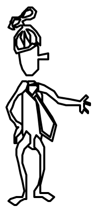
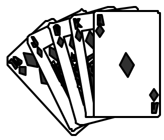
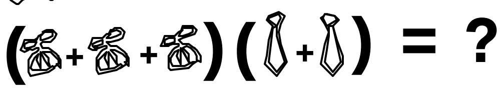
Great Theoretical Ideas In Computer Science		
Steven Rudich		CS 15-251 Spring 2005
Lecture 7	Feb 1, 2005	Carnegie Mellon University

Counting II: Recurring Problems And Correspondences

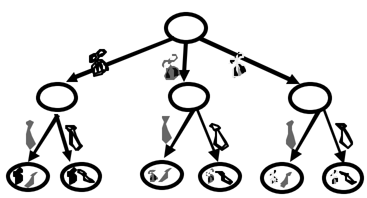





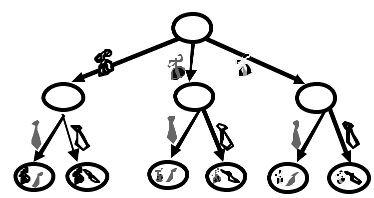
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Choice Tree



A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a "choice tree representation" of a set S, if

1) Each leaf label is in S
2) No two leaf labels are the same

Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN

there are $P_1P_2P_3..P_n$ objects in S

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of S.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

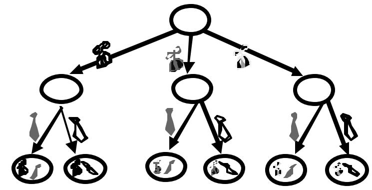
there are $P_1P_2P_3..P_n$ objects of type S.

Condition 2 of the product rule:

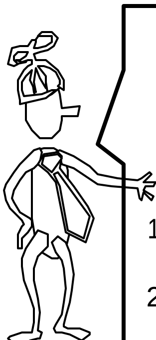
No two leaves have the same label.

Equivalently,

No object can be created in two different ways.



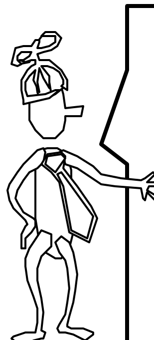
Reversibility Check:
Given an arbitrary object in S, can we reverse engineer the choices that created it?



The two big mistakes people make in associating a choice tree with a set S are:

1) Creating objects not in S

2) Creating the same object two different ways

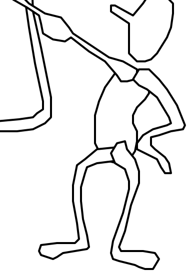


DEFENSIVE THINKING

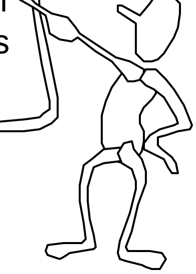
Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

The number of subsets of an n-element set is 2^n

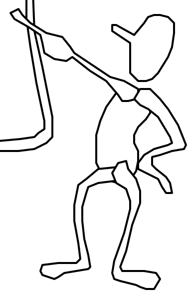


The number of permutations of n distinct objects is n!

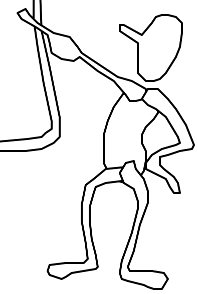


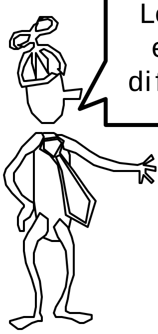
The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$



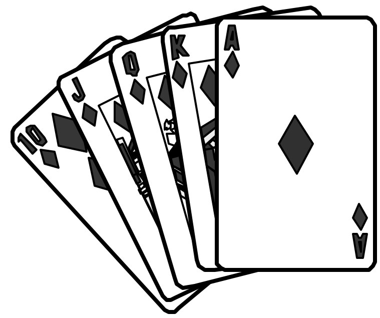
Sometimes it is easiest to count something by counting its opposite.





