Great Theoretical Ideas In Computer Science

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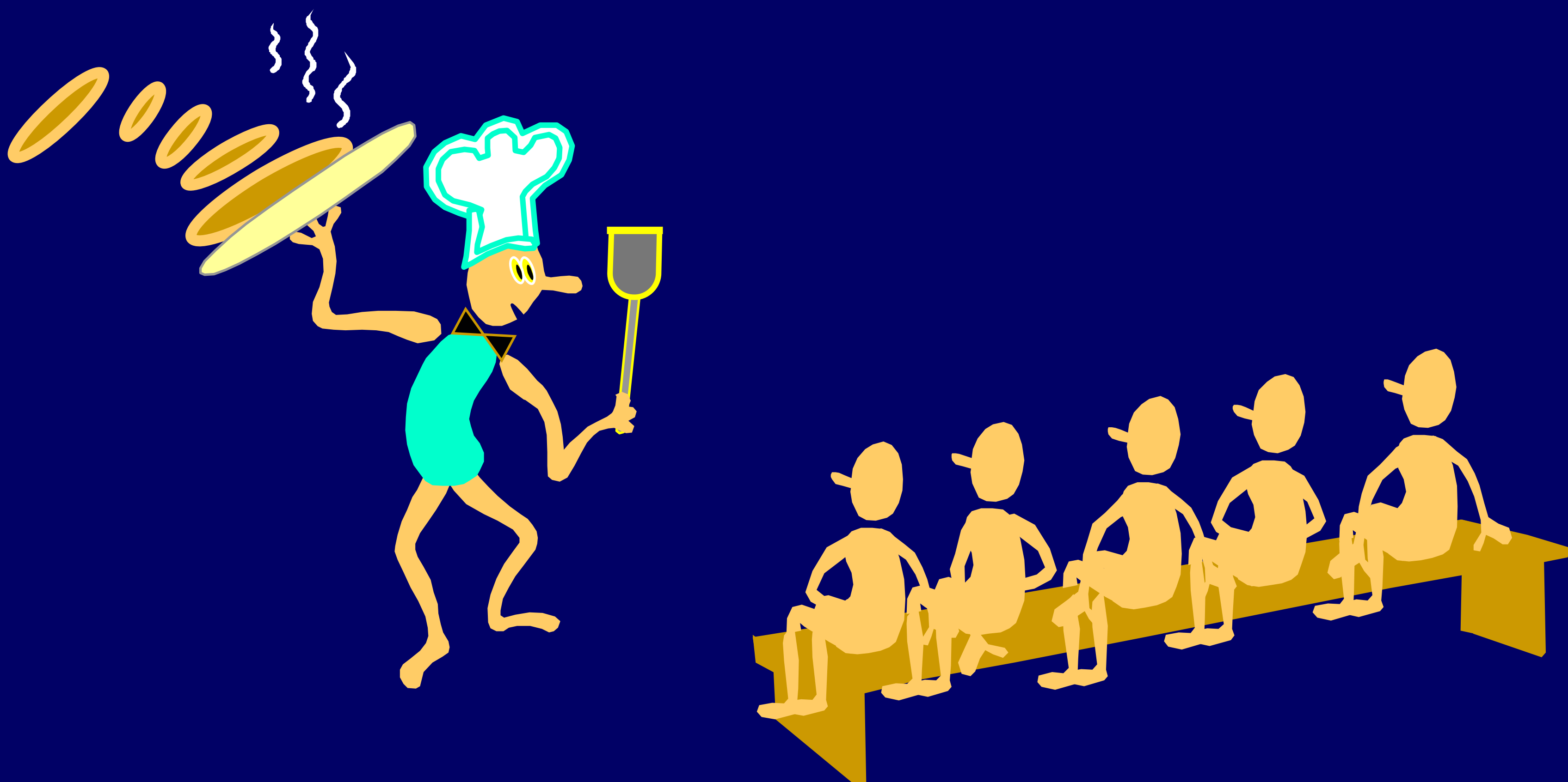
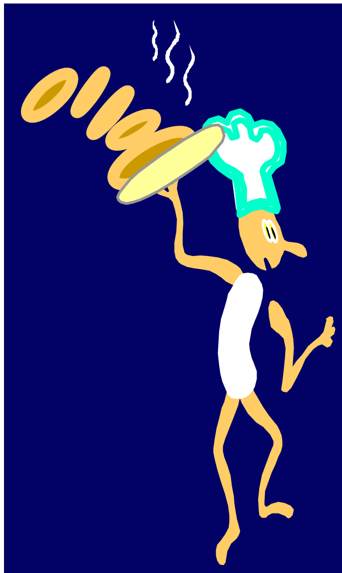
Lecture 3 Jan 18, 2005 Carnegie Mellon University

Pancakes With A Problem!




Please feel free
to ask questions!

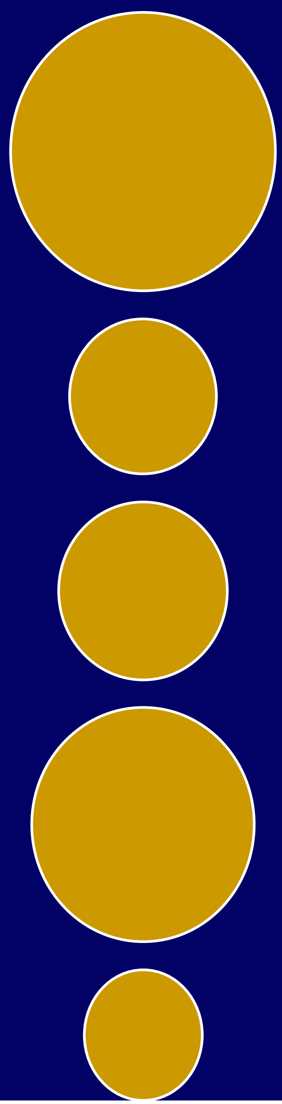


The chef at our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes.

