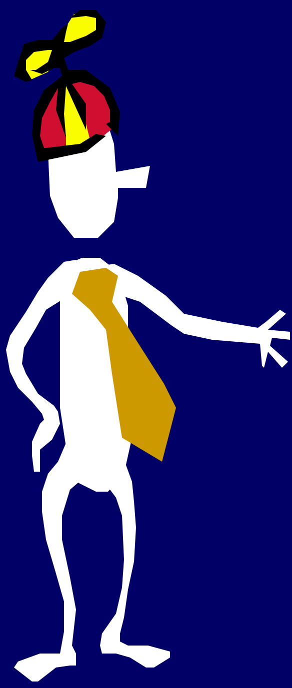
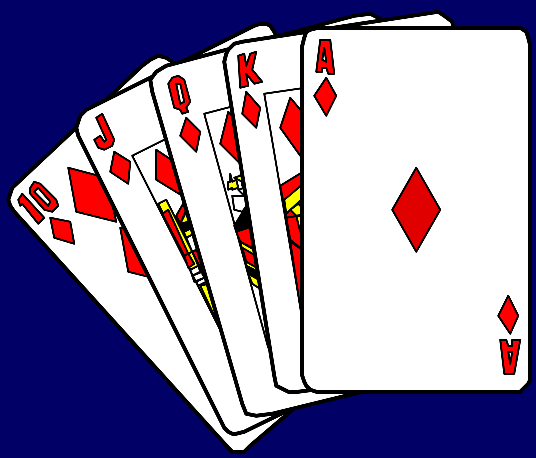
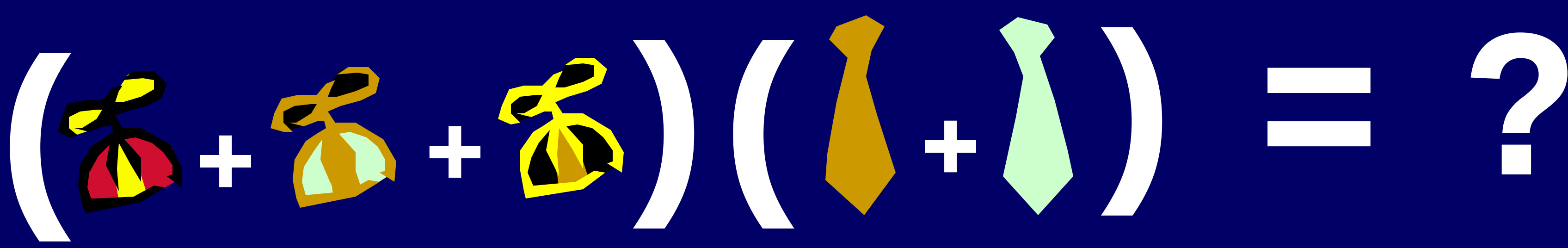
Steven Rudich CS 15-251 Spring 2005

Lecture 7 Feb 1, 2005 Carnegie Mellon University

Counting II: Recurring Problems And Correspondences



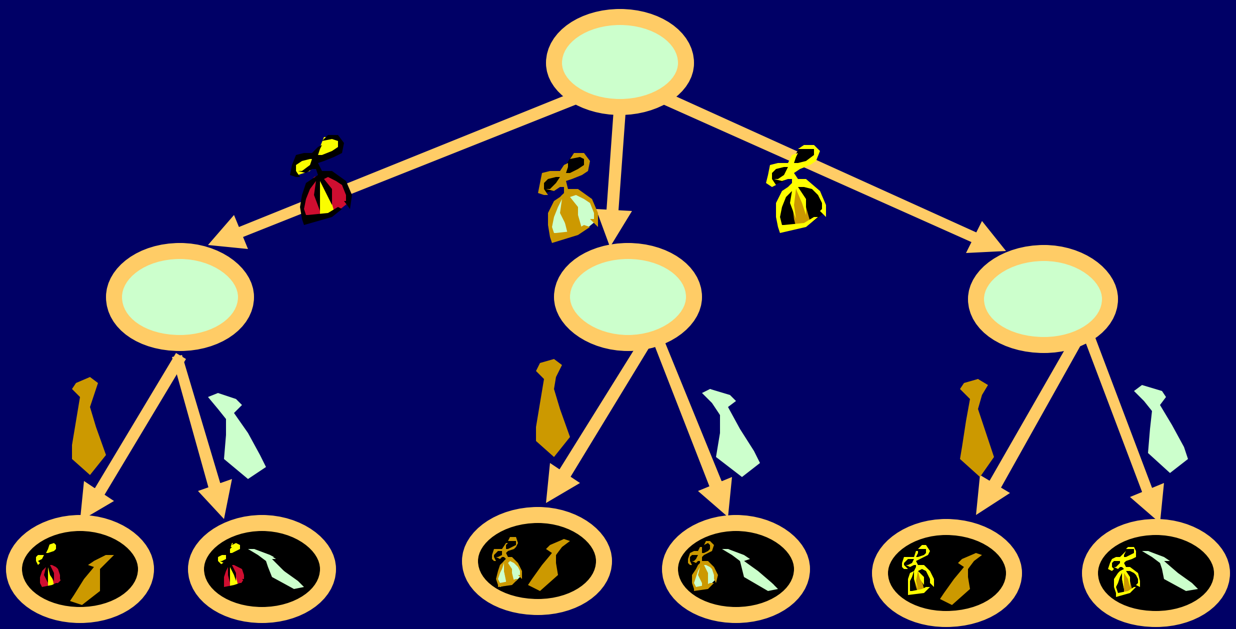




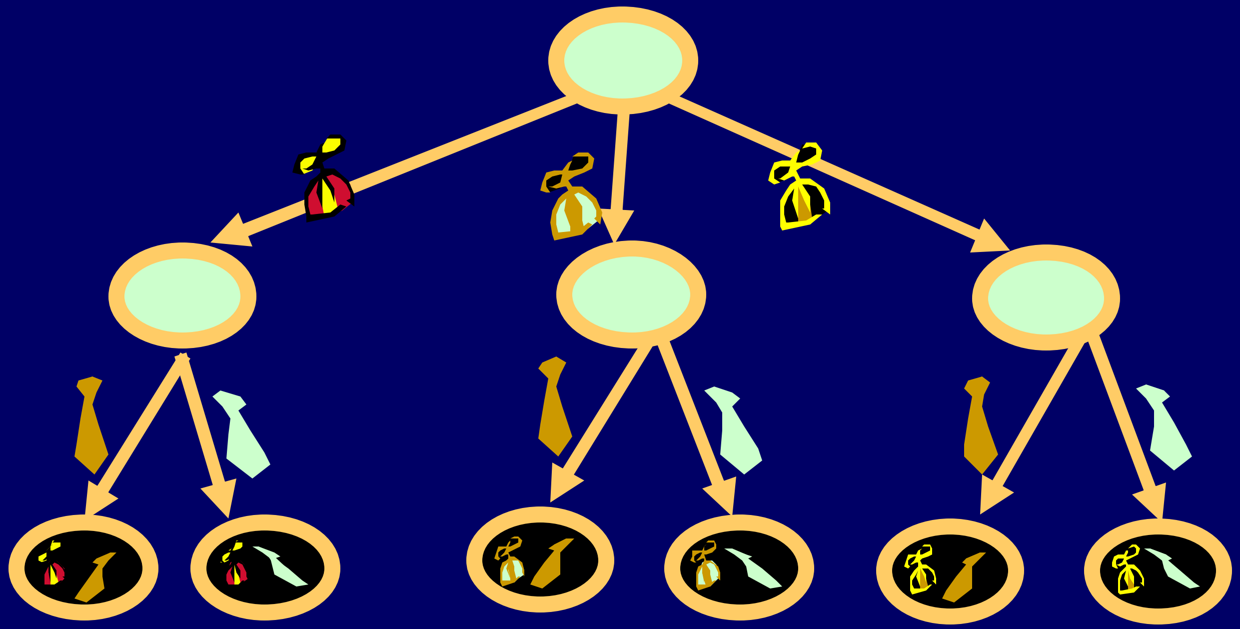
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Choice Tree



A choice tree is a rooted, directed tree with an object called a “choice” associated with each edge and a label on each leaf.



A choice tree provides a “choice tree representation” of a set S, if

1) Each leaf label is in S
2) No two leaf labels are the same

Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects in S

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of S.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

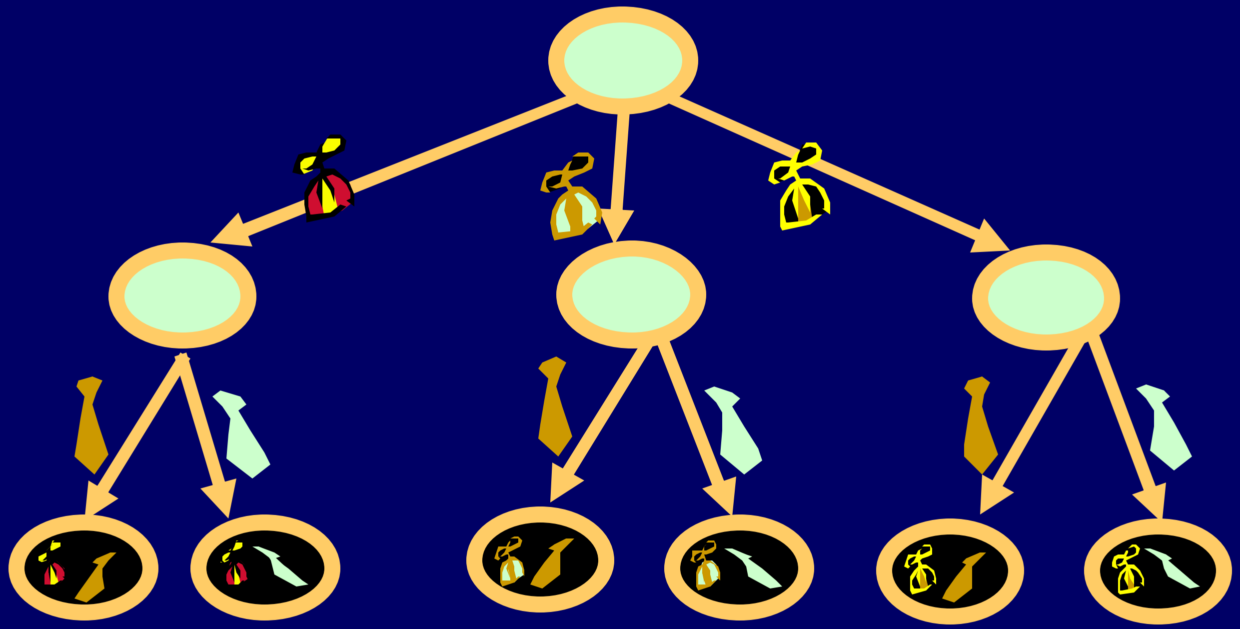
there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

Condition 2 of the product rule:

No two leaves have the same label.