Let’s use our principles to extend our reasoning to different types of objects.

Counting Poker Hands…



52 Card Deck
5 card hands

4 possible suits:
- ♥♦♣♠

13 possible ranks:
- 2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush
- A straight and a flush

4 of a kind
- 4 cards of the same rank

Full House
- 3 of one kind and 2 of another

Flush
- A flush, *but not a straight*

Straight
- A straight, *but not a flush*

3 of a kind
- 3 of the same rank, *but not a full house or 4 of a kind*

2 Pair
- 2 pairs, *but not 4 of a kind or a full house*

A Pair

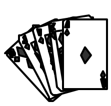
Straight Flush

4 Of A Kind

Flush

Straight

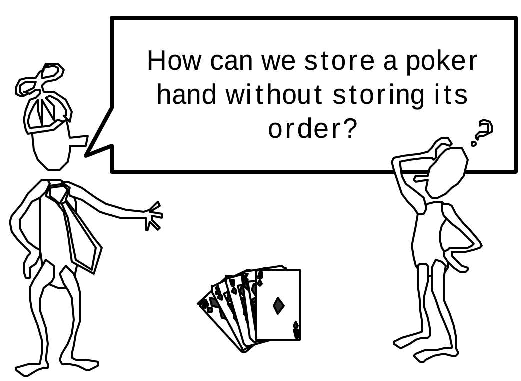
Storing Poker Hands
How many bits per hand?



I want to store a 5 card poker hand using the smallest number of bits (space efficient).

Naïve scheme:	2 bits for suit,
	4 bits for a rank,
	and hence 6 bits per card
Total:	<u>30 bits per hand</u>

How can I do better?



Order the 2,598,560 Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2,598,560) \rceil$=22 bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010
.
.
.

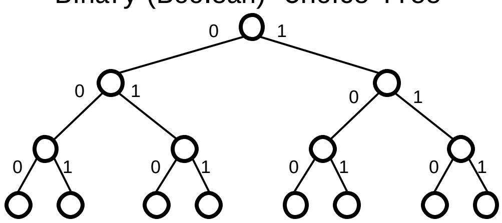
22 Bits Is OPTIMAL

2^{21} = 2097152 < 2,598,560

Thus there are more poker hands than there are 21-bit strings.

Hence, you can't have a 21-bit string for each hand.

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2. Usually the choices will be labeled 0 and 1.

22 Bits Is OPTIMAL

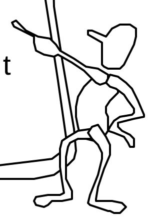
2^{21} = 2097152 < 2,598,560

A binary choice tree of depth 21 can have at most $2^2$1 leaves. Hence, there are not enough leaves for

Hence, you can't have a leaf for each hand.

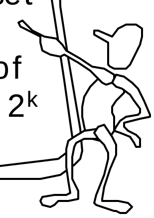
An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

Furthermore, any representation of the set will have some string of that length.



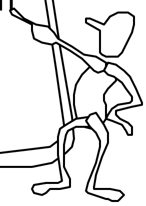
Information Counting Principle:

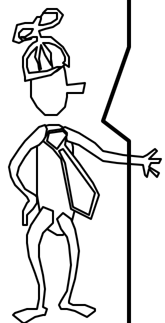
If each element of a set can be represented using k bits, the size of the set is bounded by 2^k



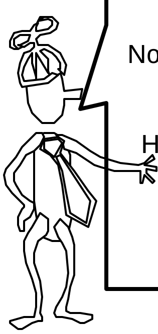
Information Counting Principle:

Let S be a set represented by a depth k binary choice tree, the size of the set is bounded by 2^k





ONGOING MEDITATION:
Let S be any set and T be a binary choice tree representation of S. We can think of each element of S being encoded by the binary sequences of choices that lead to its leaf. We can also start with a binary encoding of a set and make a corresponding binary choice tree.