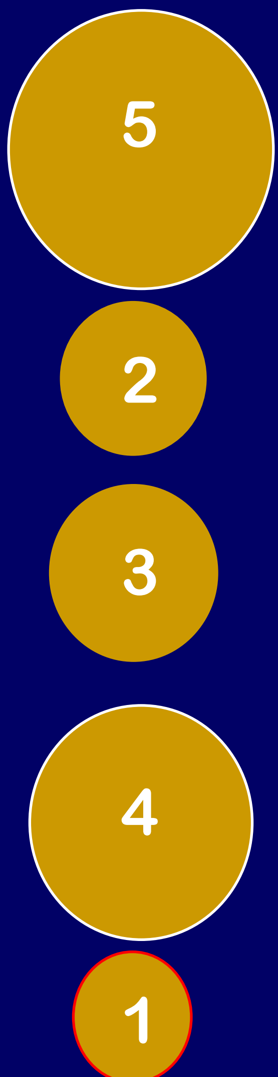
Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom).

I do this by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary.

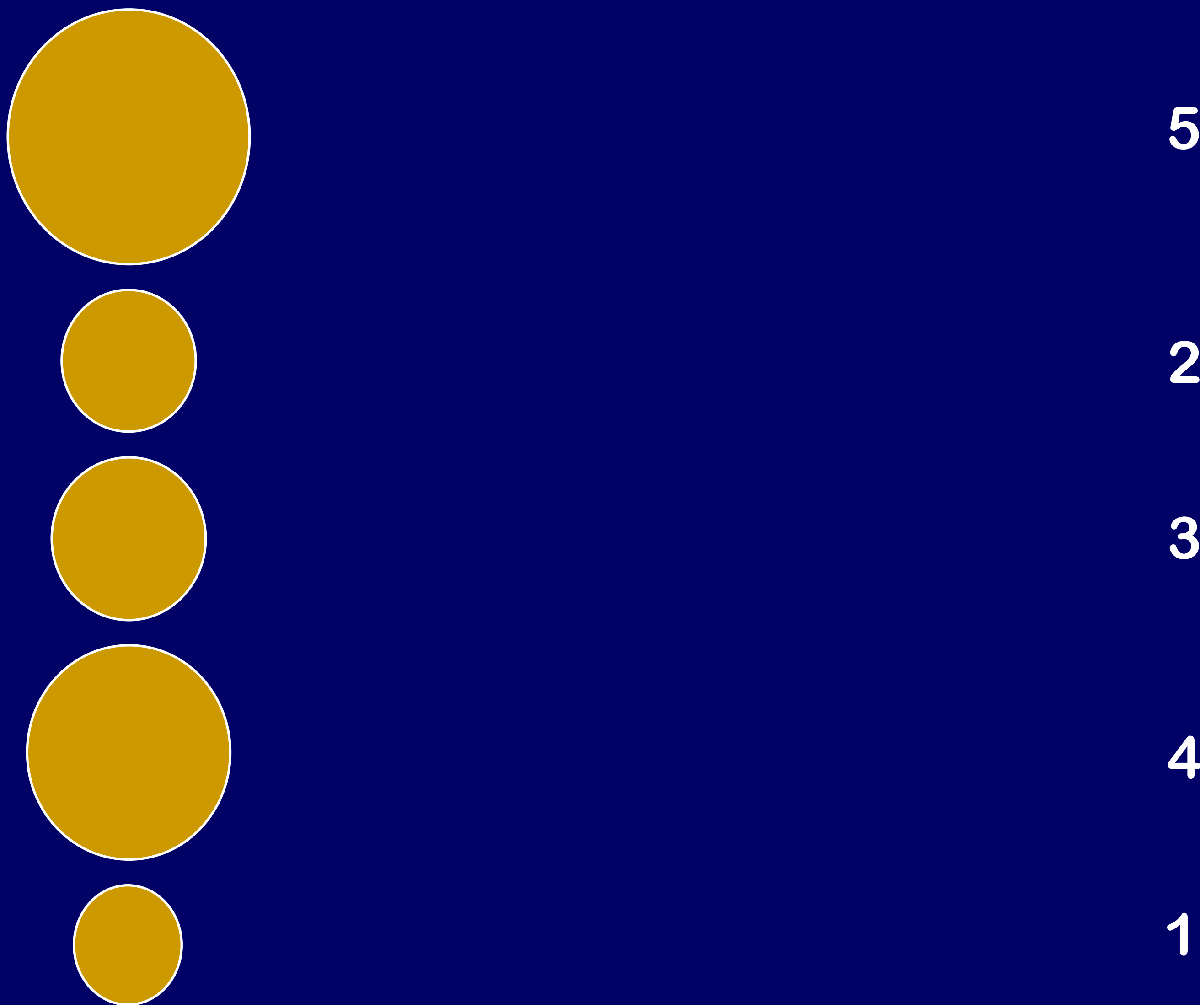
Developing A Notation: Turning pancakes into numbers