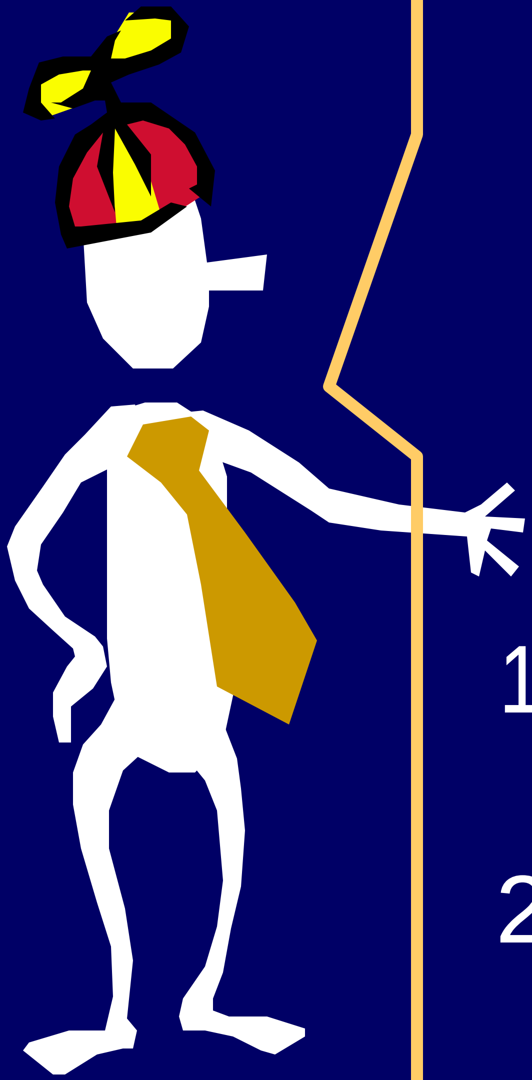
<u>Equivalently,</u>

No object can be created in two different ways.



Reversibility Check:

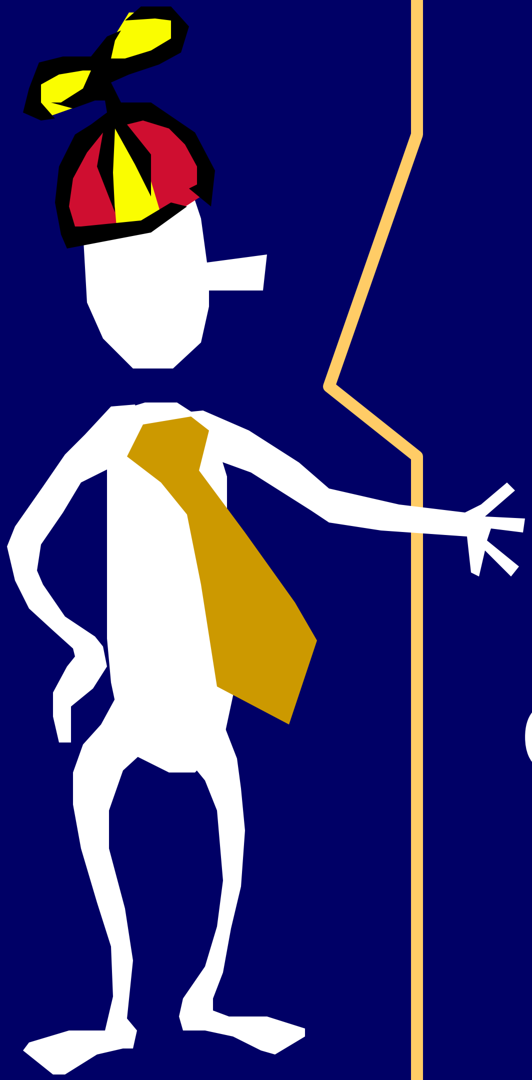
Given an arbitrary object in S, can we reverse engineer the choices that created it?



The two big mistakes people make in associating a choice tree with a set S are:

1) Creating objects not in S

2) Creating the same object two different ways



DEFENSIVE THINKING

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

The number of subsets of an n-element set is 2^n

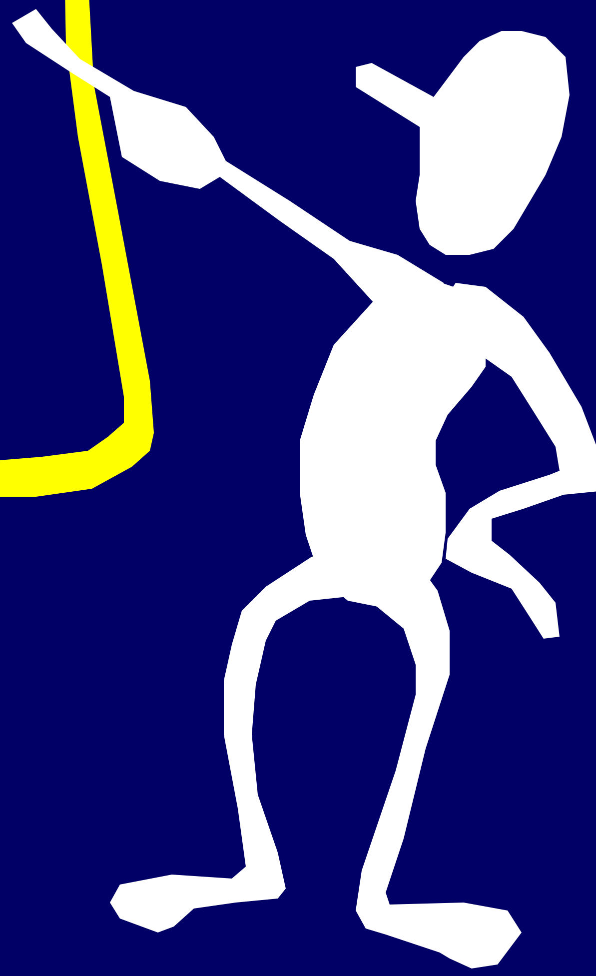
The number of permutations of n distinct objects is

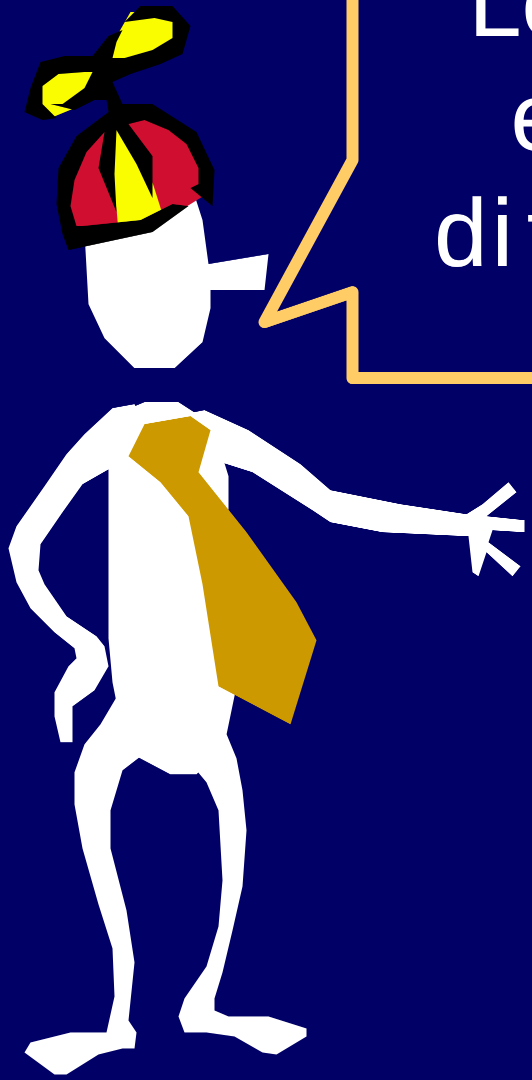
n!

The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

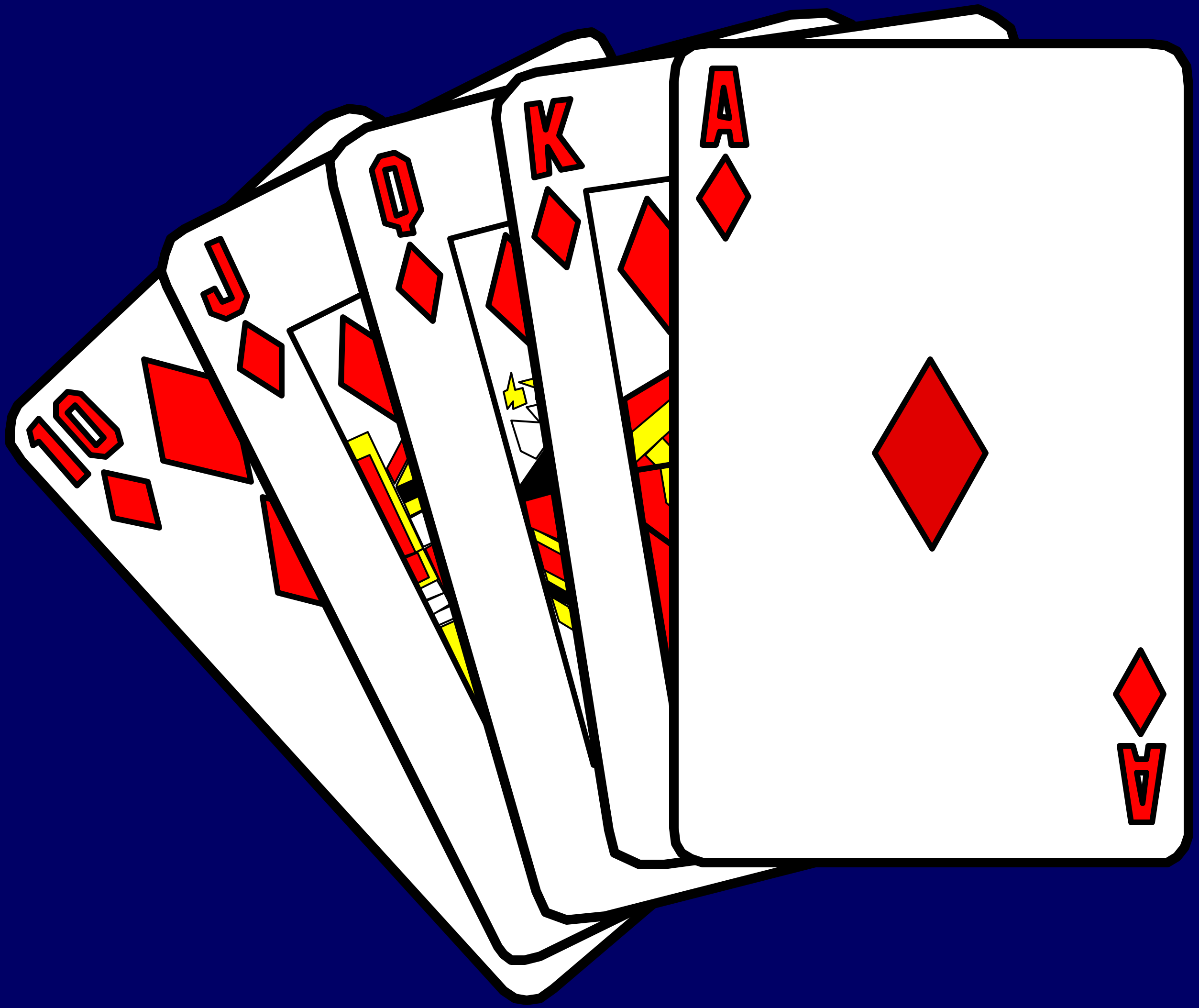
Sometimes it is easiest to count something by counting its opposite.





Let’s use our principles to extend our reasoning to different types of objects.

Counting Poker Hands...



52 Card Deck
5 card hands

4 possible suits:

- ♥ ♦ ♣ ♠

13 possible ranks:

- 2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank

Straight: 5 cards of consecutive rank

Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush

- A straight and a flush

4 of a kind

- 4 cards of the same rank

Full House

- 3 of one kind and 2 of another

Flush

- A flush, *but not a straight*

Straight

- A straight, *but not a flush*

3 of a kind

- 3 of the same rank, *but not a full house or 4 of a kind*

2 Pair

- 2 pairs, *but not 4 of a kind or a full house*

A Pair

Straight Flush

9 choices for rank of lowest card at the start of the straight.

4 possible suits for the flush.

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2598960} = 1 \text{ in } 72{,}193.33..$$

4 Of A Kind

13 choices of rank.

