Counting I: One To One
Correspondence and Choice Trees
How many seats in this auditorium?

Hint: Count without counting!
If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?
Addition Rule

Let $A$ and $B$ be two disjoint finite sets.

The size of $A \cup B$ is the sum of the size of $A$ and the size of $B$.

$$|A \cup B| = |A| + |B|$$
Corollary (by induction)

Let $A_1, A_2, A_3, \ldots, A_n$ be disjoint, finite sets.

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} \left| A_i \right|
\]
Suppose I roll a white die and a black die.
\( S \equiv \text{Set of all outcomes where the dice show different values.} \)

\[ |S| = ? \]

\( A_i \equiv \text{set of outcomes where the black die says } i \text{ and the white die says something else.} \)

\[ |S| = \bigcup_{i=1}^{6} |A_i| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30 \]
\[ S \equiv \text{Set of all outcomes where the dice show different values.} \]
\[ |S| = ? \]

\[ T \equiv \text{set of outcomes where dice agree.} \]

\[ |S \cup T| = \# \text{of outcomes} = 36 \]
\[ |S| + |T| = 36 \quad |T| = 6 \]
\[ |S| = 36 - 6 = 30 \]
$S \equiv \text{Set of all outcomes where the black die shows a smaller number than the white die.} \quad |S| = ?$

$A_i \equiv \text{set of outcomes where the black die says } i \text{ and the white die says something larger.}$

\[
S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6
\]
\[
|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15
\]
$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. \quad \quad \quad \quad \quad \quad |S| = ?

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

\[ |S| + |L| = 30 \]

It is clear by symmetry that $|S| = |L|$.

Therefore $|S| = 15$
It is clear by symmetry that $|S| = |L|$. 
Pinning down the idea of symmetry by exhibiting a correspondence.

Let’s put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.
Pinning down the idea of symmetry by exhibiting a correspondence.

Let’s put each outcome in $S$ in correspondence with an outcome in $L$ by swapping the color of the dice.

Each outcome in $S$ gets matched with exactly one outcome in $L$, with none left over.

Thus: $|S| = |L|$. 
Let \( f: A \rightarrow B \) be a function from a set \( A \) to a set \( B \).

\( f \) is 1-1 if and only if
\[
\forall x, y \in A, \ x \neq y \implies f(x) \neq f(y)
\]

\( f \) is onto if and only if
\[
\forall z \in B \ \exists x \in A \ \ f(x) = z
\]
Let $f: A \rightarrow B$ be a function from a set $A$ to a set $B$.

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$f$ is onto if and only if
\[ \forall z \in B \ \exists x \in A \ f(x) = z \]
Let's restrict our attention to finite sets.
$\exists \text{1-1 } f: A \rightarrow B \Rightarrow |A| \leq |B|$
\exists \text{ onto } f: A \rightarrow B \Rightarrow |A| \geq |B|
$\exists$ 1-1 onto $f : A \rightarrow B \Rightarrow |A| = |B|$
1-1 Onto Correspondence
(just “correspondence” for short)
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

It’s one of the most important mathematical ideas of all time!
Question: How many n-bit sequences are there?

\[\begin{array}{ccc}
000000 & \leftrightarrow & 0 \\
000001 & \leftrightarrow & 1 \\
000010 & \leftrightarrow & 2 \\
000011 & \leftrightarrow & 3 \\
\vdots & \leftrightarrow & \vdots \\
1...11111 & \leftrightarrow & 2^{n-1}
\end{array}\]

\(2^n\) sequences
$S = \{a,b,c,d,e\}$ has many subsets.

$\{a\}, \{a,b\}, \{a,d,e\}, \{a,b,c,d,e\}, \{e\}, \emptyset, ...$

The empty set is a set with all the rights and privileges pertaining thereto.
Question: How many subsets can be formed from the elements of a 5-element set?

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<tr>
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<td>0</td>
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\{b \ c \ e\}

1 means “TAKE IT”
0 means “LEAVE IT”
Question: How many subsets can be formed from the elements of a 5-element set?

Each subset corresponds to a 5-bit sequence (using the “take it or leave it” code)

<table>
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\[ S = \{a_1, a_2, a_3, \ldots, a_n\} \]
\[ b = b_1b_2b_3\ldots b_n \]

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<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(\ldots)</td>
<td>(a_n)</td>
</tr>
<tr>
<td>(b_1)</td>
<td>(b_2)</td>
<td>(b_3)</td>
<td>(\ldots)</td>
<td>(b_n)</td>
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\[ f(b) = \{a_i \mid b_i=1\} \]
f is 1-1: Any two distinct binary sequences b and b' have a position i at which they differ. Hence, f(b) is not equal to f(b') because they disagree on element $a_i$.

\[
f(b) = \{ a_i \mid b_i = 1 \} \]
Let $S$ be a subset of $\{a_1, \ldots, a_n\}$. Let $b_k = 1$ if $a_k$ in $S$; $b_k = 0$ otherwise. $f(b_1 b_2 \ldots b_n) = S$. 