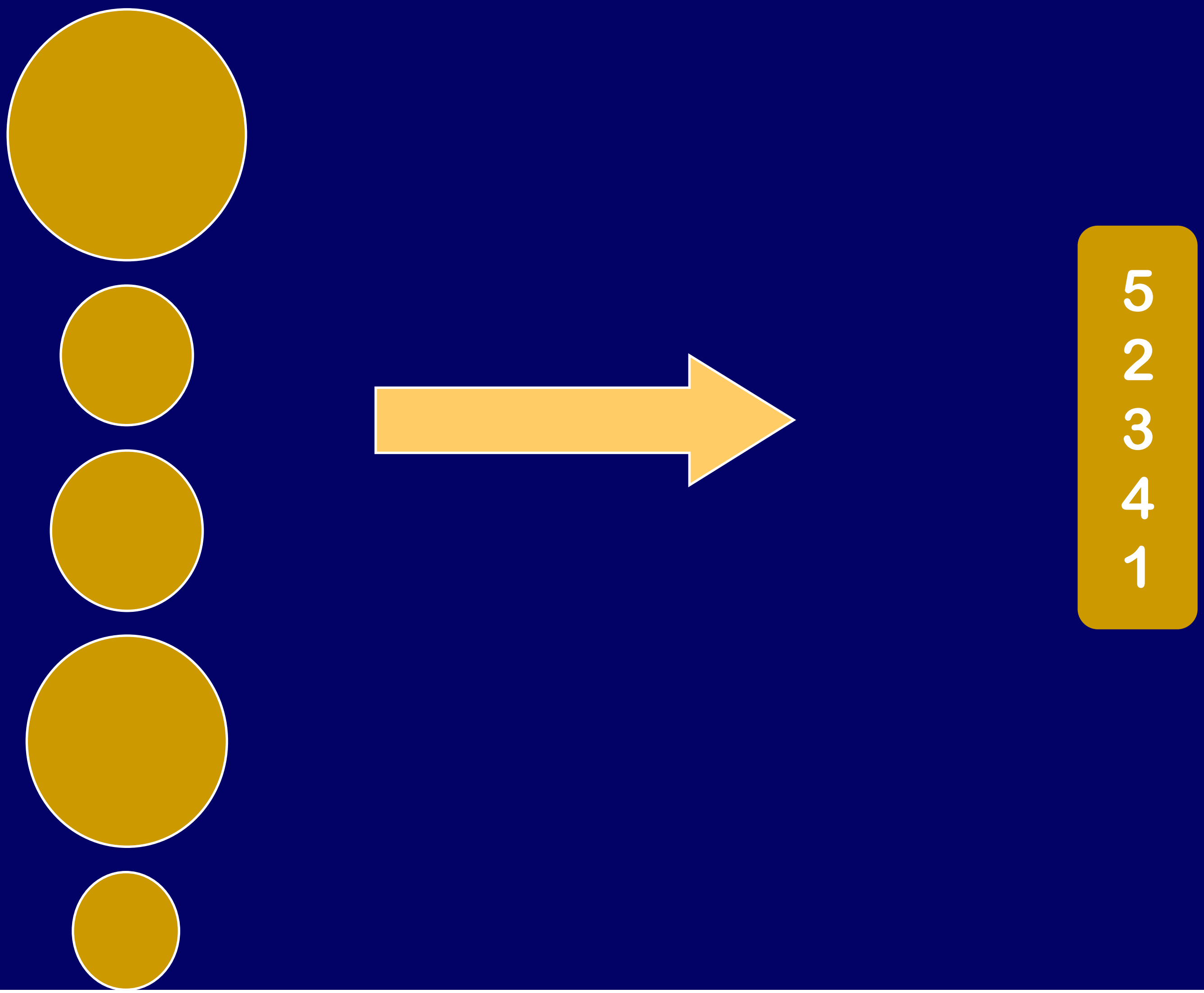
Developing A Notation:
Turning pancakes into numbers



Developing A Notation: Turning pancakes into numbers

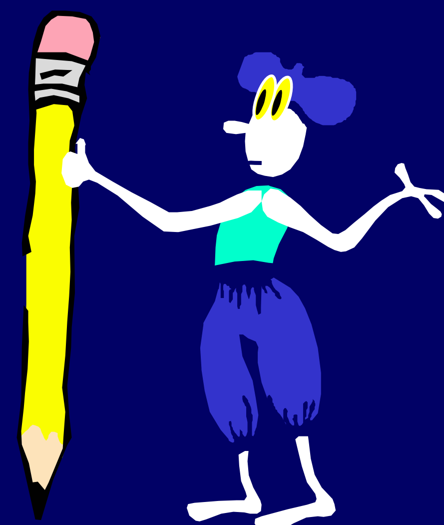


Developing A Notation: Turning pancakes into numbers

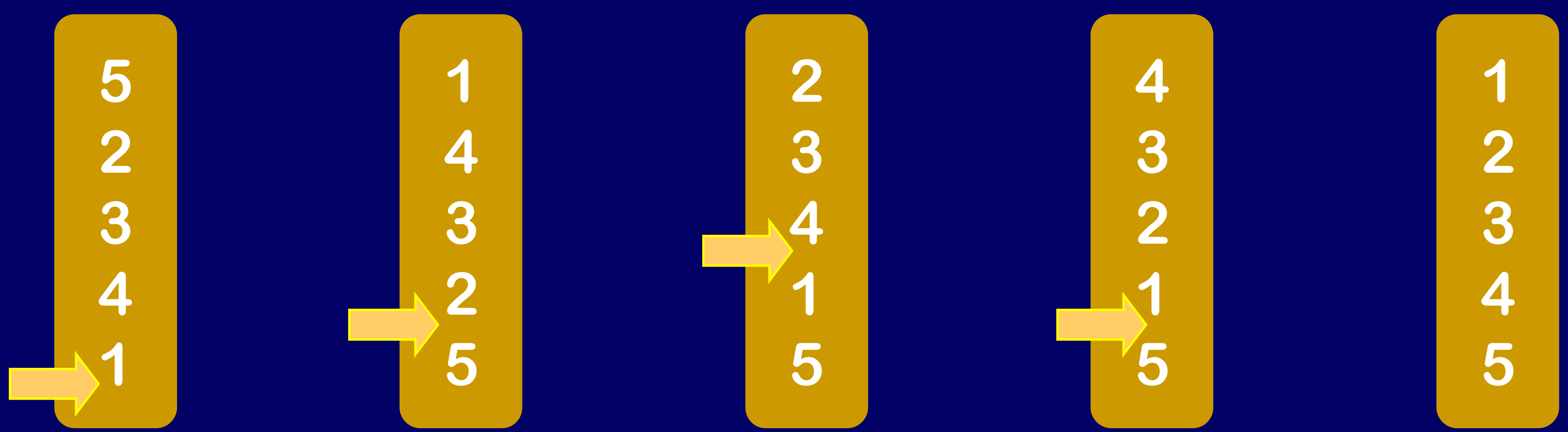


How do we sort this stack?
How many flips do we need?

