+1}b^{n-i} \dots \quad + a^nb^1 + a^{n+1}b^0$$

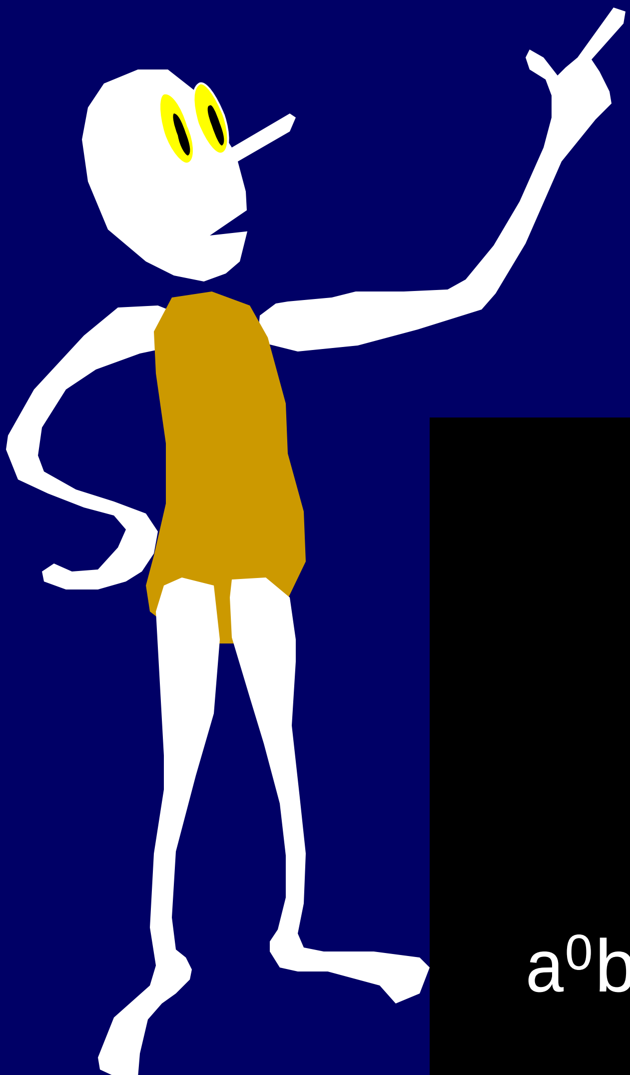
$$- a^0b^{n+1} \quad - a^1b^n \dots \quad a^{i+1}b^{n-i} \dots \quad - a^{n-1}b^2 \ - a^nb^1$$

$$= \quad - \ b^{n+1} \quad + \quad a^{n+1}$$

$$= \quad a^{n+1} - b^{n+1}$$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + \dots + c_k X^k + \dots$$



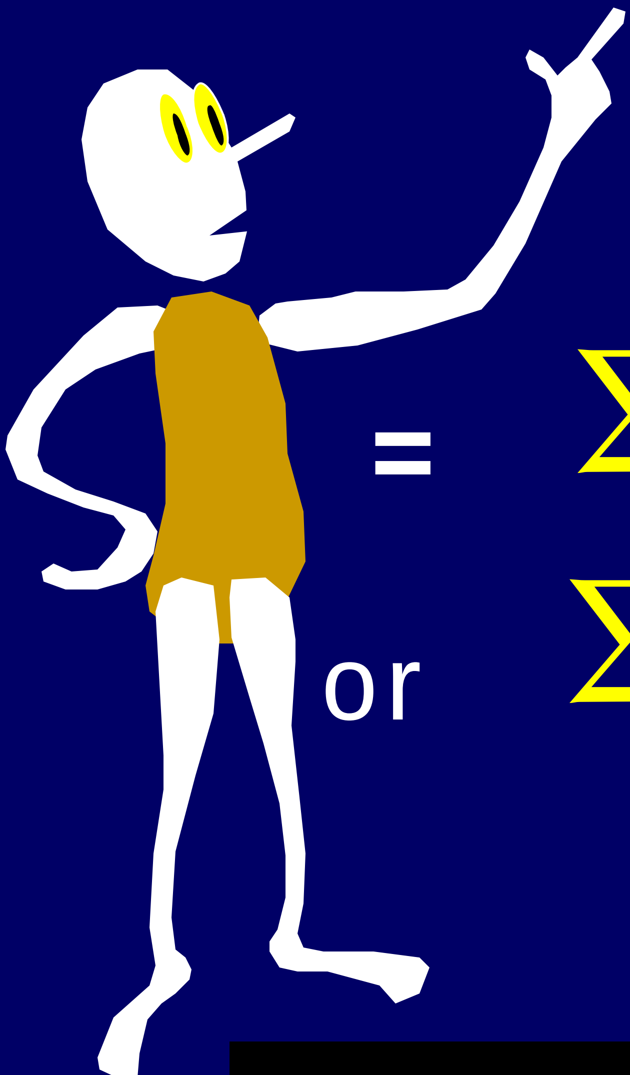
if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\ (1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$= \frac{1}{(1 - aX)(1-bX)}$$