48 choices for remaining card.

$$13 \times 48 = 624$$

$$\frac{624}{2598960} = 1 \text{ in } 4165$$

Flush

4 choices of suit.

$\binom{13}{5}$ choices of set of 5 ranks.

= 5148

\- 36 Straight Flushes

= 5112

$$\frac{5112}{2598960} = 1 \text{ in } 508.4$$

Straight

9 choices of lowest rank in the straight.

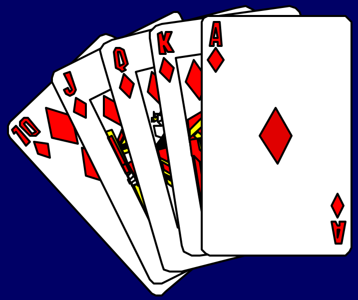
4^5 choices of suits to each card in sequence.

=9216

- 36 Straight Flushes

= 9180

$$\frac{9180}{2598960} = 1 \text{ in } 283.11$$



Storing Poker Hands
How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient).

Naïve scheme: 2 bits for suit,

4 bits for a rank,

and hence 6 bits per card

Total: <u>30 bits per hand</u>

How can I do better?


How can we store a poker hand without storing its order?

Order the 2,598,560 Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2,598,560) \rceil$ =22 bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.

.

.

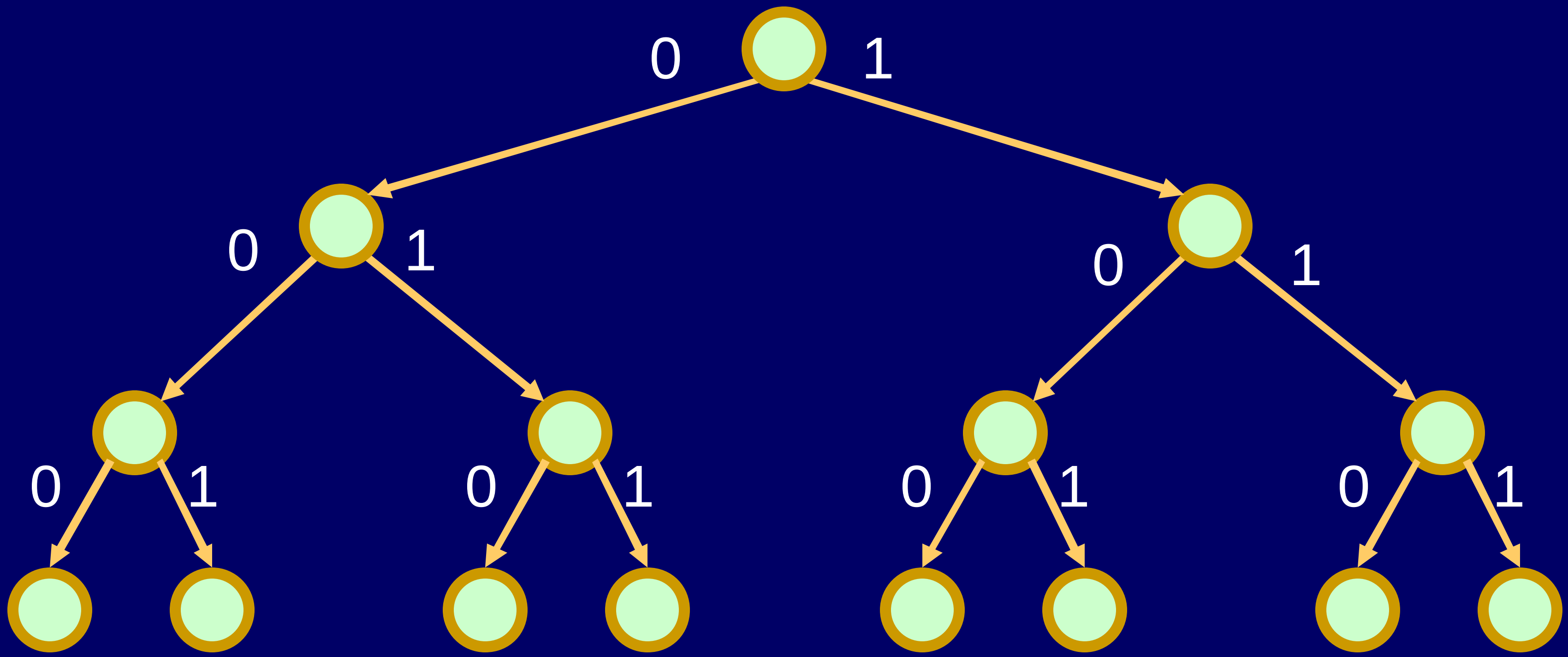
22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2{,}598{,}560$

Thus there are more poker hands than there are 21-bit strings.

Hence, you can't have a 21-bit string for each hand.

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2. Usually the choices will be labeled 0 and 1.

22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2,598,560$

A binary choice tree of depth 21 can have at most 2^{21} leaves. Hence, there are not enough leaves for

Hence, you can't have a leaf for each hand.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

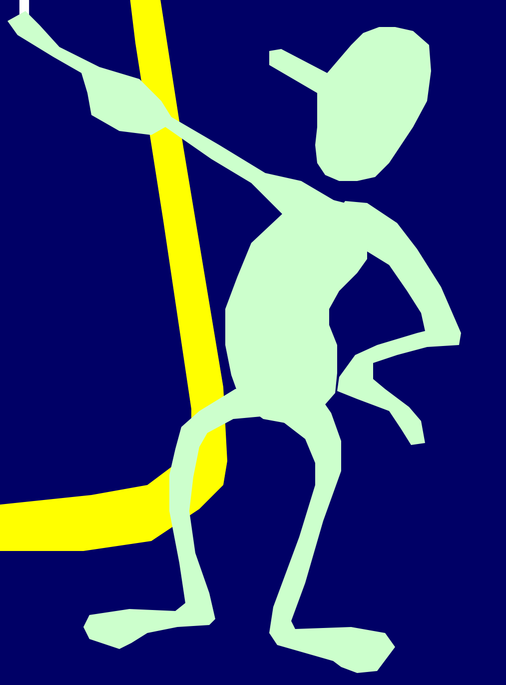
Furthermore, any representation of the set will have some string of that length.

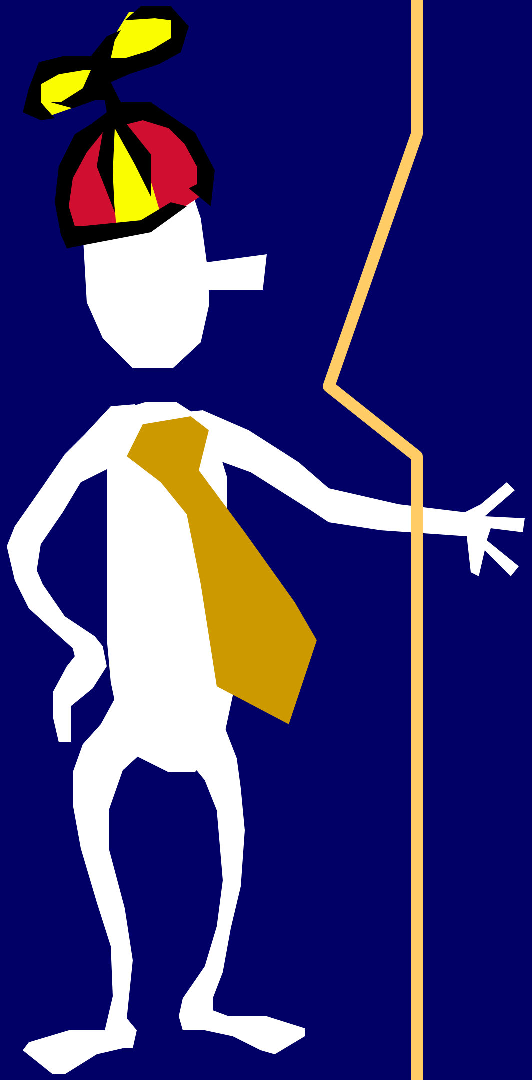
Information Counting Principle:

If each element of a set can be represented using k bits, the size of the set is bounded by 2^k

Information Counting Principle:

Let S be a set represented by a depth k binary choice tree, the size of the set is bounded by 2^k





ONGOING MEDITATION:

Let S be any set and T be a binary choice tree representation of S. We can think of each element of S being encoded by the binary sequences of choices that lead to its leaf. We can also start with a binary encoding of a set and make a corresponding binary choice tree.

Now, for something completely different…

How many ways to rearrange the letters in the word **“SYSTEMS”**?

SYSTEMS

1) 7 places to put the Y, 6 places to put the T, 5 places to put the E, 4 places to put the M, and the S’s are forced.

   7 X 6 X 5 X 4 = 840

2) $\binom{7}{3}$ choices of positions for the S's

   4 choices for the Y

   3 choices for the T

   2 choices for the E

   1 choice for the M

$$\frac{7!}{3!4!} \times 4 \times 3 \times 2 \times 1 = \frac{7!}{3!} = 840$$

SYSTEMS

3) Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$.

$$\frac{7!}{3!} = 840$$

Arrange n symbols

r_1 of type 1, r_2 of type 2, ..., r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{r_k}{r_k}$$

$$= \frac{n!}{r_1!\,(n-r_1)!}\,\frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!}\,\frac{(n-r_1-r_2)!}{r_3!\,(n-r_1-r_2-r_3)!}\cdots 1$$

$$= \frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$

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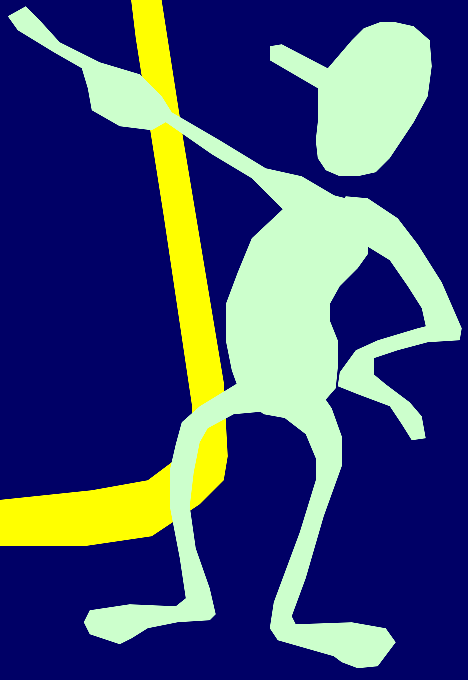
$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

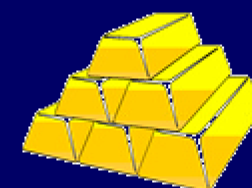
Remember:

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, ..., r_k of type k is:

$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

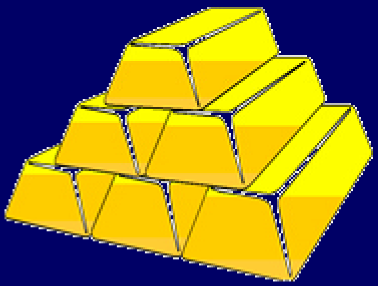
In general, the i^{th} pirate gets the number of G's after the $i\text{-}1^{st}$ / and before the i^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \; , x_2 , x_3 , x_4 , x_5 \geq 0$$

Think of X_k as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer solutions to the following equations?

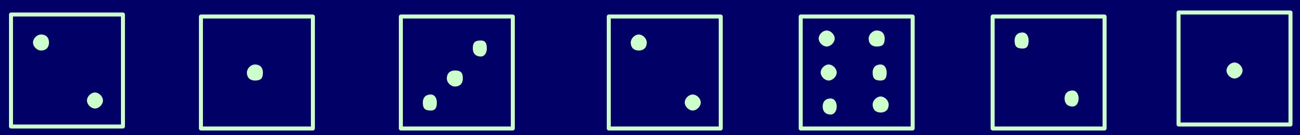
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1 \;, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Identical/Distinct Dice

Suppose that we roll seven dice.