5
2
3
4
1



4 Flips Are Sufficient



Algebraic Representation

X = The smallest number of flips required to sort:

5
2
3
4
1

$? \leq X \leq ?$

Lower Bound

Upper Bound

Algebraic Representation

X = The smallest number of flips required to sort:

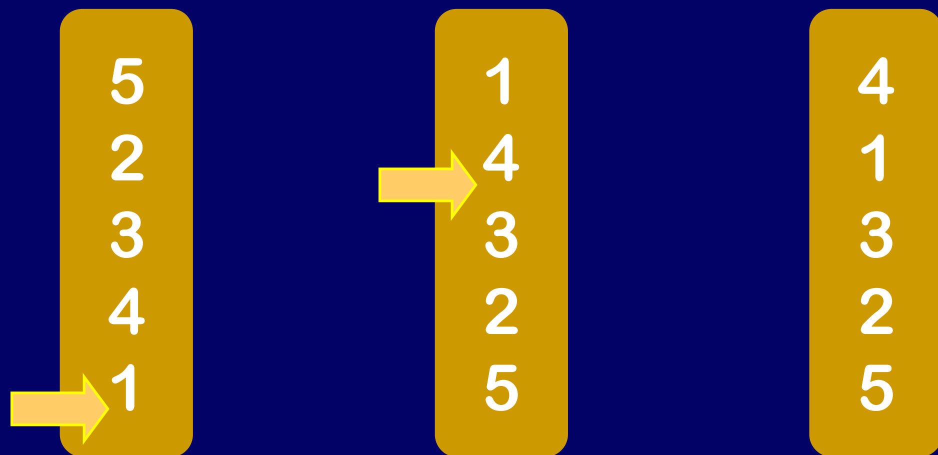
5
2
3
4
1

$? \leq X \leq 4$

Lower Bound

Upper Bound

4 Flips Are Necessary

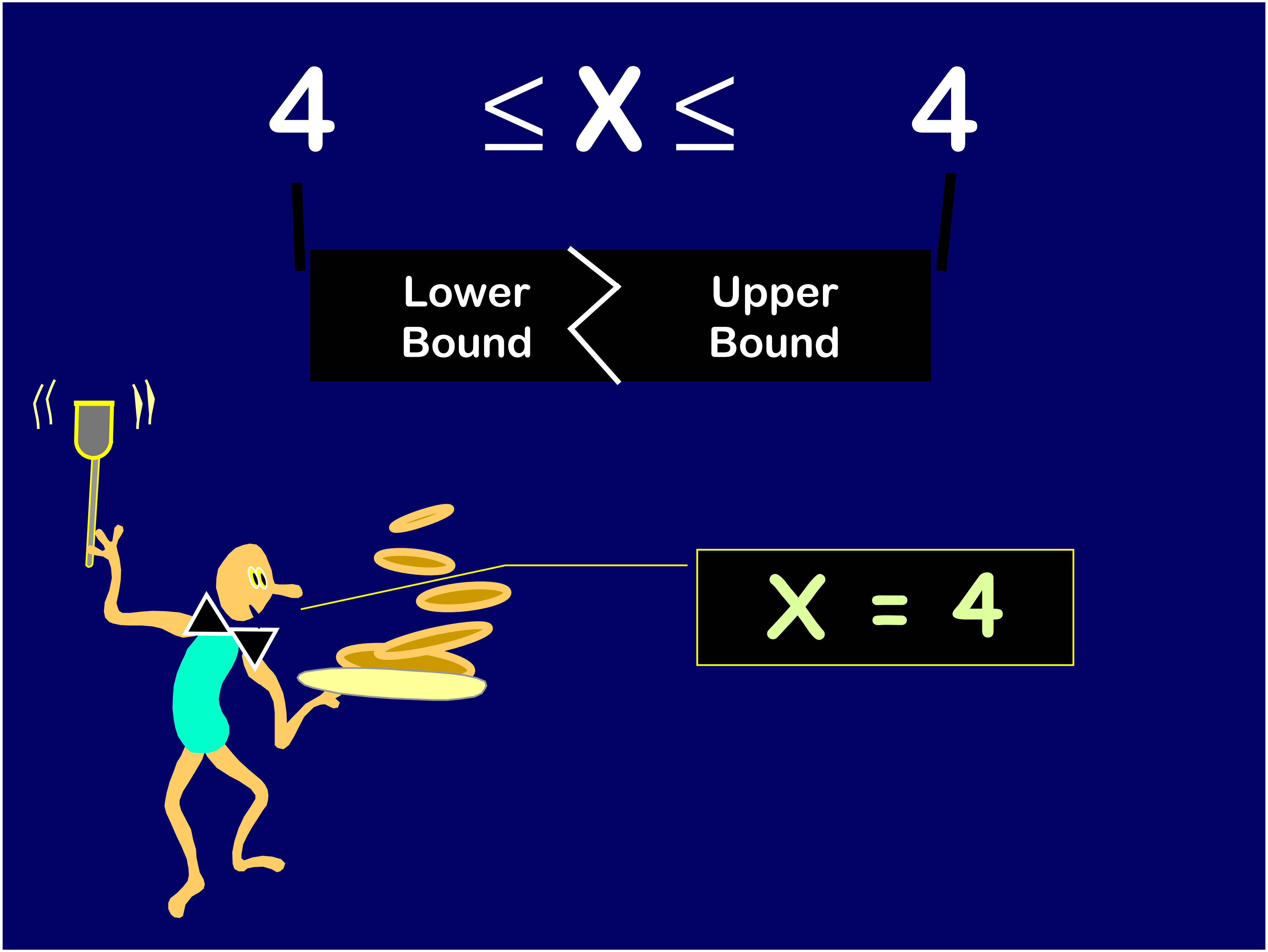


Flip 1 has to put 5 on bottom

Flip 2 must bring 4 to top.