$$= \sum_{n=0..\ 1} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

or

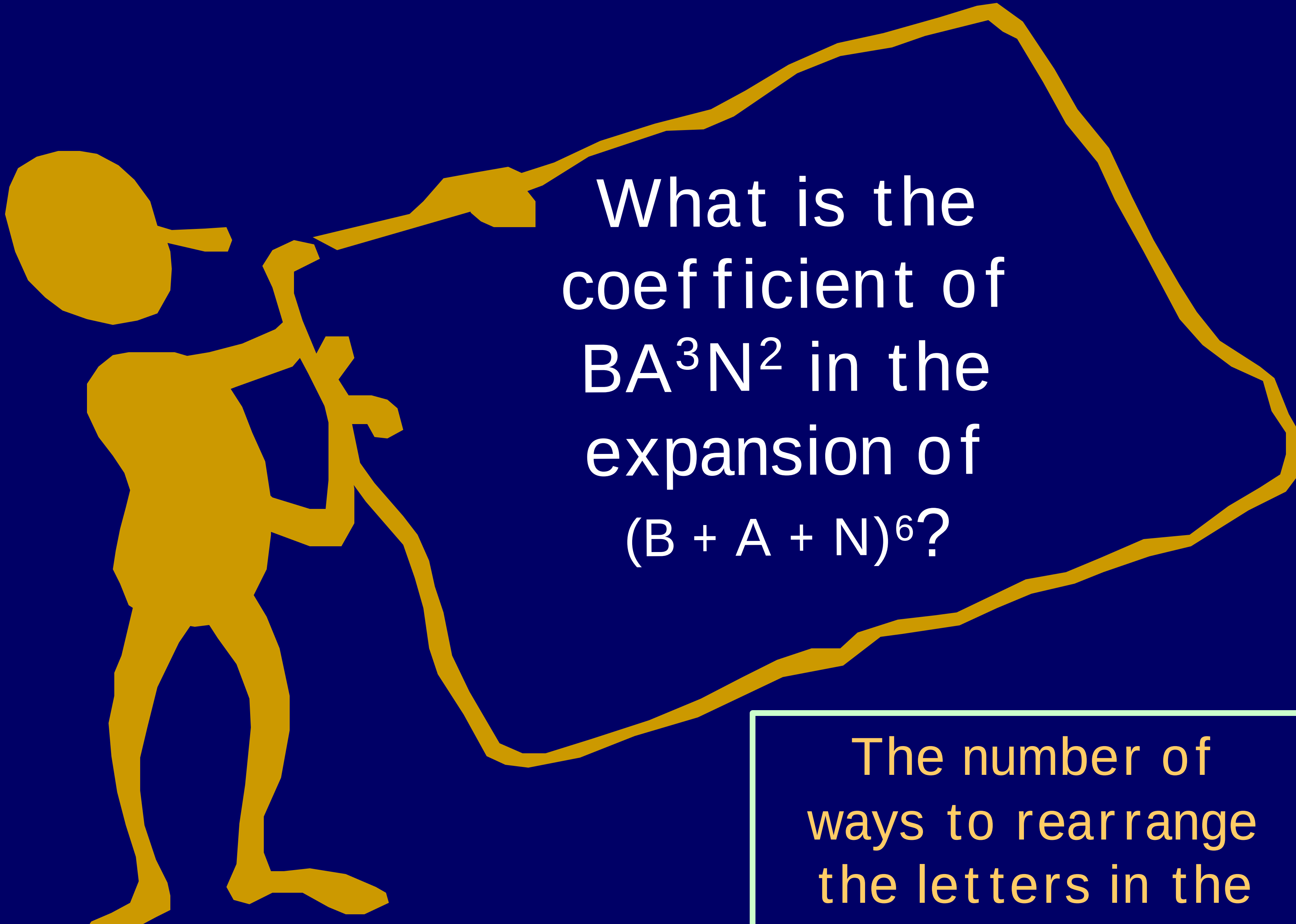
$$\sum_{n=0..\ 1} (n+1)a^n\ X^n$$

when a=b

Geometric Series (Quadratic Form)

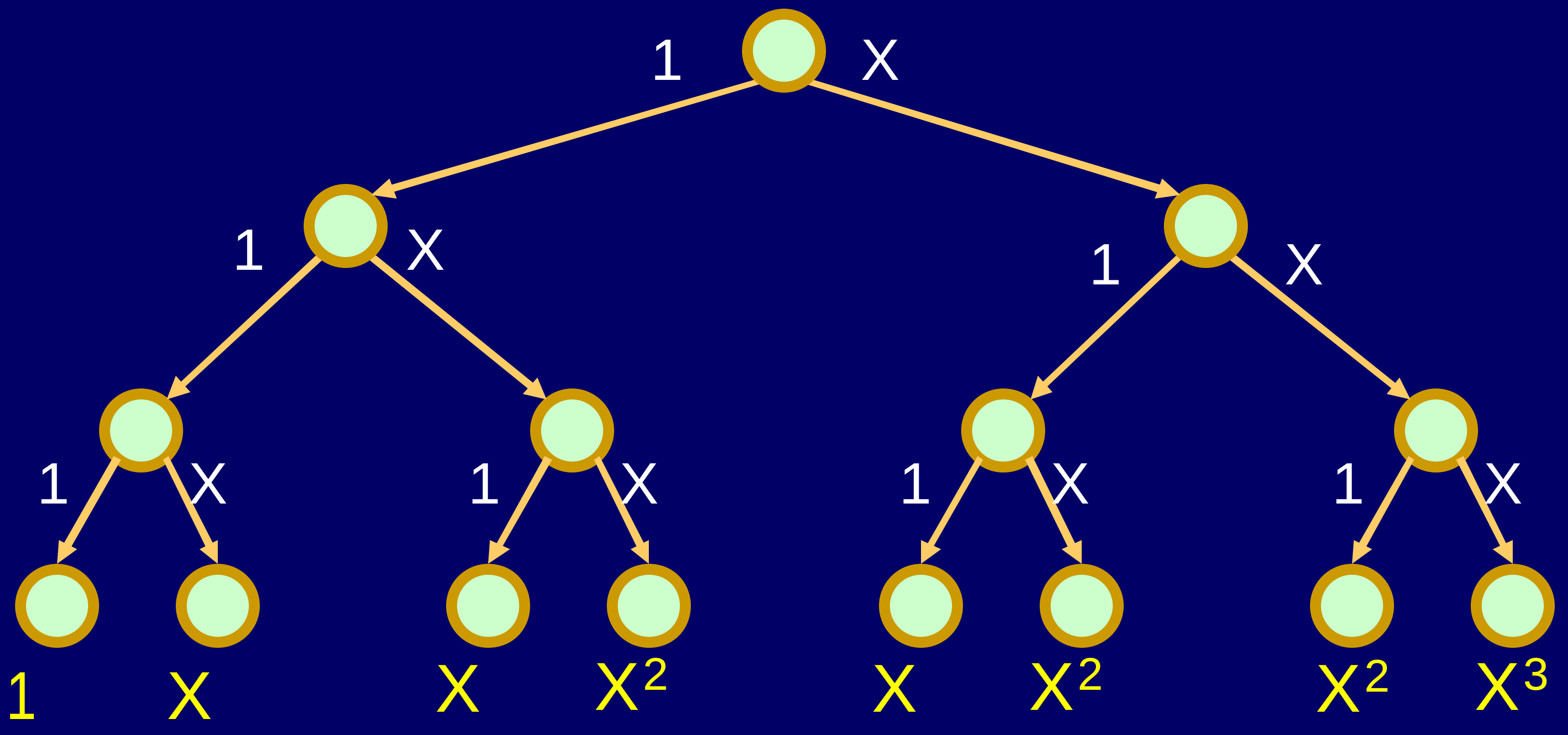
Previously, we saw that

Polynomials Count!



The number of ways to rearrange the letters in the word BANANA.

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or "Generating form"

"Additive form" or "Expanded form"

Power Series Representation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

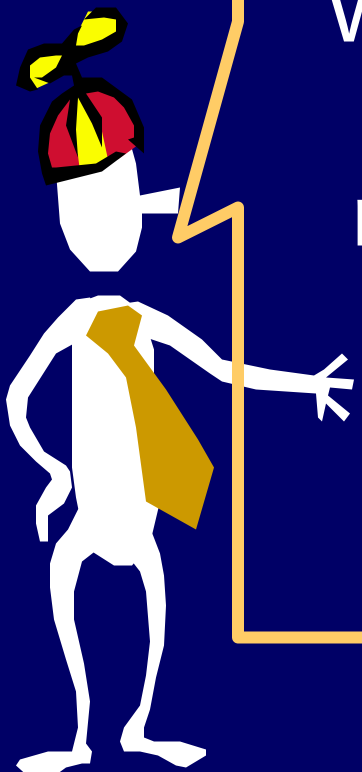
Let *x*= *-1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