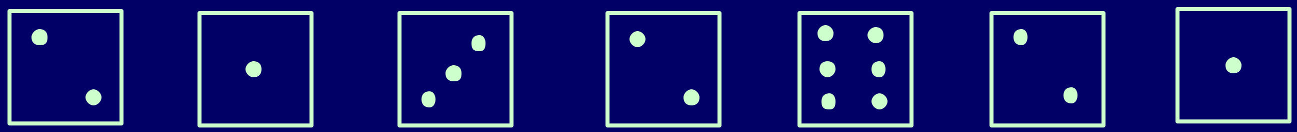
How many different outcomes are there, if order matters?

$$6^7$$

What if order doesn't matter?
(E.g., Yahtzee)

$$\binom{12}{7}$$

7 Identical Dice



How many different outcomes?

Corresponds to 6 pirates
and 7 bars of gold!

Let X_k be the number of dice showing k.
The kth pirate gets X_k gold bars.

$$\binom{6+7-1}{7}$$

Multisets

A <u>multiset</u> is a set of elements, each of which has a *multiplicity*.

The <u>size</u> of the multiset is the sum of the multiplicities of all the elements.

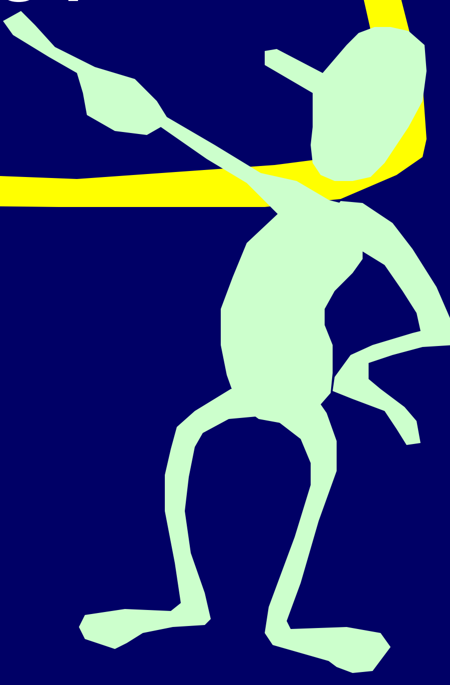
Example:

{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

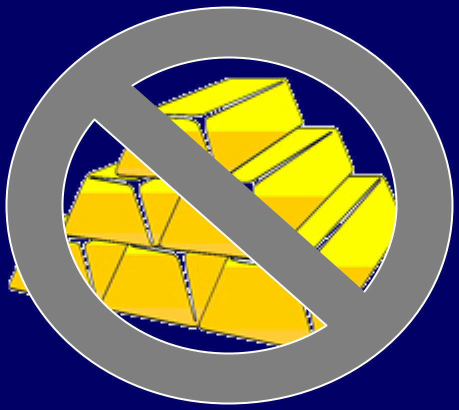
Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose a multiset of size k from n types of elements



Back to the pirates



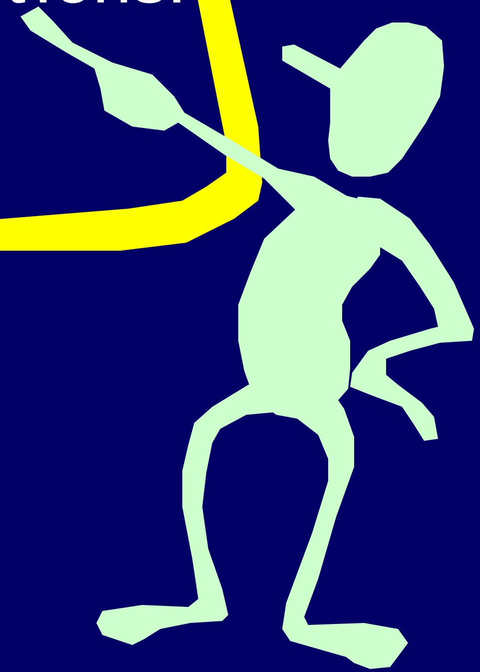
How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

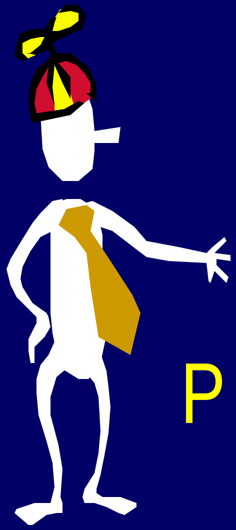
$$\binom{5+20-1}{20} = \binom{24}{20} = \binom{24}{4}$$

$$x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \dots, x_{n-1}, x_n \geq 0$$

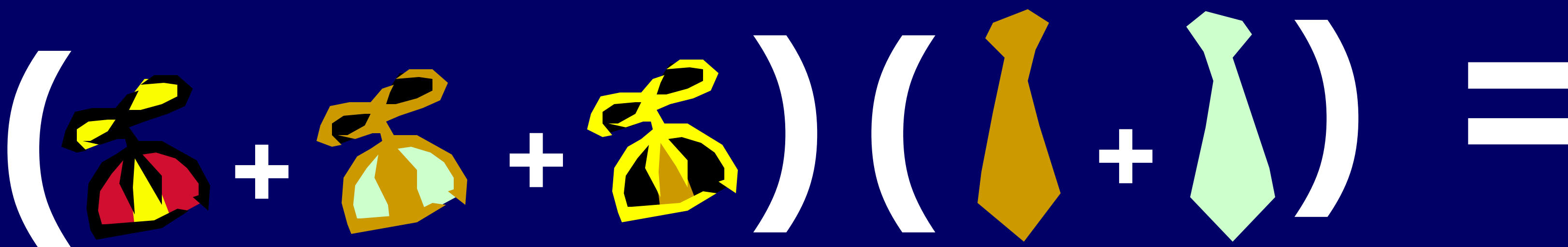
has $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ integer solutions.