Now, for something completely different…

How many ways to rearrange the letters in the word **"SYSTEMS"**?

SYSTEMS

1) 7 places to put the Y, 6 places to put the T, 5 places to put the E, 4 places to put the M, and the S's are forced.

   7 X 6 X 5 X 4 = 840

2)

SYSTEMS

3) Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$.

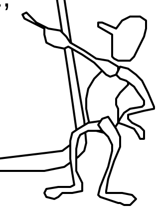
Arrange n symbols
r_1 of type 1, r_2 of type 2, …, r_k of type k

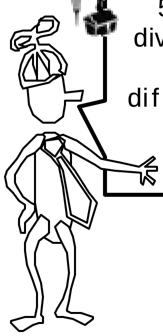
CARNEGIEMELLON

Remember:

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$





5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

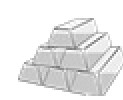
In general, the i^{th} pirate gets the number of G's after the $i\text{-}1^{st}$ / and before the i^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

How many integer solutions to the following equations?

Think of X_k as being the number of gold bars that are allotted to pirate k.

$\binom{24}{4}$

How many integer solutions to the following equations?

Identical/Distinct Dice

Suppose that we roll seven dice.

⚁ ⚀ ⚂ ⚁ ⚅ ⚁ ⚀

How many different outcomes are there, if order matters?

6^7

What if order doesn't matter?
(E.g., Yahtzee)

$\binom{12}{7}$

7 Identical Dice

⚁ ⚀ ⚂ ⚁ ⚅ ⚁ ⚀

How many different outcomes?

Corresponds to 6 pirates
and 7 bars of gold!

Let X_k be the number of dice showing k.
The k^{th} pirate gets X_k gold bars.

$\binom{6+7-1}{7}$

Multisets

A <u>multiset</u> is a set of elements, each of which has a *multiplicity*.

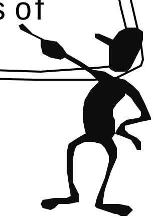
The <u>size</u> of the multiset is the sum of the multiplicities of all the elements.

Example:
{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

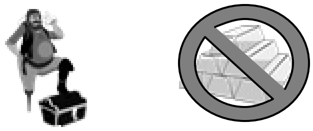
Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

There are ways to choose a multiset of size k from n types of elements

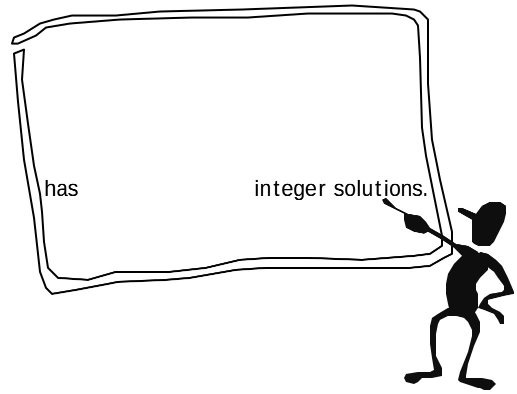


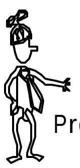
Back to the pirates



How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

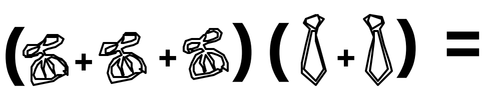
$$\binom{5+20-1}{20} = \binom{24}{20} = \binom{24}{4}$$