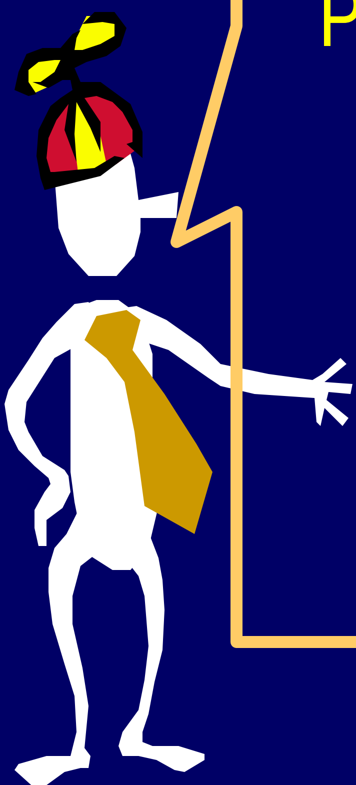
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



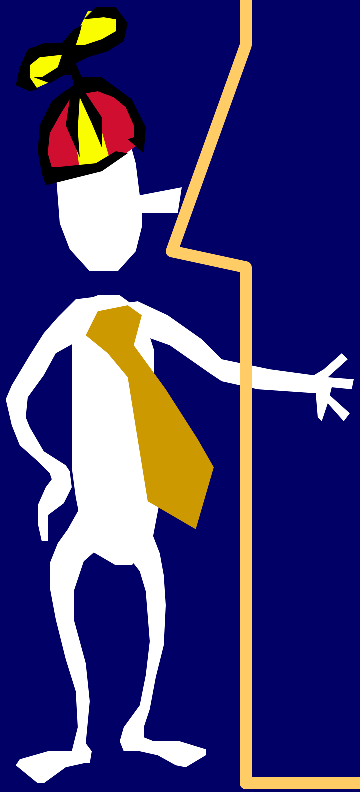
We could discover new identities by substituting in different numbers for X. One cool idea is to try complex roots of unity, however, the lecture is going in another direction.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called “algebraic” arguments. Proofs that build a 1-1 onto correspondence are called “combinatorial” arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd n. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

Uh oh. Complementing maps evens to evens!

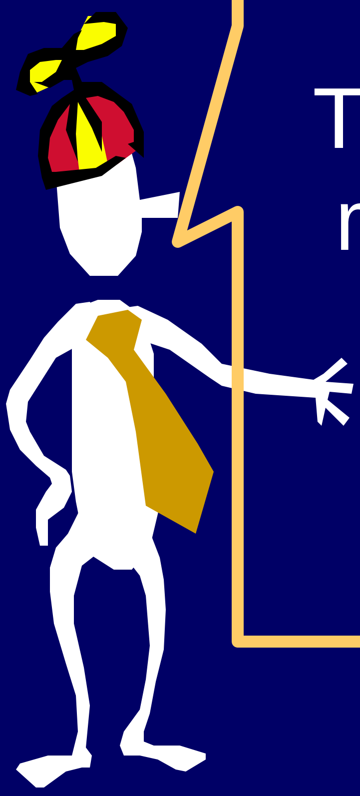
A correspondence that works for all *n*

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{k},\ldots\binom{n}{n}$$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:

Pascal(n,0) = Pacal (n,n) = 1;

Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1 \qquad \binom{1}{1}=1$$

$$\binom{2}{0}=1 \qquad \binom{2}{1}=2 \qquad \binom{2}{2}=1$$

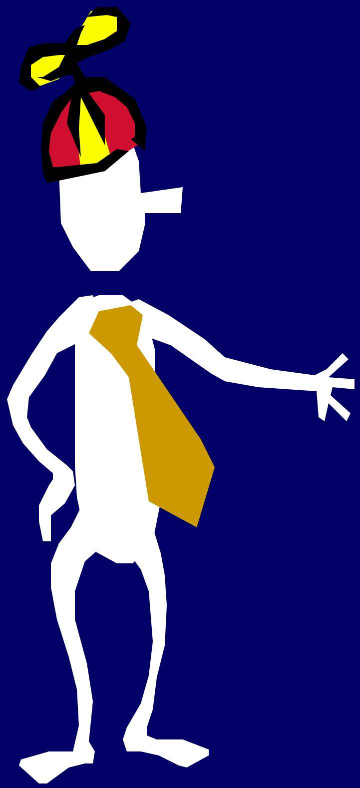
$$\binom{3}{0}=1 \qquad \binom{3}{1}=3 \qquad \binom{3}{2}=3 \qquad \binom{3}{3}=1$$

Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . .

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	=1
1 + 1	=2
1 + 2 + 1	=4
1 + 3 + 3 + 1	=8
1 + 4 + 6 + 4 + 1	=16
1 + 5 + 10 + 10 + 5 + 1	=32
1 + 6 + 15 + 20 + 15 + 6 + 1	=64

⋰ ⋮ ⋱

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

Summing on 1st Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n\cdot(n+1)}{2}$$

Summing on kth Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

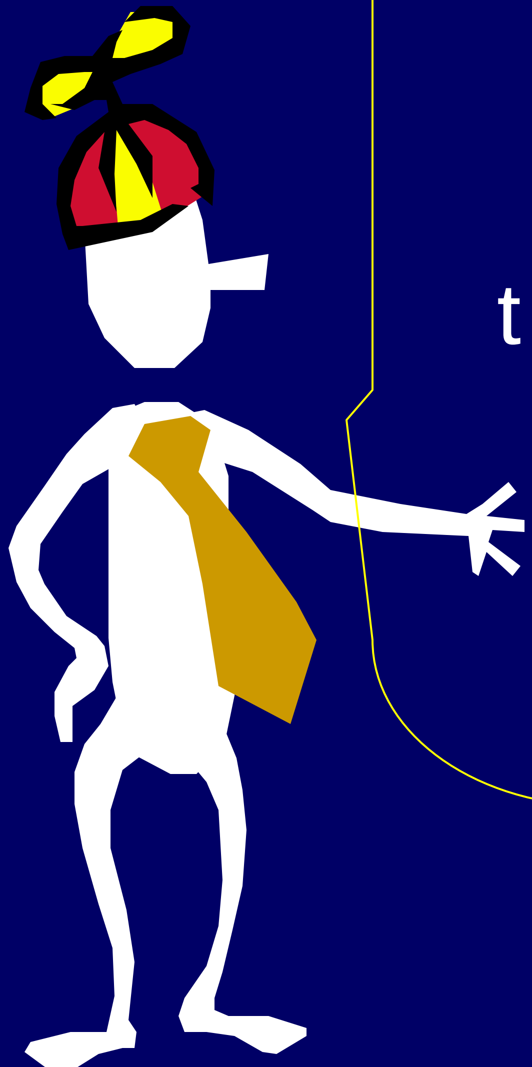
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
=2
=3
=5
=8
=13