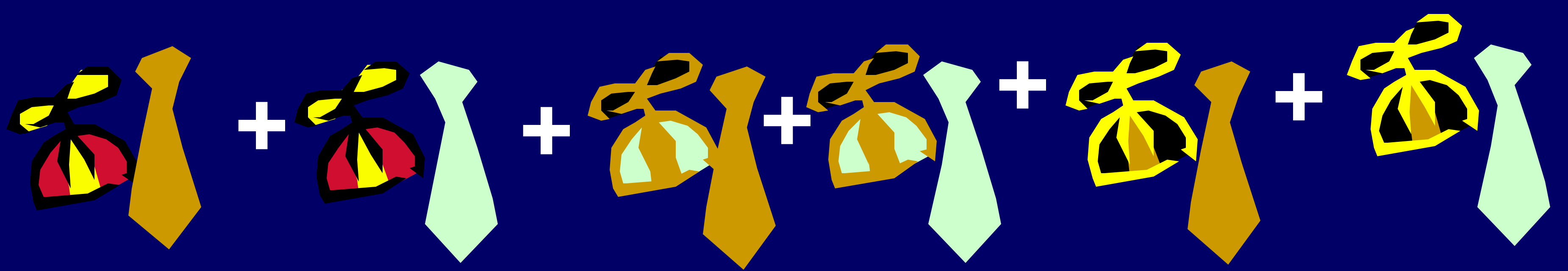


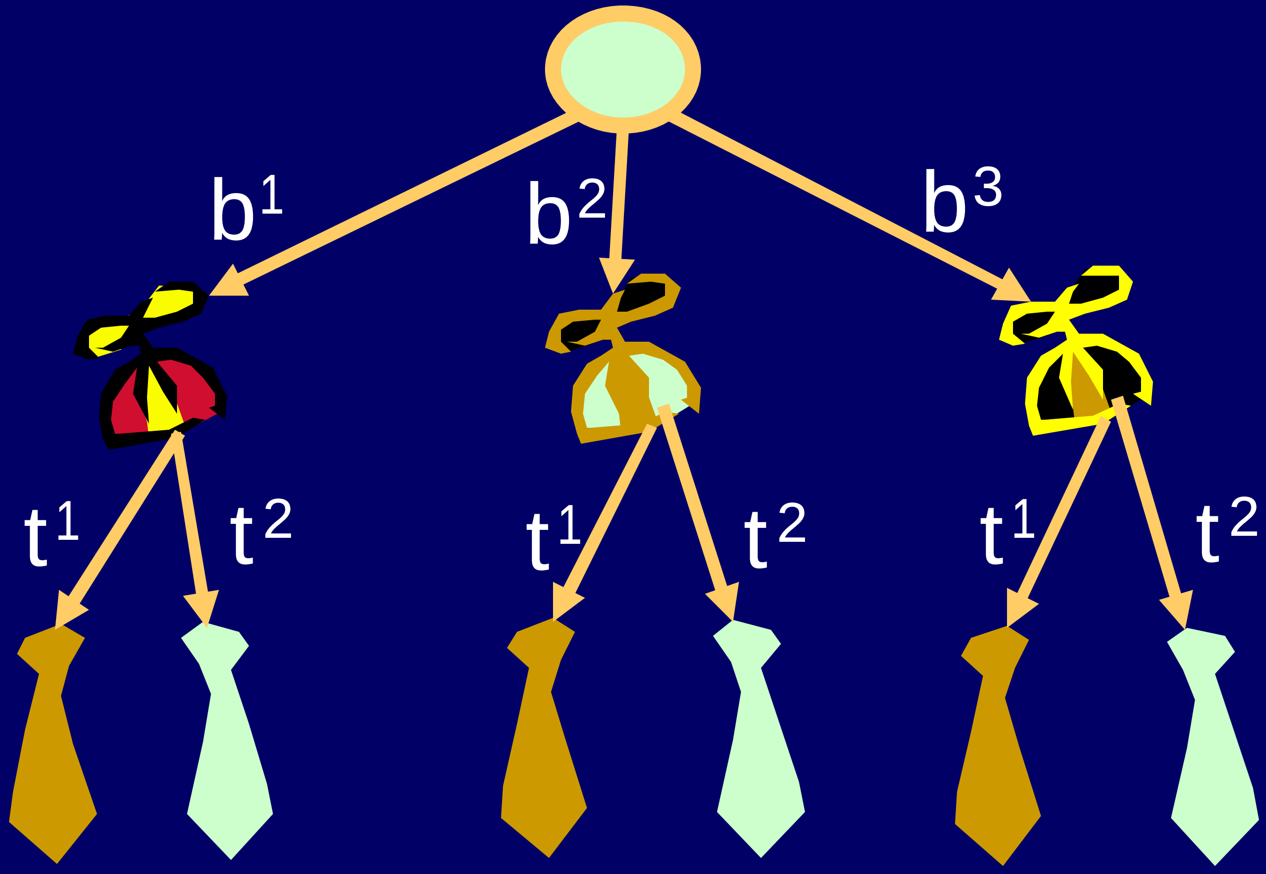


POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

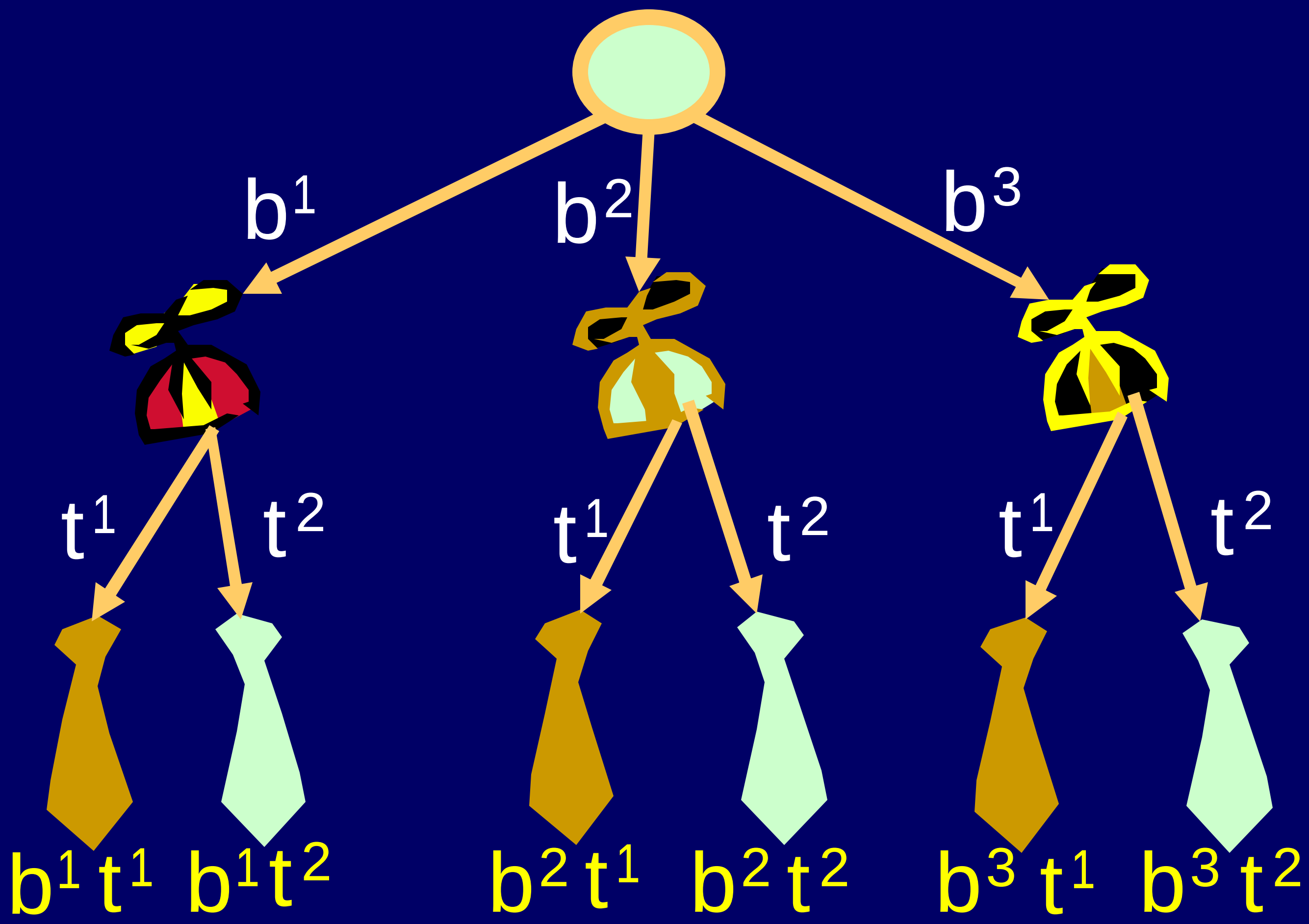
Products of Sum = Sums of Products



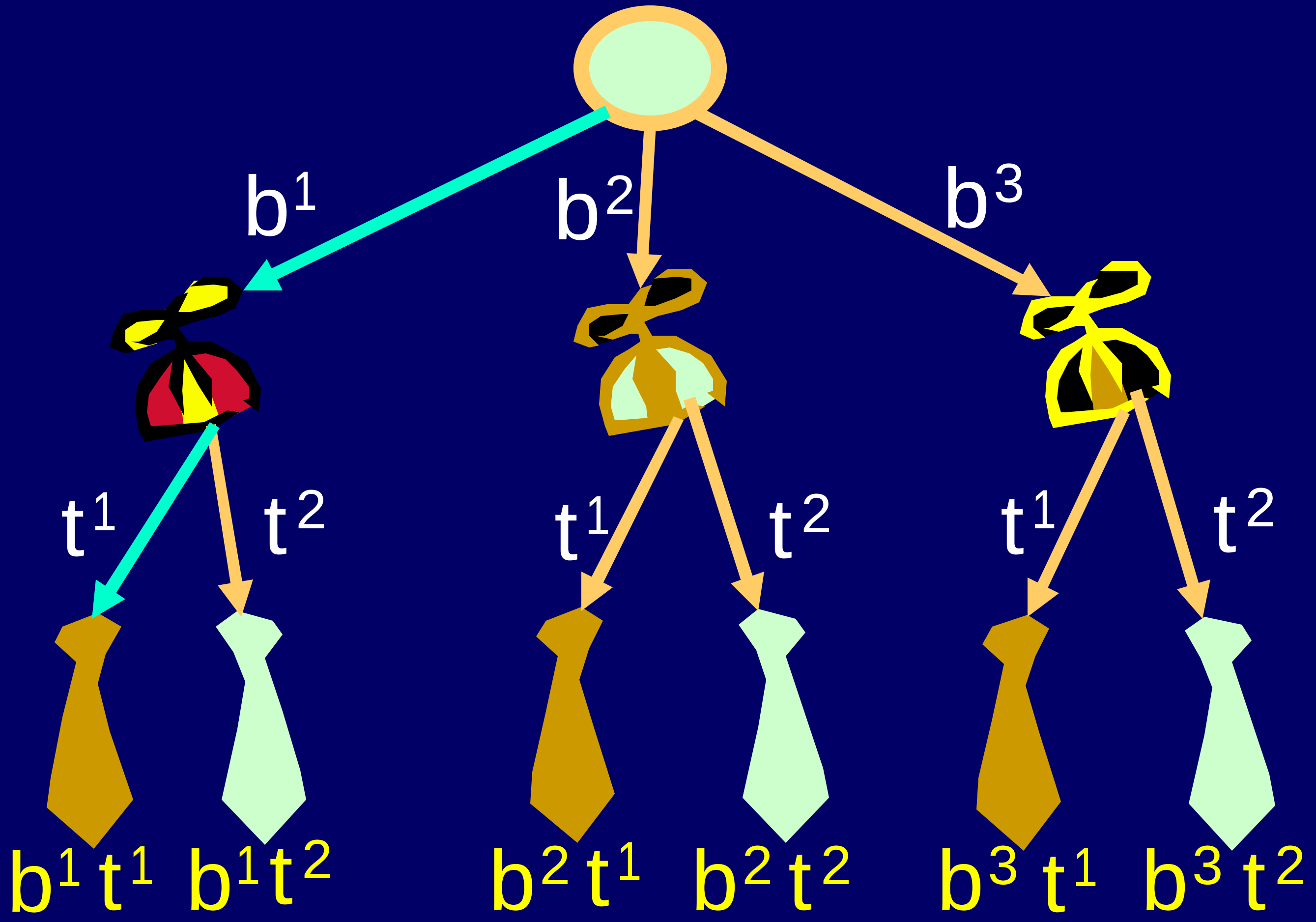