$$? \leq X \leq 4$$

Lower Bound



4 ≤ X ≤ 4
Lower Bound
Upper Bound
X = 4

5^{th} Pancake Number

$P_5 =$ The number of flips required to sort the worst case stack of 5 pancakes.

$$? \leq P_5 \leq ?$$

Upper Bound

Lower Bound

5^{th} Pancake Number

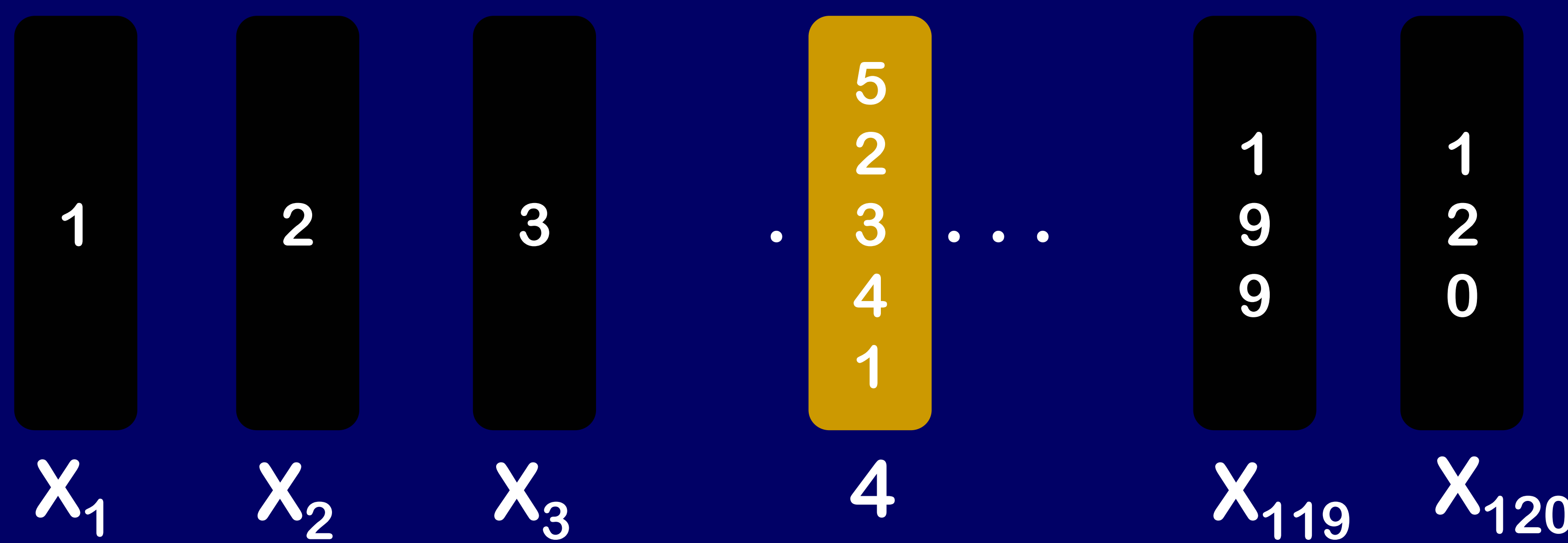
$P_5 =$ The number of flips required to sort the worst case stack of 5 pancakes.