1
1 1
1² 2² 1²
1² 3² 3² 1²
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

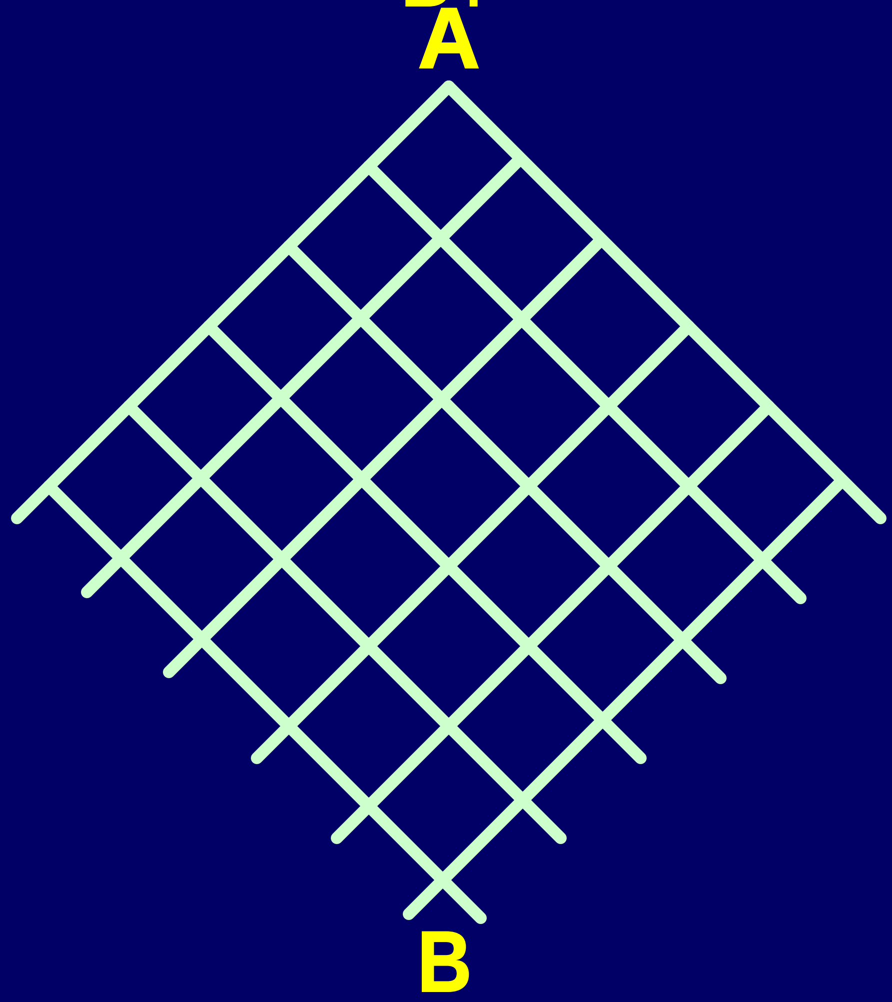
Al-Karaji Squares

1		=0
1 1		=1
1 2 + 2*1		=4
1 3 + 2*3 1		=9
1 4 + 2*6 4 1		=16
1 5 + 2*10 10 5 1		=25
1 6 + 2*15 20 15 6 1		=36



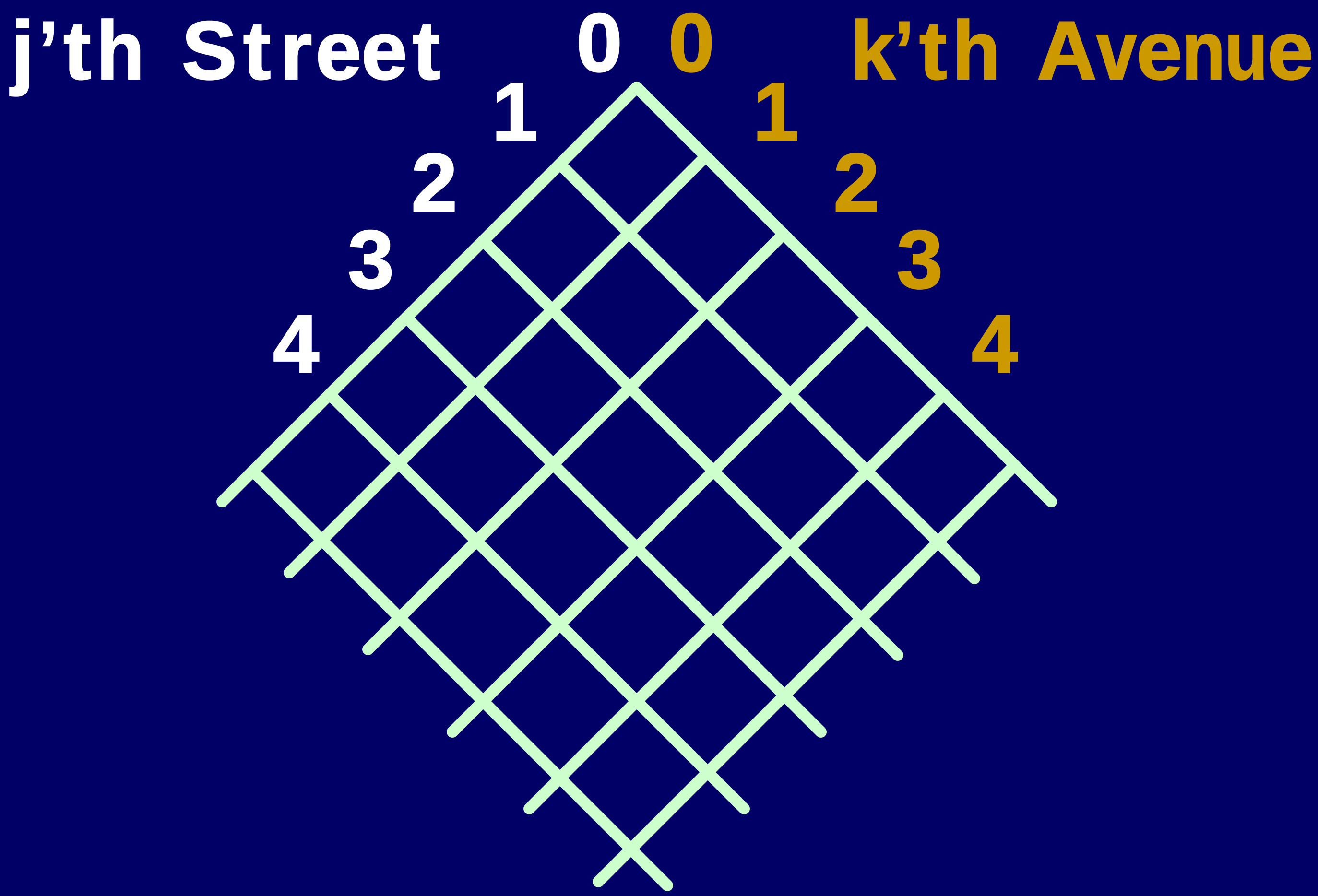
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?