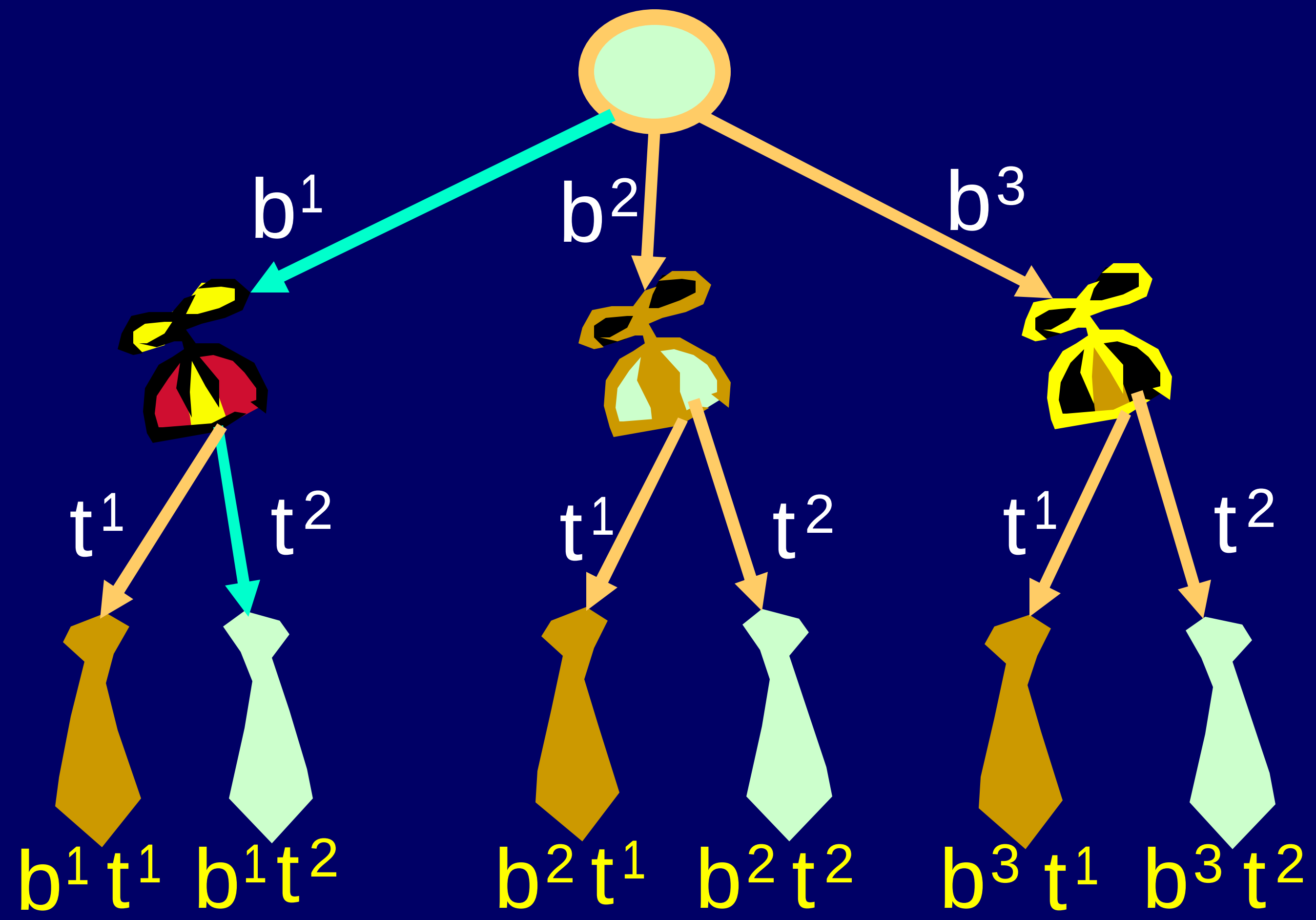
b1
b2
b3
t1
t2
t1
t2
t1
t2



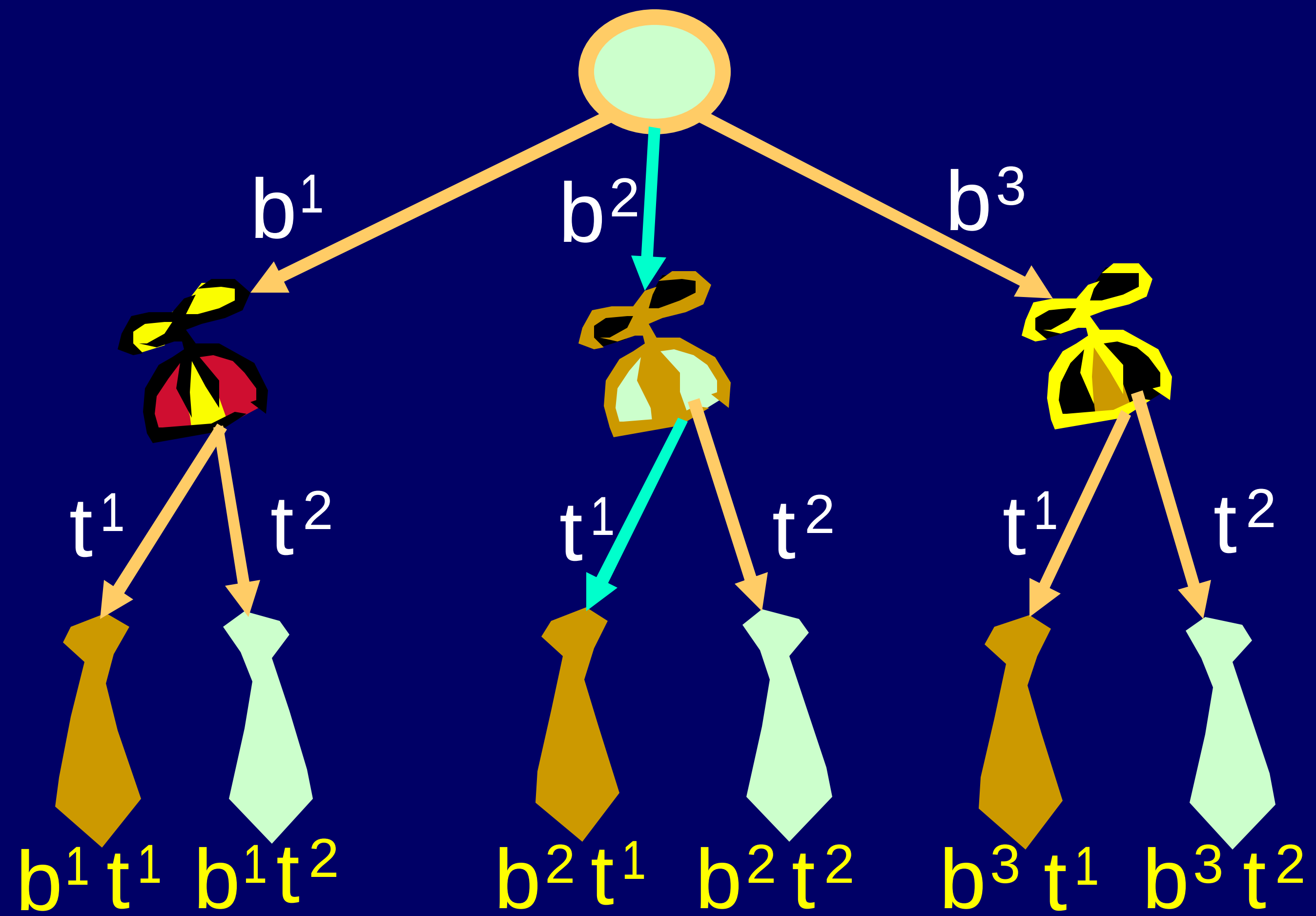
$(b^1 + b^2 + b^3)(t^1 + t^2) =$



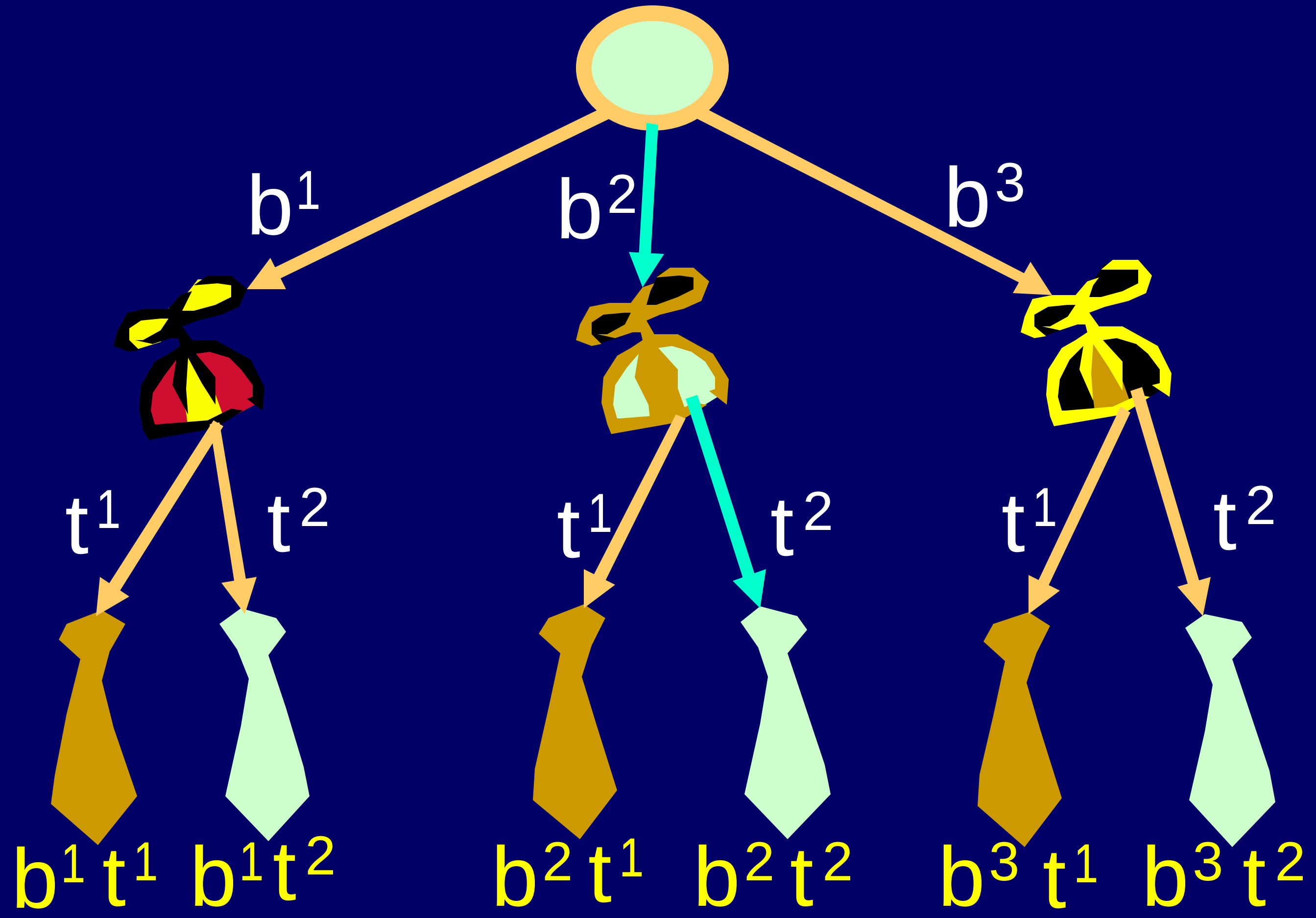
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1 t^1 +$$