$$4 \leq P_5 \leq ?$$

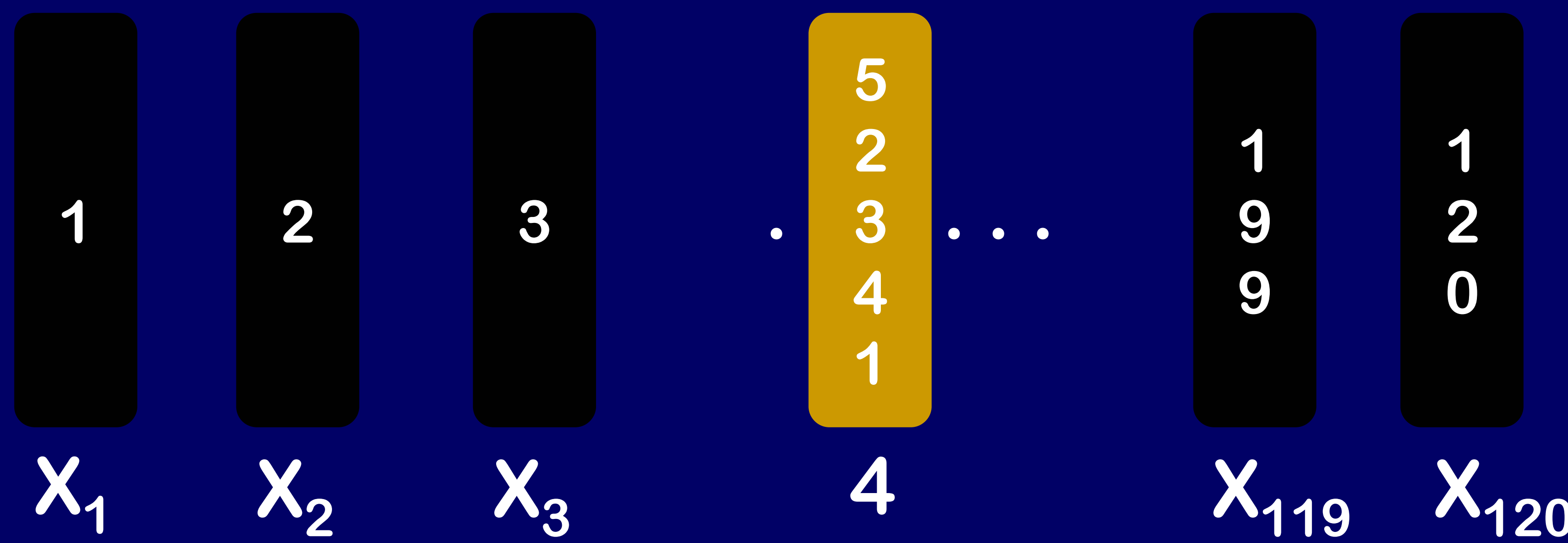
Upper Bound

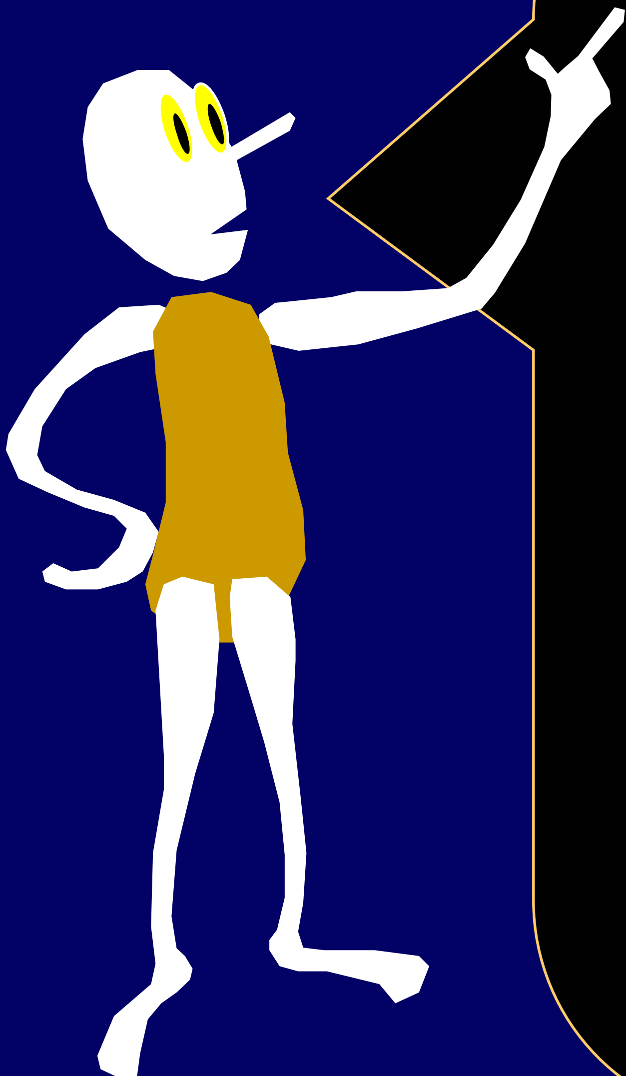
Lower Bound

The 5th Pancake Number: The MAX of the X's



P_5 = MAX over s $\in$ stacks of 5
of MIN # of flips to sort s





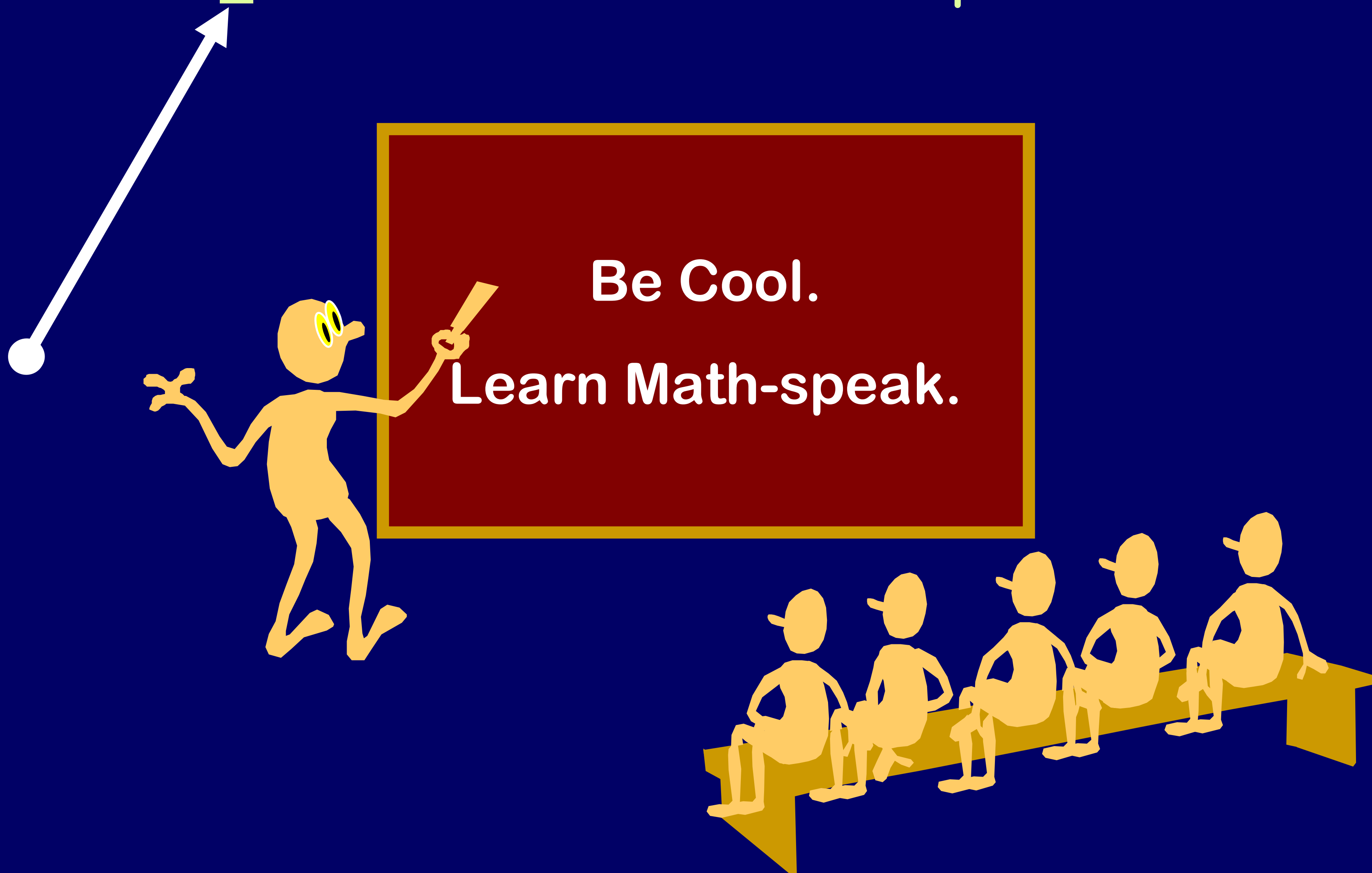
P_n

MAX over $s \in$ stacks
of n pancakes of
MIN # of flips to sort s

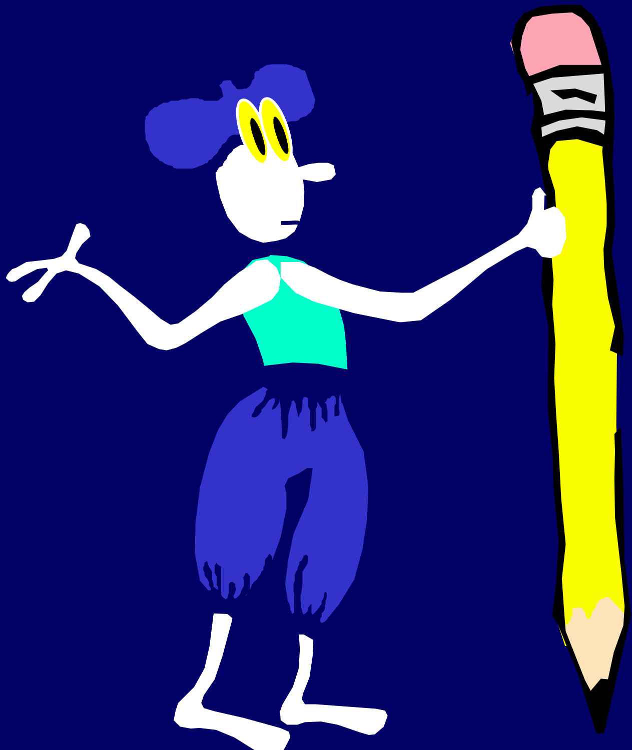
Or,

The number of flips
required to sort a
worst-case stack of n
pancakes.

P_n = The number of flips required to sort a worst-case stack of n pancakes.