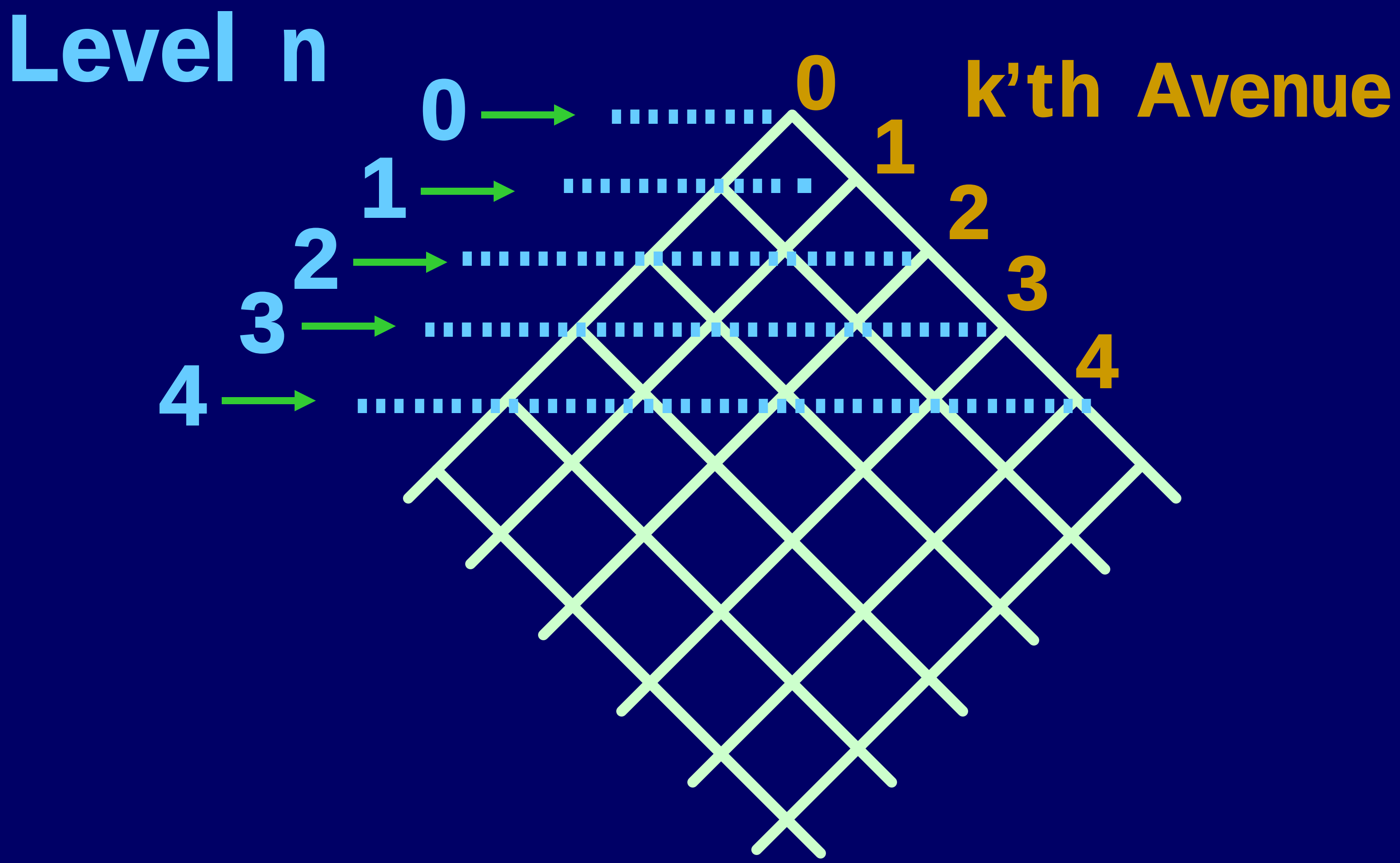
$$\binom{10}{5}$$

Manhattan



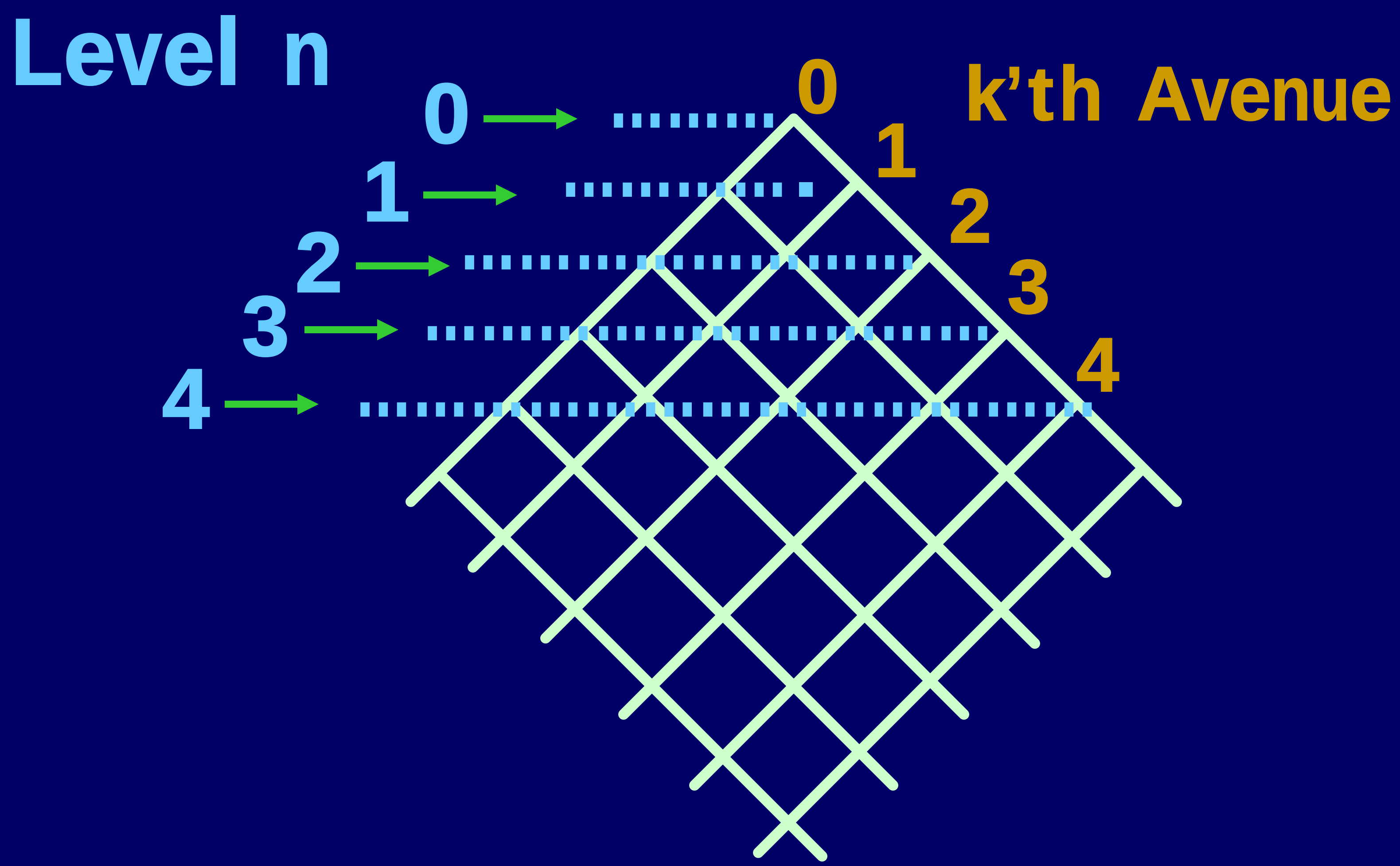
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