$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 +$

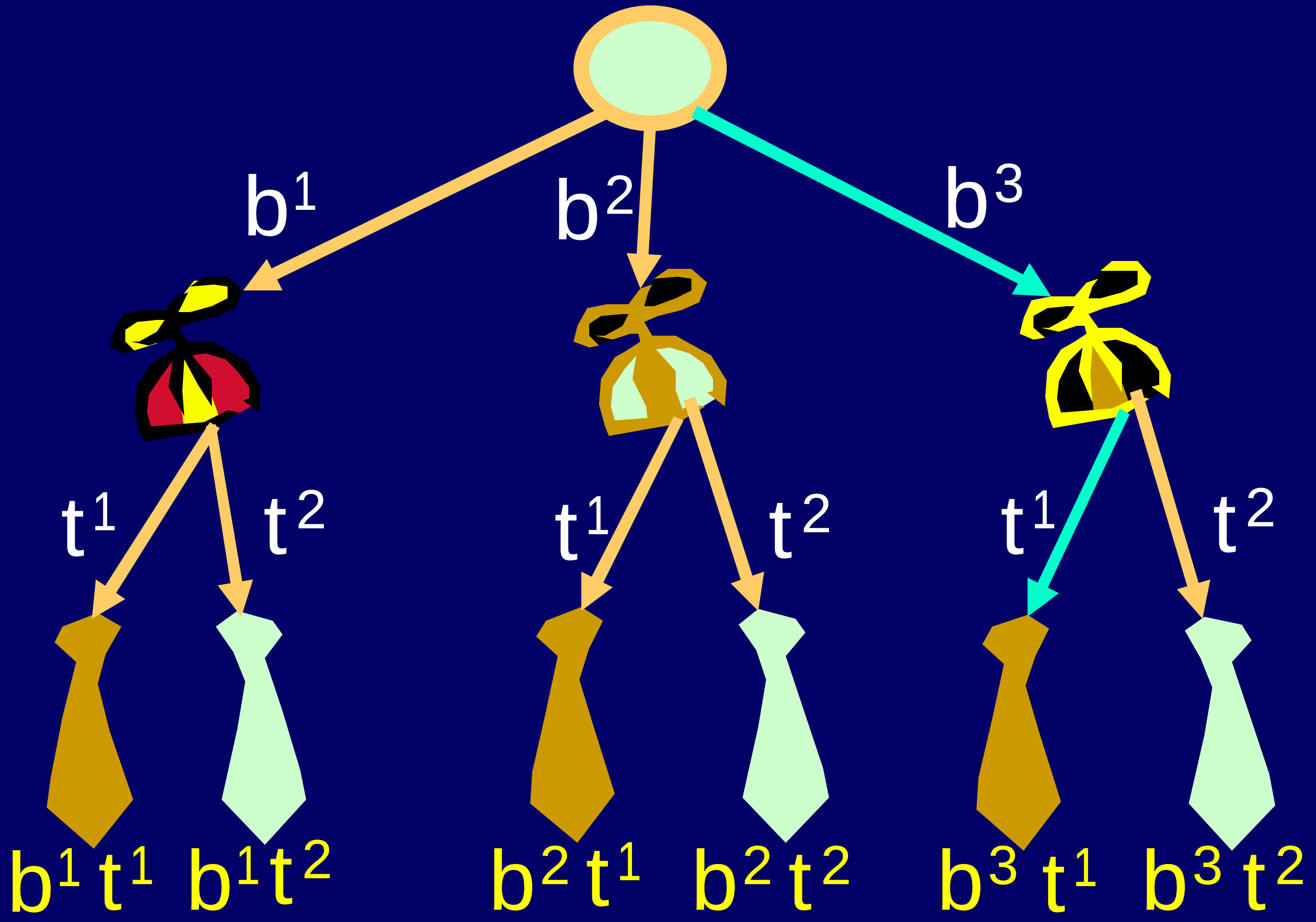


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 +$$

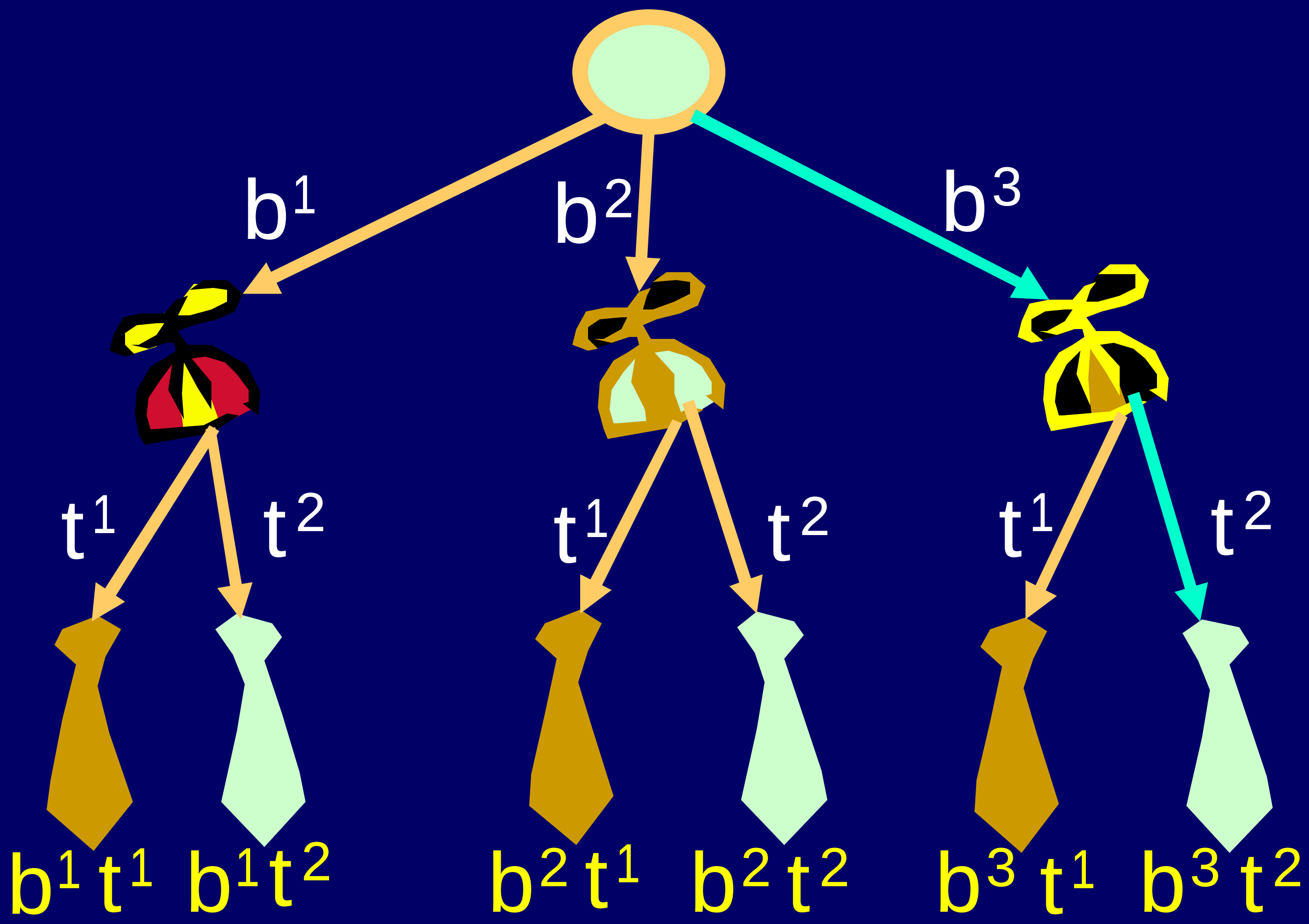


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$

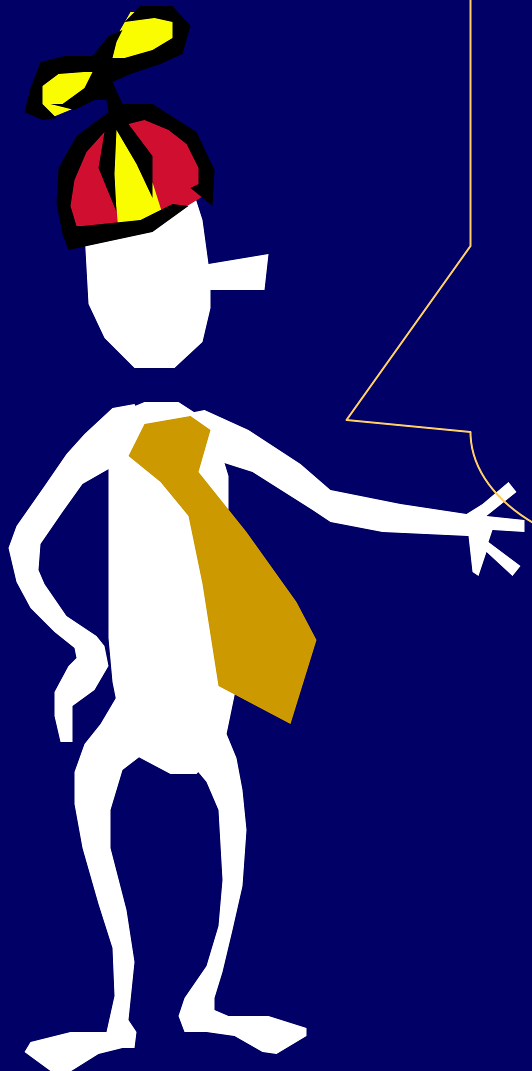
$$+$$



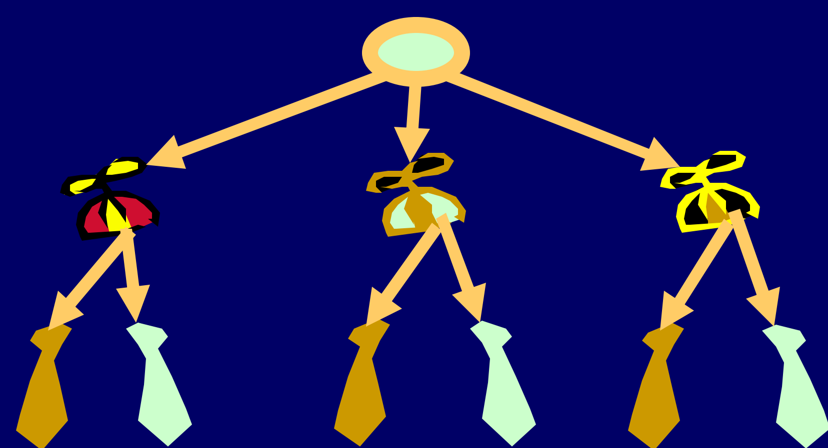
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$
$$+ b^3t^1 +$$



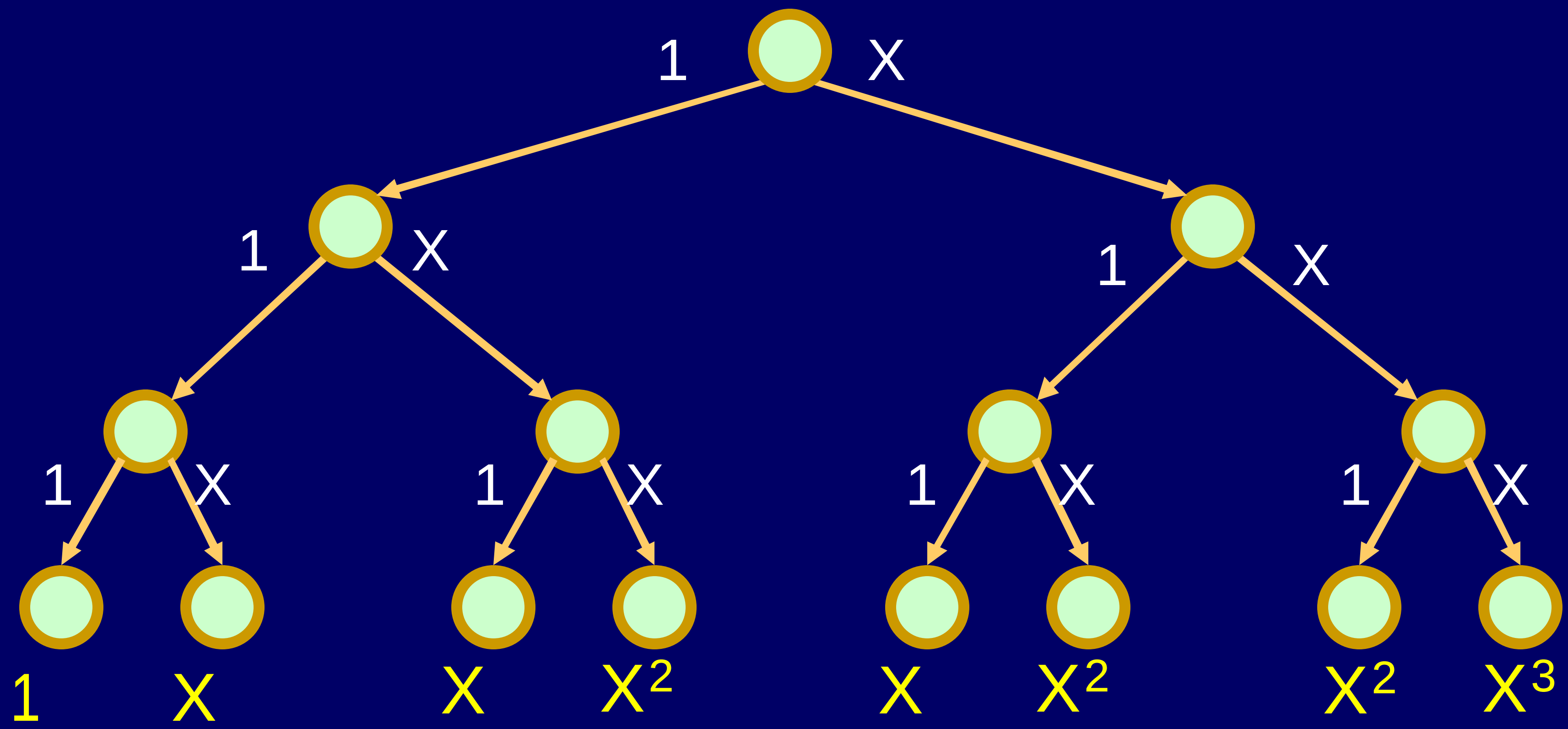
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$



There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!



Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

What is a closed form expression for c_k?

$$(1+X)^n = c_0 + c_1 X + c_2 X^2 + \ldots + c_n X^n$$

What is a closed form expression for c_n?

$$(1+X)^n$$

$$= \underbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}_{n \text{ times}}$$

After multiplying things out, but *before combining like terms*, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with *exactly k X's*.