$f$ is onto: 

$$f(b) = \{ a_i \mid b_i = 1 \}$$
The number of subsets of an \( n \)-element set is \( 2^n \).
Let $f: A \rightarrow B$ be a function from a set $A$ to a set $B$.

$f$ is 1-1 if and only if
\[ \forall x, y \in A, \quad x \neq y \Rightarrow f(x) \neq f(y) \]

$f$ is onto if and only if
\[ \forall z \in B \quad \exists x \in A \quad f(x) = z \]
Let $f: \mathbb{A} \rightarrow \mathbb{B}$ be a function from a set $\mathbb{A}$ to a set $\mathbb{B}$.

$f$ is a 1 to 1 correspondence iff
\[ \forall z \in \mathbb{B} \; \exists \text{ exactly one } x \in \mathbb{A} \; \text{s.t.} \; f(x) = z \]

$f$ is a $k$ to 1 correspondence iff
\[ \forall z \in \mathbb{B} \; \exists \text{ exactly } k \; x \in \mathbb{A} \; \text{s.t.} \; f(x) = z \]
To count the number of horses in a barn, we count the number of hoofs and then divide by 4.
If Finite set A has a k to 1 correspondence to finite set B, then \( \#B = \#A/k \)
I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?
A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many items on the menu?
• $5 + 6 + 3 + 7 = 21$

How many ways to choose a complete meal?
• $5 \times 6 \times 3 \times 7 = 630$
A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many ways to order a meal if I might not have some of the courses?

- $6 \times 7 \times 4 \times 8 = 1344$
Hobson’s restaurant has only 1 appetizer, 1 entree, 1 salad, and 1 dessert.

$2^4$ ways to order a meal if I might not have some of the courses.

Same as number of subsets of the set \{Appetizer, Entrée, Salad, Dessert\}
Leaf Counting Lemma

Let $T$ be a depth $n$ tree when each node at depth $0 \leq i \leq n-1$ has $P_{i+1}$ children. The number of leaves of $T$ is given by:

$$P_1 P_1 P_2 \ldots P_n$$
We can use a “choice tree” to represent the construction of objects of the desired type.
Label each leaf with the object constructed by the choices along the path to the leaf.
2 choices for first bit
2 choices for second bit
2 choices for third bit
...
2 choices for the $n^{th}$
A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.
A choice tree provides a “choice tree representation” of a set $S$, if

1) Each leaf label is in $S$
2) No two leaf labels are the same
We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.
Product Rule

IF $S$ has a choice tree representation with $P_1$ possibilities for the first choice, $P_2$ for the second, and so on,

THEN

there are $P_1P_2P_3...P_n$ objects in $S$

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of $S$. 
Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with \( P_1 \) possibilities for the first choice, \( P_2 \) for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

there are \( P_1 P_2 P_3 \ldots P_n \) objects of type S.
How many different orderings of deck with 52 cards?

What type of object are we making?

• Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;
51 possible choices for the second card;
50 possible choices for the third card;

... 

1 possible choice for the 52^\text{cond} card.
How many different orderings of deck with 52 cards?

By the product rule:

\[ 52 \times 51 \times 50 \times \ldots \times 3 \times 2 \times 1 = 52! \]

52 “factorial” orderings
A permutation or arrangement of $n$ objects is an ordering of the objects.

The number of permutations of $n$ distinct objects is $n!$.
How many sequences of 7 letters are there?

$26^7$
How many sequences of 7 letters contain at least two of the same letter?

$$26^7 - 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20$$
Sometimes it is easiest to count the number of objects with property $Q$, by counting the number of objects that do not have property $Q$. 
A formalization

Let $S(x): \Sigma^* \rightarrow \{\text{True, False}\}$ be any predicate.

We can associate $S$ with the set:

$\text{OBJECTS}_S = \{x \in \Sigma^* \mid S(x)\}$

the “object space” $S$ (or objects of type $S$)

When $\text{OBJECTS}_S$ is finite, let us define

$\#\text{OBJECTS}_S = \text{the size of } \text{OBJECTS}_S$

In fact, define $\#S$ as $\#\text{OBJECTS}_S$
Object property $Q$ on object space $S$

Consider $Q(x): \text{OBJECTS}_S \rightarrow \{\text{True, False}\}$

Define: $Q(x): \text{OBJECTS}_S \rightarrow \{\text{True, False}\}$
As $\text{Input}(x)$; return $\text{NOT } Q(x)$

Proposition: $\#Q = \#S - \#(: Q)$
How many of our objects have property Q in object space S?

\[ \#Q = \#\text{OBJECTS}_S - \#(: Q) \]
Helpful Advice:

In logic, it can be useful to represent a statement in the contrapositive.