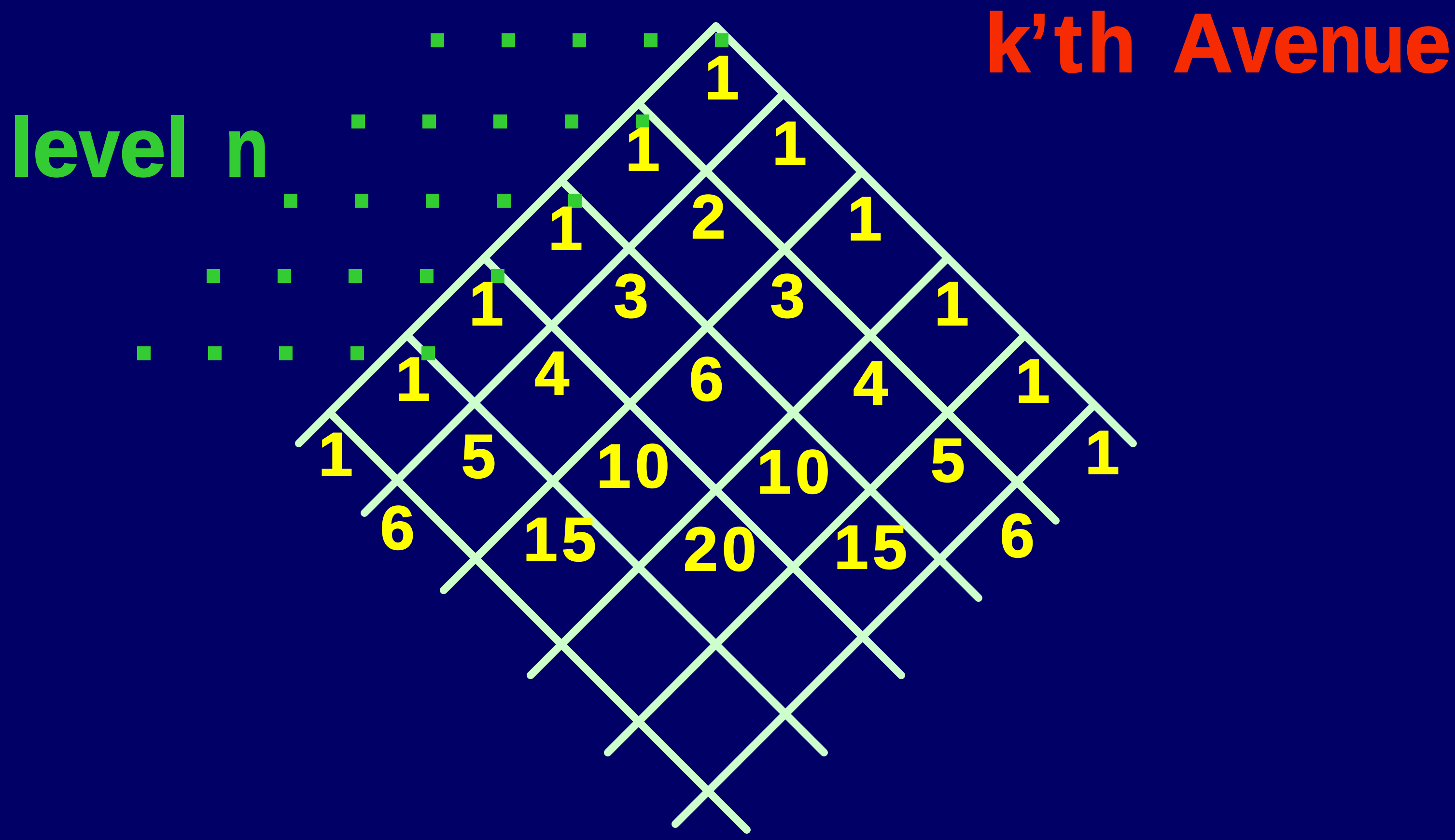


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

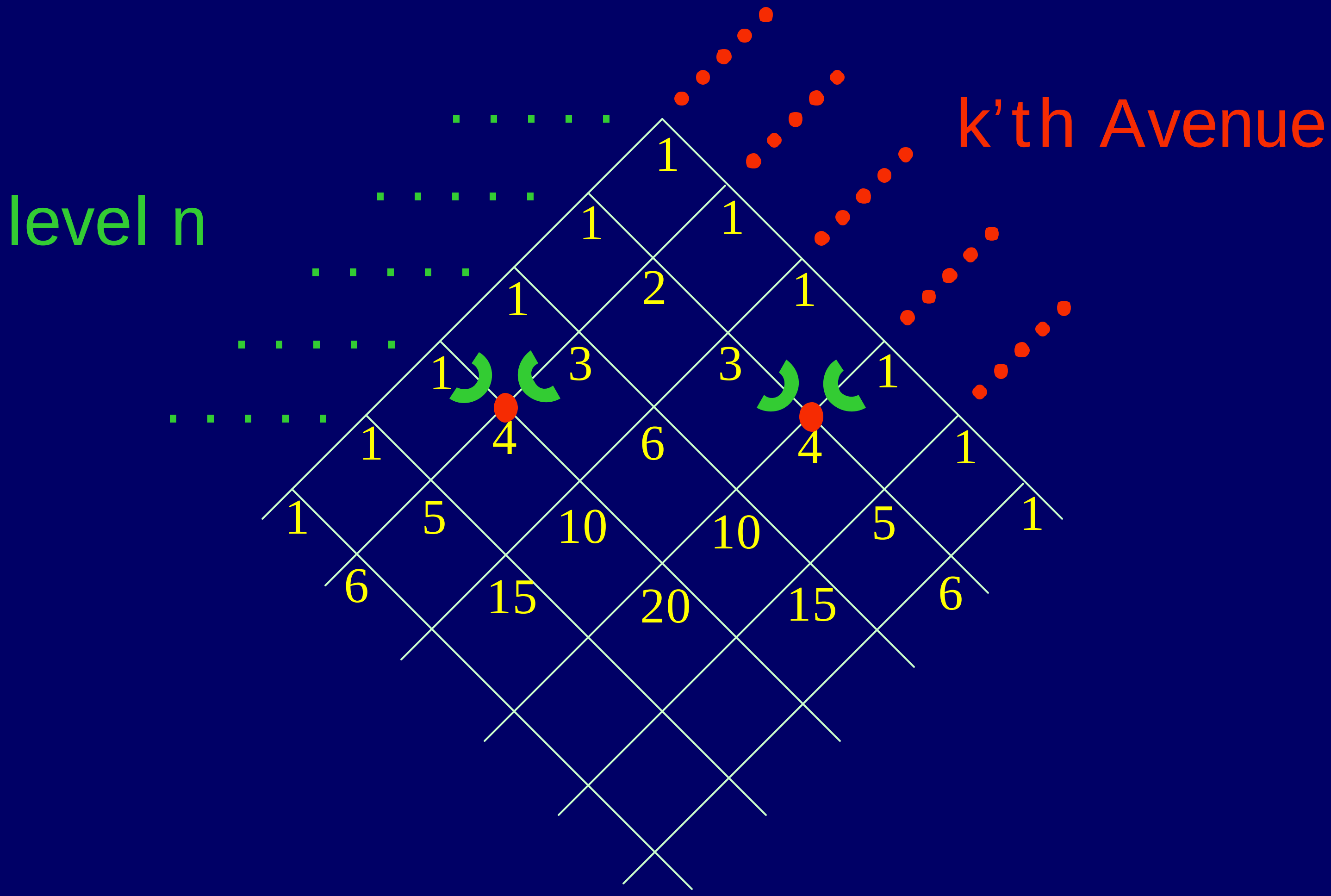
Manhattan



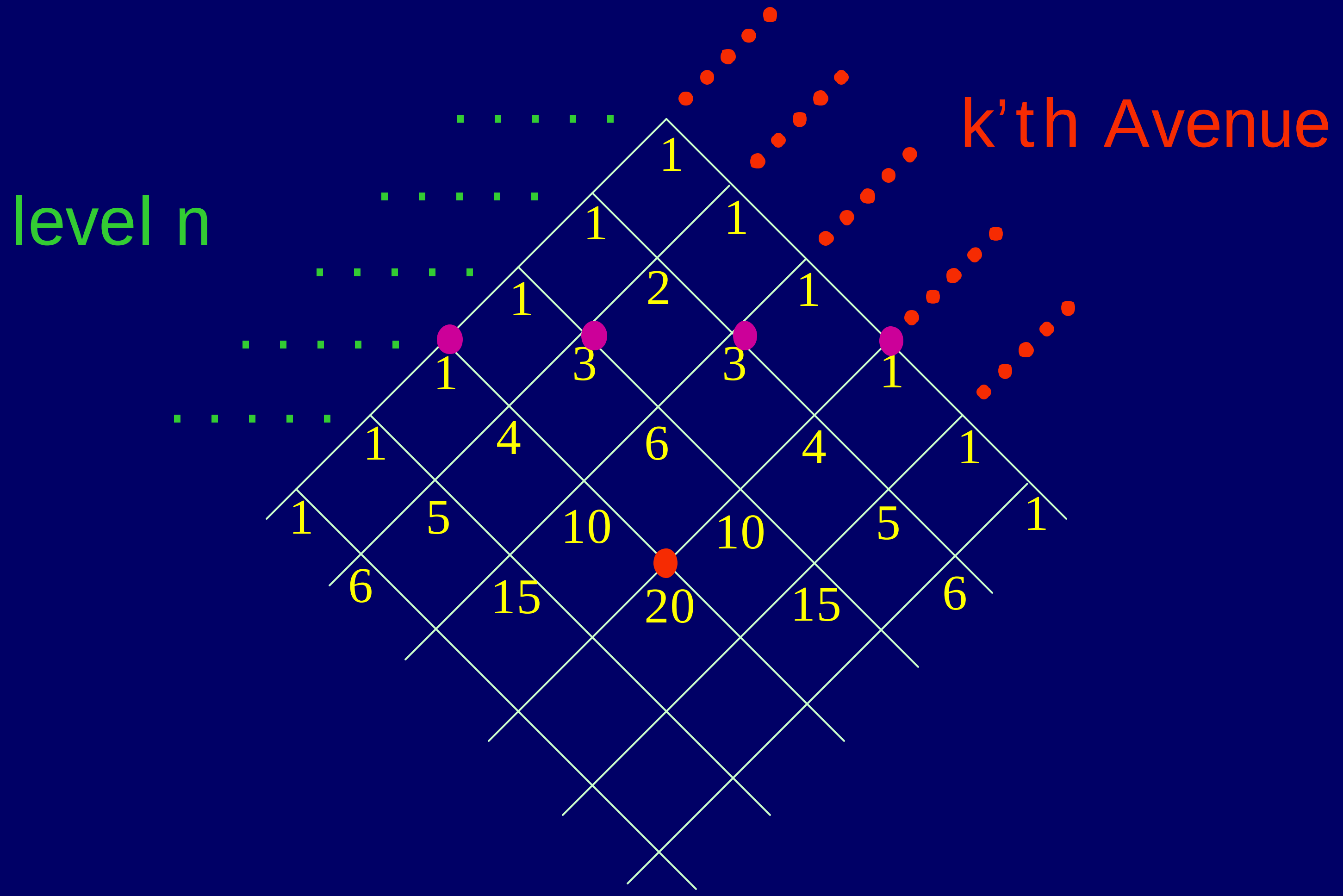