$$c_k = \binom{n}{k}$$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

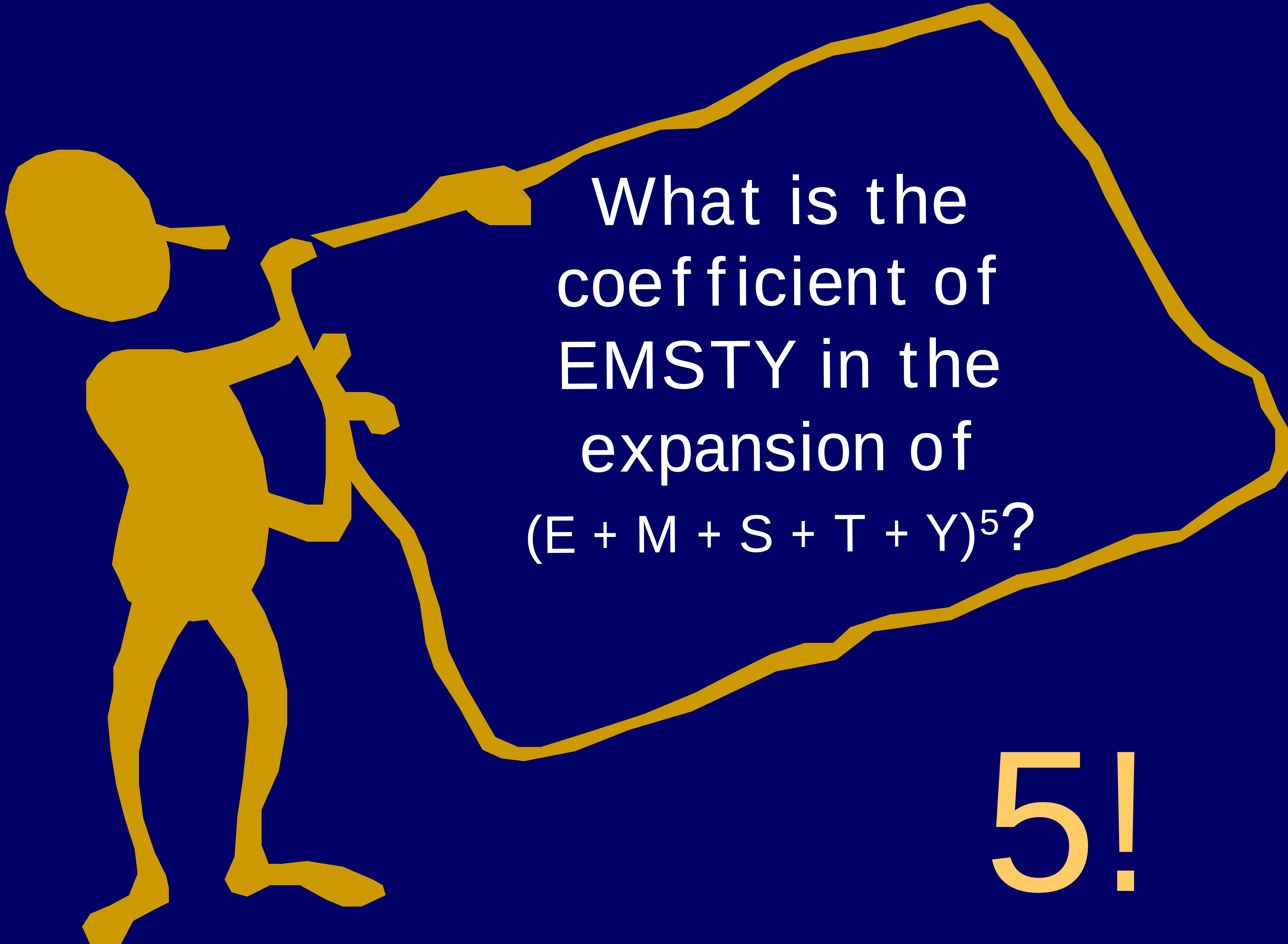
$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

The Binomial Formula

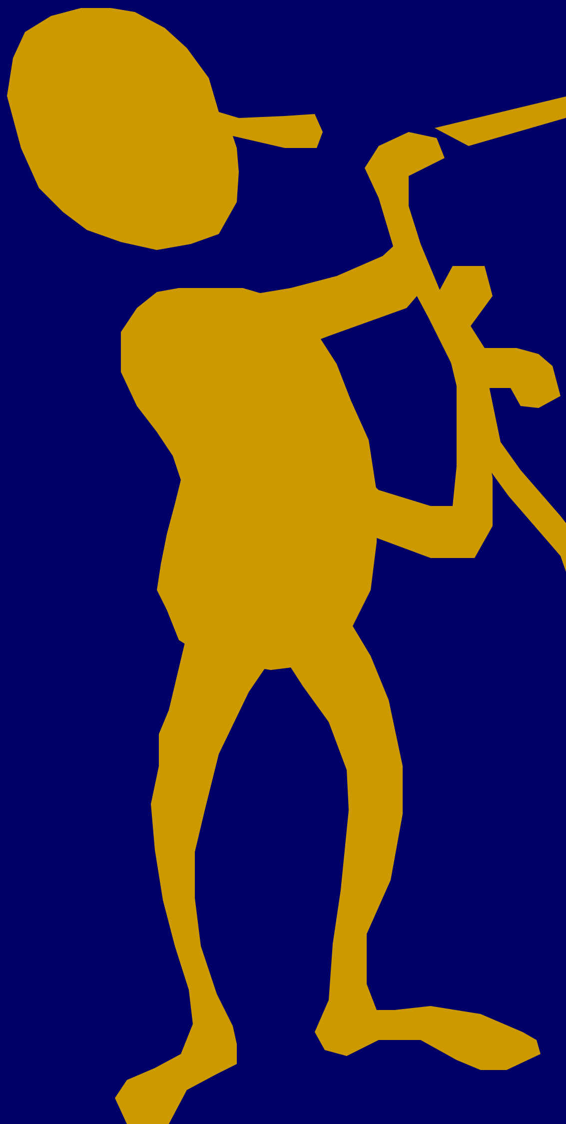
$$(X+Y)^n = \binom{n}{0}X^0Y^n + \binom{n}{1}X^1Y^{n-1} + \binom{n}{2}X^2Y^{n-2} + \ldots + \binom{n}{k}X^kY^{n-k} + \ldots + \binom{n}{n}X^nY^0$$

The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{k=n} \binom{n}{k} X^k Y^{n-k}$$



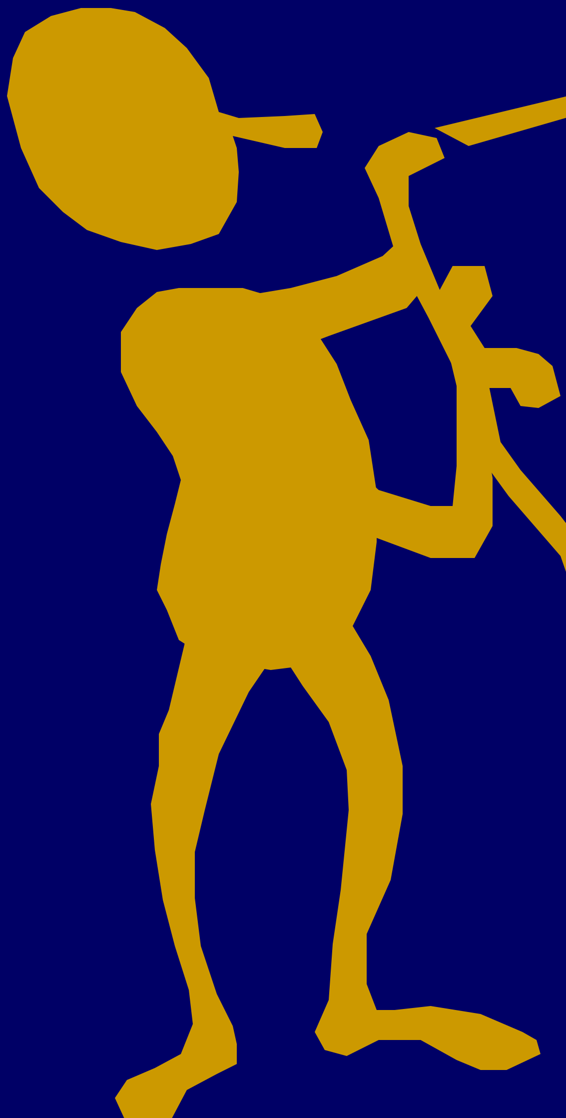
5!



What is the coefficient of EMS^3TY in the expansion of

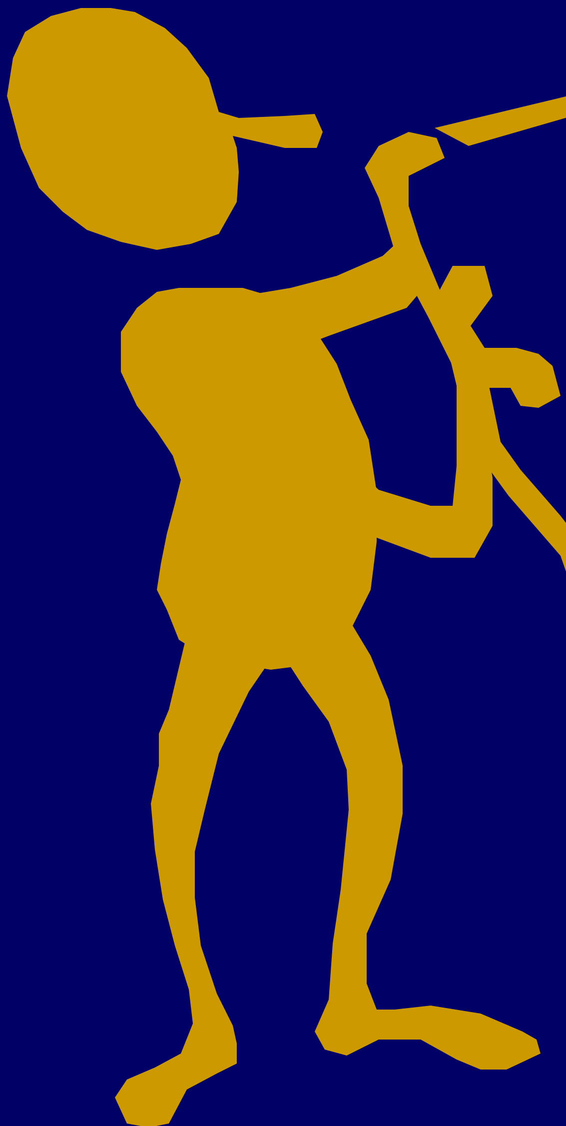
$(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS.



What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA.



What is the coefficient of $X_1^{r_1} X_2^{r_2} X_3^{r_3} \dots X_k^{r_k}$ in the expansion of $(X_1+X_2+X_3+\dots+X_k)^n$?

$$\frac{n!}{r_1!\, r_2!\, r_3! \dots r_k!}$$

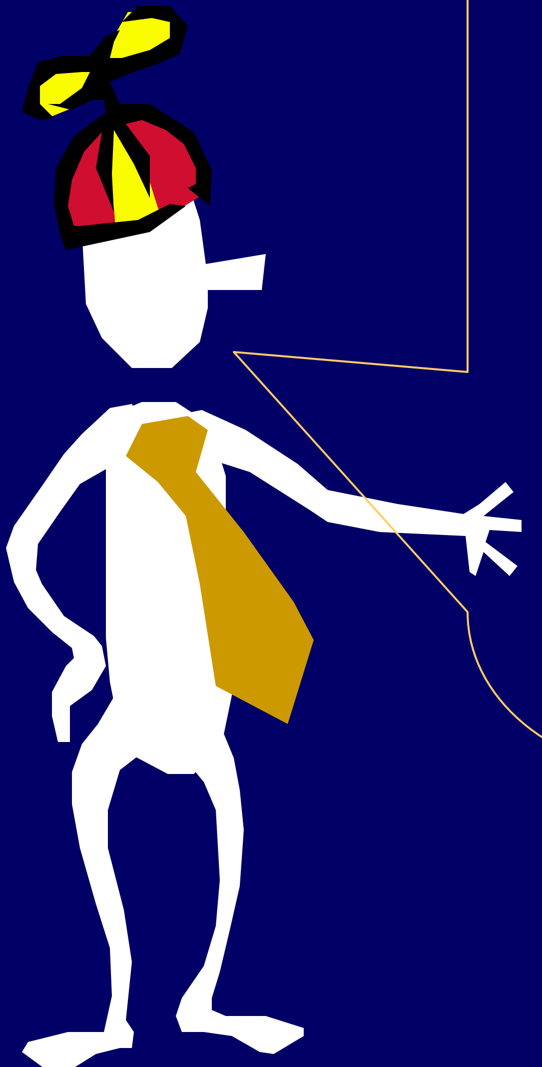
Multinomial Coefficients

$$\binom{n}{r_1; r_2; \ldots; r_k} \equiv \begin{cases} 0 \text{ if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \ldots r_k!} \end{cases}$$

$$\binom{n}{k; n-k} = \binom{n}{k}$$

The Multinomial Formula

$$\left(X_1 + X_2 + \ldots + X_k\right)^n = \sum_{\substack{r_1, r_2, \ldots, r_k \\ \sum r_i = n}} \binom{n}{r_1; r_2; \ldots; r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$



There is much, much more to be said about how polynomials encode counting questions!

References

Applied Combinatorics, by Alan Tucker