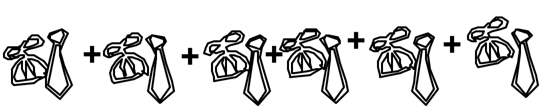


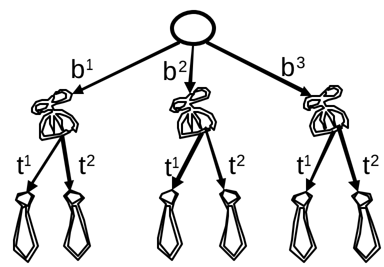


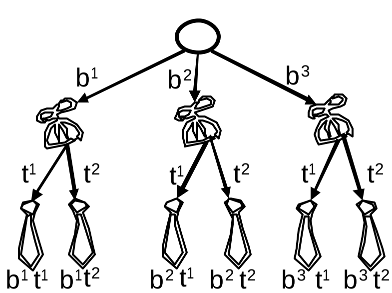
POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

Products of Sum = Sums of Products

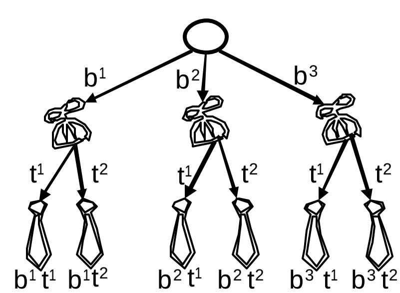




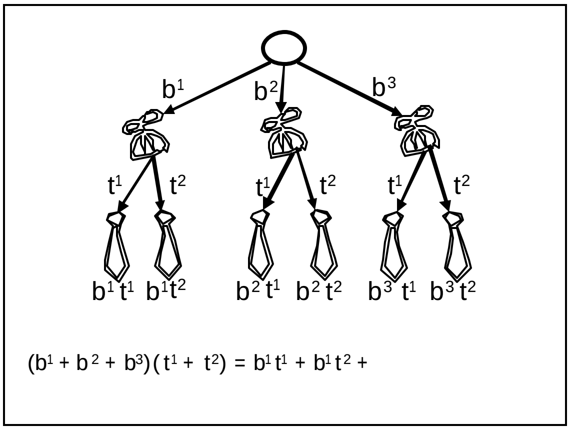


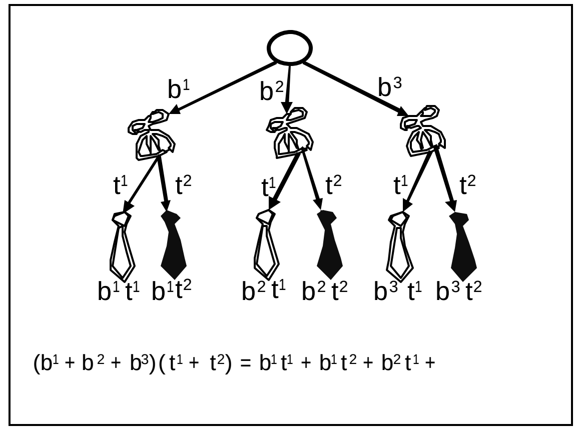


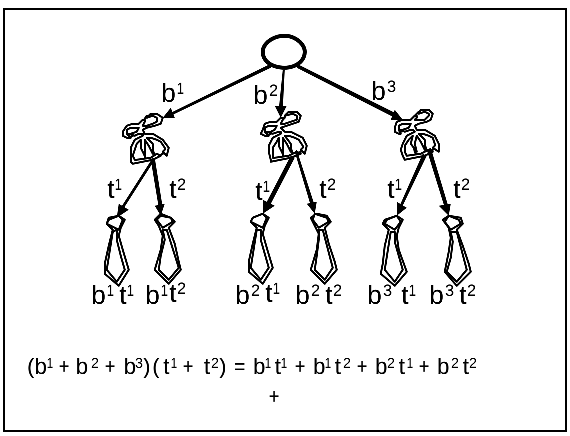
$(b^1 + b^2 + b^3)(t^1 + t^2) =$

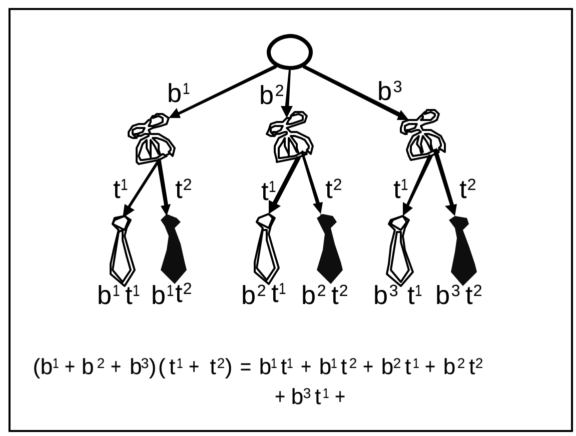


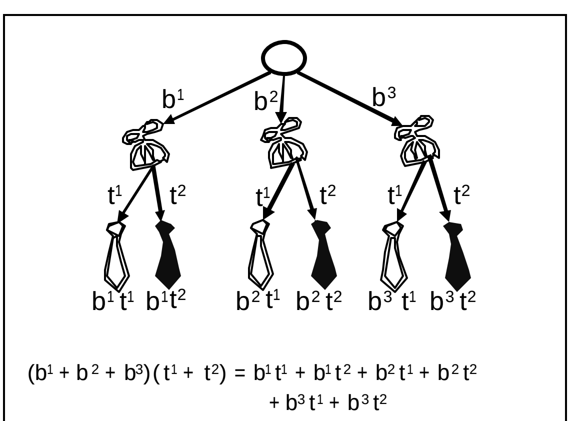
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 +$

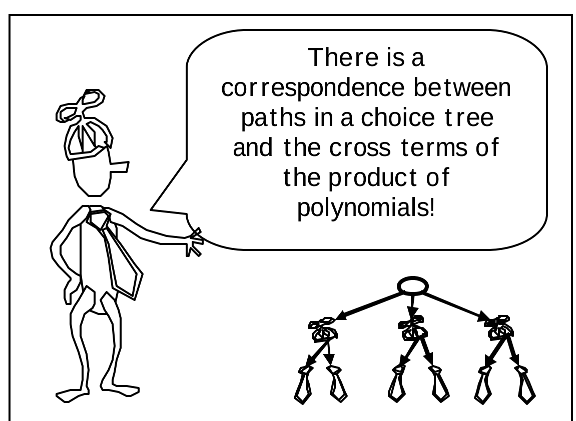

b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 +$


b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 +$

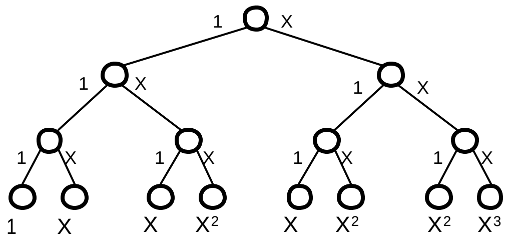

b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$
$+$


b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$
$+ b^3t^1 +$


b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$
$+ b^3t^1 + b^3t^2$


There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

What is a closed form expression for c_k?

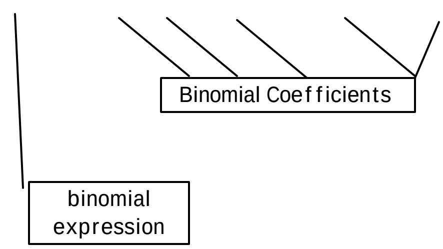
What is a closed form expression for c_n?

n times

After multiplying things out, but *before combining like terms*, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with *exactly k X's*.