There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



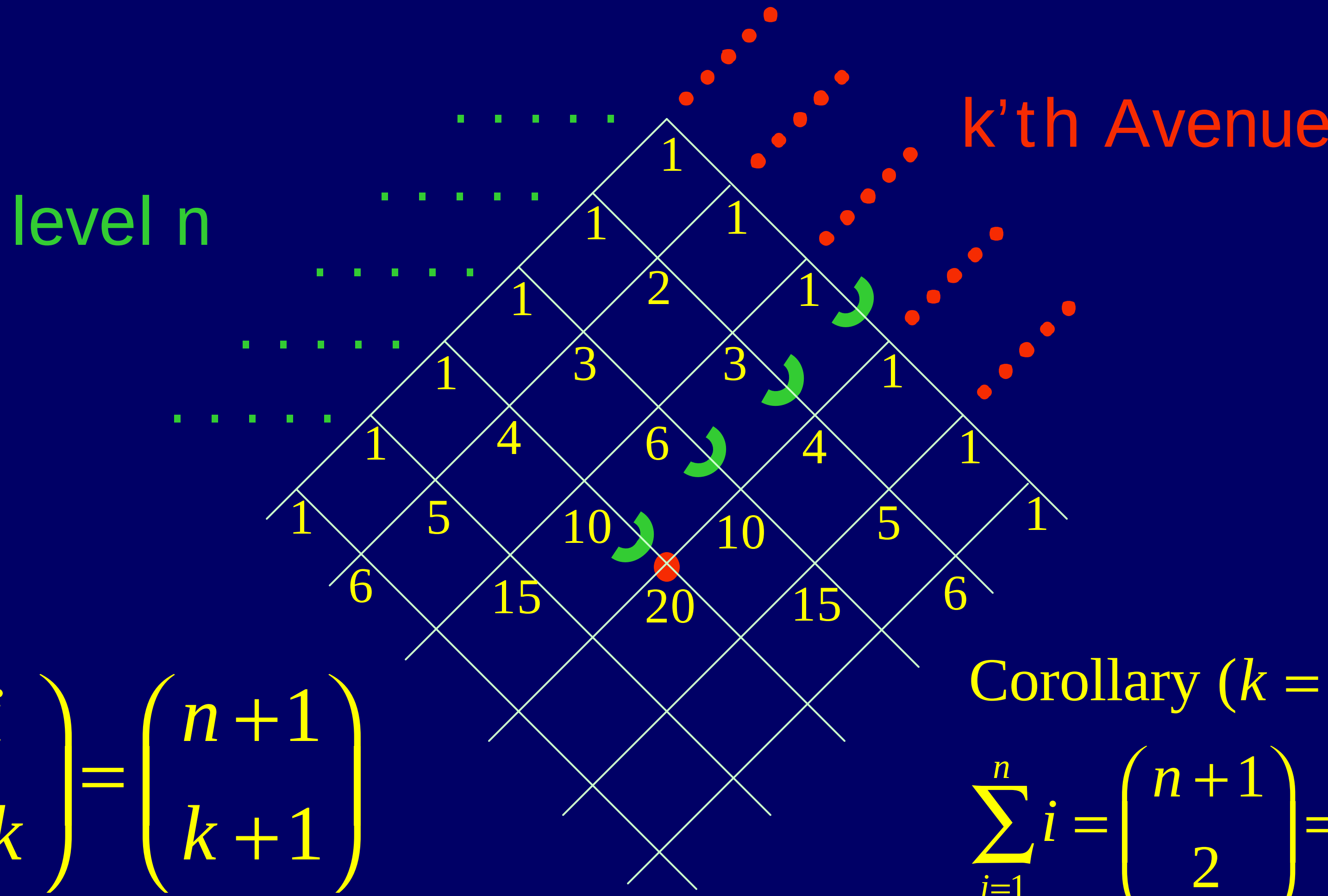
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$$0! = 1$$ (empty product = 1)