In counting, it can be useful to represent a set in terms of its complement.
If 10 horses race, how many orderings of the top three finishers are there?

$$10 \times 9 \times 8 = 720$$
The number of ways of ordering, permuting, or arranging \( r \) out of \( n \) objects.

\( n \) choices for first place, \( n-1 \) choices for second place, \( \ldots \)

\[
n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))\]

\[
= \frac{n!}{(n-r)!}
\]
Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?
• $52 \times 51$

How many unordered pairs?
• $\frac{52 \times 51}{2} \leftarrow$ divide by overcount
  
  Each unordered pair is listed twice on a list of the ordered pairs, but we consider the ordered pairs to be the same.
Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?
• $52 \times 51$

How many unordered pairs?
• $52 \times 51 / 2 \leftarrow \text{divide by overcount}$

We have a 2 to 1 map from ordered pairs to unordered pairs. Hence: the #unordered pairs = (#ordered pairs)/2
Ordered Versus Unordered

From a deck of 52 cards how many **ordered** 5 card sequences can be formed?

• $52 \times 51 \times 50 \times 49 \times 48$

How many orderings of 5 cards?

• $5!$

How many **unordered** 5 card hands? pairs?

• $\frac{52\times51\times50\times49\times48}{5!} = 2,598,960$
A combination or choice of \( r \) out of \( n \) objects is an (unordered) set of \( r \) of the \( n \) objects.

The number of \( r \) combinations of \( n \) objects:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]
The number of subsets of size $r$ that can be formed from an $n$-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
How many 8 bit sequences have 2 0's and 6 1's?

Tempting, but incorrect:
8 ways to place first 0 times
7 ways to place second 0

Violates condition 2 of product rule!
Choosing position i for the first 0 and then position j for the second 0 gives the same sequence as choosing position j for the first 0 and position i for the second.
How many 8 bit sequences have 2 0’s and 6 1’s?

1) Choose the set of 2 positions to put the 0’s. The 1’s are forced.

\[
\binom{8}{2} \times 1 = \binom{6}{2}
\]

2) Choose the set of 6 positions to put the 1’s. The 0’s are forced.

\[
\binom{8}{6} \times 1 = \binom{2}{0}
\]
Symmetry in the formula:

\[ \binom{n}{r} \frac{n!}{r!(n-r)!} = \binom{n}{n-r} \]
How many hands have at least 3 aces?

\[ \binom{4}{3} = 4 \text{ ways of picking 3 of the 4 aces.} \]

\[ \binom{49}{2} = 1176 \text{ ways of picking 2 cards from the remaining 49 cards.} \]

\[ 4 \times 1176 = 4704 \]
How many hands have at least 3 aces?

How many hands have exactly 3 aces?

\[ \binom{4}{3} = 4 \text{ ways of picking 3 of the 4 aces.} \]

\[ \binom{48}{2} = 1128 \text{ ways of picking 2 cards non-ace cards.} \]

\[ 4 \times 1128 = 4512 \]

How many hands have exactly 4 aces?

\[ \binom{4}{4} = 1 \text{ way of picking 4 of the 4 aces.} \]

48 ways of picking one of the remaining cards

\[ 4512 + 48 = 4560 \]
4704 ≠ 4560

At least one of the two counting arguments is not correct.
Four different sequences of choices produce the same hand

\[
\begin{align*}
\binom{4}{3} &= 4 \text{ ways of picking 3 of the 4 aces.} \\
\binom{49}{2} &= 1176 \text{ ways of picking 2 cards from the remaining 49 cards.}
\end{align*}
\]

\[4 \times 1176 = 4704\]
Is the other argument correct? How do I avoid fallacious reasoning?
The Sleuth’s Criterion

Condition (2) of the product rule:

For any object it should be possible to reconstruct the sequence of choices which lead to it.
1) Choose 3 of 4 aces
2) Choose 2 of the remaining cards

Sleuth can’t determine which cards came from which choice.
Is the other argument correct? How do I avoid fallacious reasoning?
1) Choose 3 of 4 aces
2) Choose 2 non-ace cards

A♣ Q♠ A♦ A♥ K♦

Sleuth reasons:

The aces came from the first choice and the non-aces came from the second choice.
1) Choose 4 of 4 aces
2) Choose 1 non-ace

Sleuth reasons:

The aces came from the first choice and the non-ace came from the second choice.