The Binomial Formula



The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

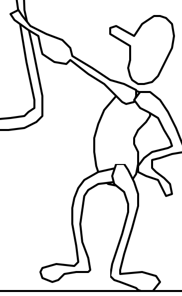
$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

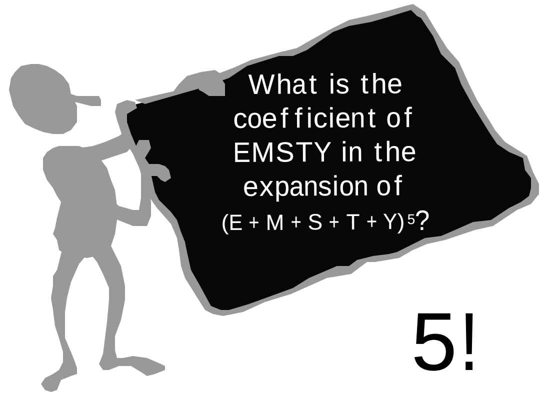
$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

The Binomial Formula

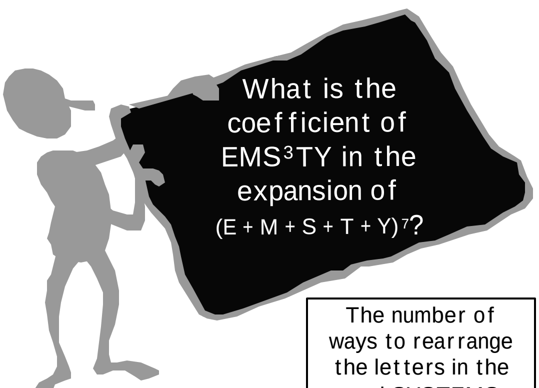
The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{k=n}\binom{n}{k}X^kY^{n-k}$$

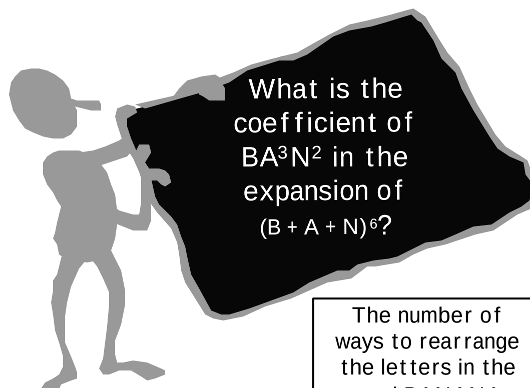




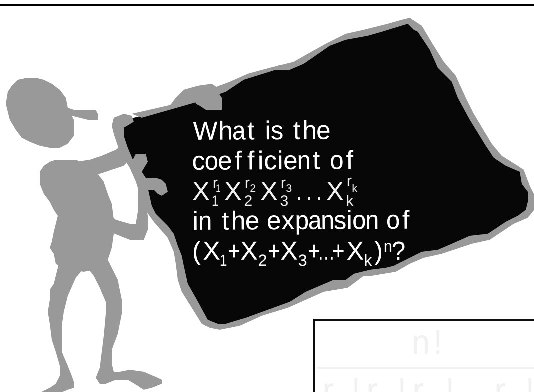
5!



The number of ways to rearrange the letters in the word SYSTEMS.



The number of ways to rearrange the letters in the word BANANA.



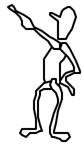
$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$

Multinomial Coefficients

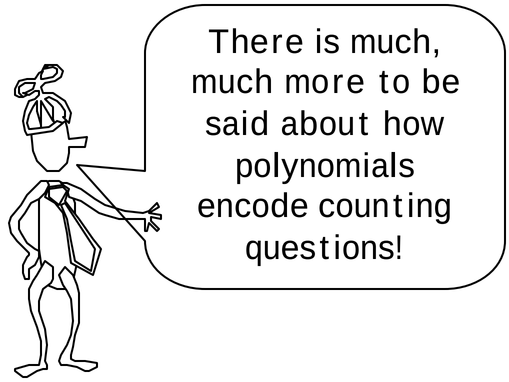
$$\binom{n}{r_1;r_2;\ldots;r_k} = \begin{cases} 0 \text{ if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula



$$(x_1+x_2+\ldots+x_k)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i=n}} \binom{n}{r_1;r_2;\ldots;r_k} x_1^{r_1}x_2^{r_2}x_3^{r_3}\ldots x_k^{r_k}$$



References

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