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k = 1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= (1 \cdot 0 + 1) + (2 \cdot 1 + 2) + (3 \cdot 2 + 3) + \cdots + (n(n-1) + n)$$

$$= 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable V! can be thought of as:

< *, *, *, *, *, *, ... >

0 1 2 3 4 5 . . . .

Vector Programs

Let k stand for a scalar constant

<k> will stand for the vector <k,0,0,0,...>

<0> = <0,0,0,0,....>

<1> = <1,0,0,0,...>

V ! +T ! means to add the vectors position-wise.

<4,2,3,...> + <5,1,1,....> = <9,3,4,...>

Vector Programs

RIGHT(V!) means to shift every number in V! one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, ...>) = <0,1,2,3,. ...>

Vector Programs

Example:	Stare
V ! := <6>;	V ! = <6,0,0,0,..>
V ! := RIGHT(V !) + <42>;	V ! = <42,6,0,0,..>
V ! := RIGHT(V !) + <2>;	V ! = <2,42,6,0,..>
V ! := RIGHT(V !) + <13>;	V ! = <13,2,42,6,.>

V ! = < 13, 2, 42, 6, 0, 0, 0, . . . >

Vector Programs

Example:

V ! := <1>;

Loop n times:

V ! := V ! + RIGHT(V !);

Stare

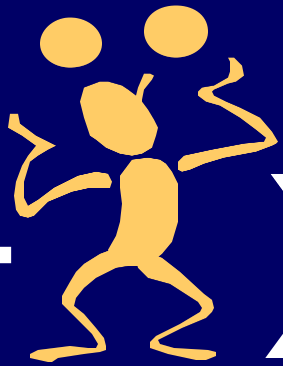
V ! = <1,0,0,0,..>

V ! = <1,1,0,0,..>

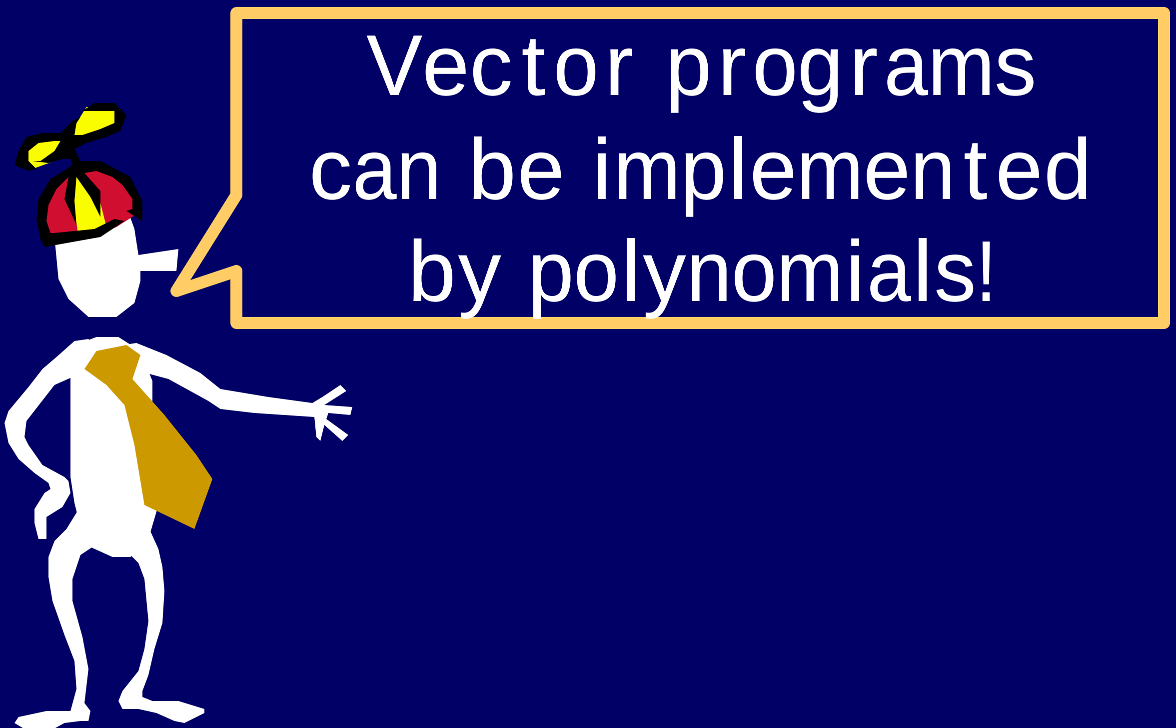
V ! = <1,2,1,0,..>

V ! = <1,3,3,1,.>

V ! = n^{th} row of Pascal's triangle.


$$x^1 + x^2 + x^3$$



Programs -----> Polynomials

The vector V! = < a_0, a_1, a_2, . . . > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector V! = < a_0, a_1, a_2, . . . > will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

V ! = < a_0, a_1, a_2, . . . >

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

<0>is represented by 0

<k>is represented by k

V ! + T ! is represented by $(P_V + P_T)$

RIGHT(V !) is represented by $(P_V\ X)$

Vector Programs

Example:

V ! := <1>; P_V := 1;

Loop n times:

V ! := V ! + RIGHT(V !); **P_V := P_V + P_V X;**

V ! = n^th row of Pascal's triangle.

Vector Programs

Example:

V ! := <1>; $P_V := 1;$

Loop n times:

V ! := V ! + RIGHT(V !); $P_V := P_V (1 + X);$

V ! = nth row of Pascal's triangle.

Vector Programs

Example:

V ! := <1>;

Loop n times:

V ! := V ! + RIGHT(V !);

$P_V = (1+X)^n$

V ! = n^th row of Pascal's triangle.