What is P_n for small n?



Can you do
n = 0, 1, 2, 3 ?

Initial Values Of P_n

n	0	1	2	3
P_n	0	0	1	3

$P_3 = 3$

1
3
2   requires 3 Flips, hence $P_3 \geq 3$.

ANY stack of 3 can be done in 3 flips.
Get the big one to the bottom ($\leq$ 2 flips).
Use $\leq$ 1 more flip to handle the top two.

Hence, $P_3 \leq 3$.

n^{th} Pancake Number

P_n = Number of flips required to sort a worst case stack of n pancakes.

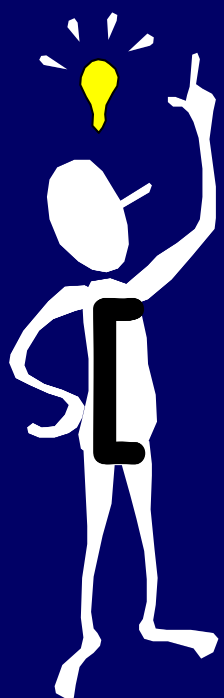
$$? \leq P_n \leq ?$$

Upper Bound

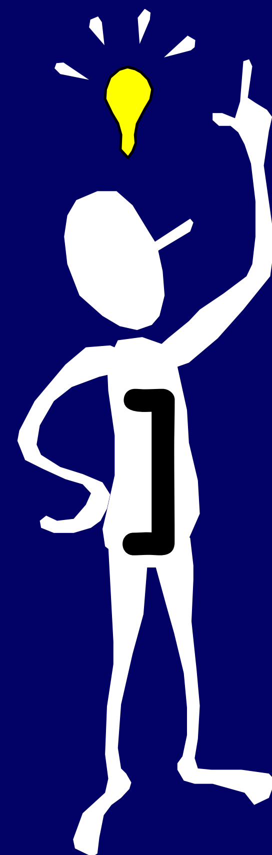
Lower Bound

Bracketing:

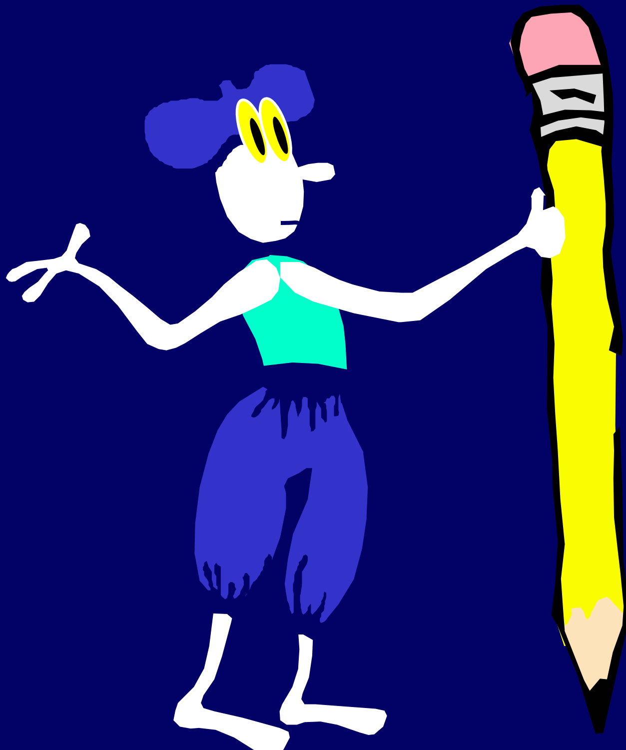
What are the best lower and upper bounds that I can prove?



$\leq f(x) \leq$



$? \leq P_n \leq ?$



Take a few minutes to try and prove bounds on P_n, for $n>3$.

Bring To Top Method

Bring biggest to top. Place it on bottom. Bring next largest to top. Place second from bottom. And so on...

Upper Bound On P_n:
Bring To Top Method For n Pancakes

If n=1, no work - we are done.

Else: flip pancake n to top and then
flip it to position n.

Now use: Bring To Top Method For n-1 Pancakes

Total Cost: at most 2(n-1) = 2n –2 flips.

Better Upper Bound On P_n:
Bring To Top Method For n Pancakes

If n=2, use one flip and we are done.

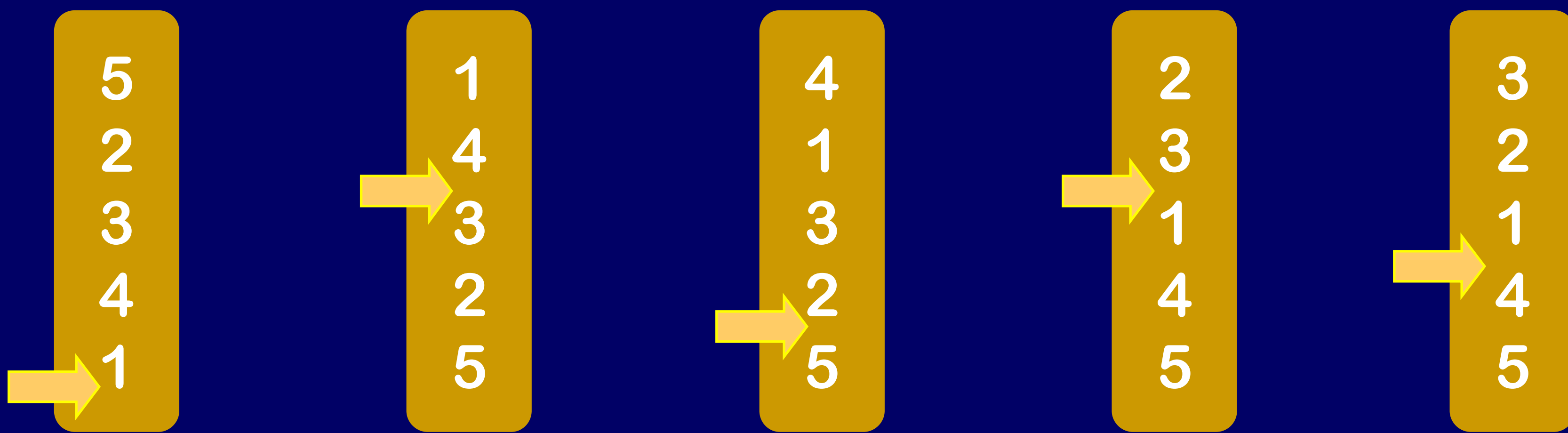
Else: flip pancake n to top and then
flip it to position n.

Now use:

Bring To Top Method
For n-1 Pancakes

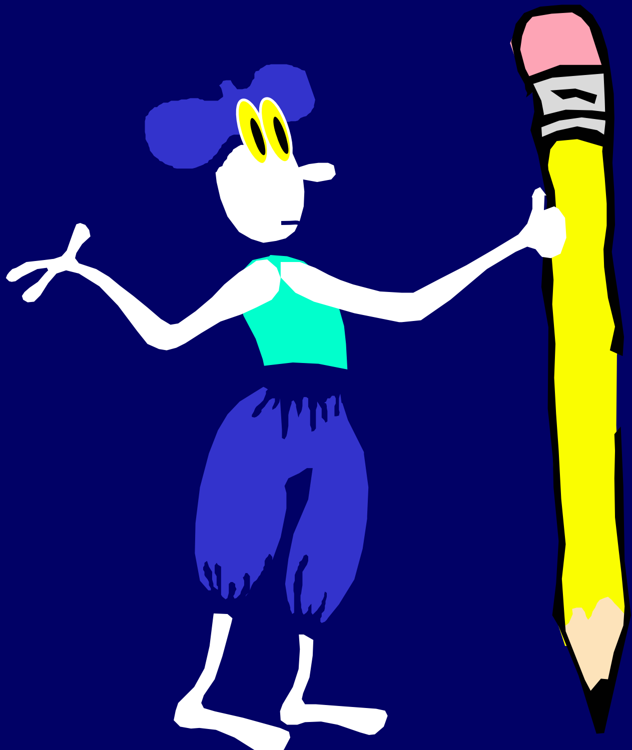
Total Cost: at most 2(n-2) + 1 = 2n –3 flips.

Bring to top not always optimal for a particular stack



5 flips, but can be done in 4 flips

$? \le P_n \le 2n - 3$

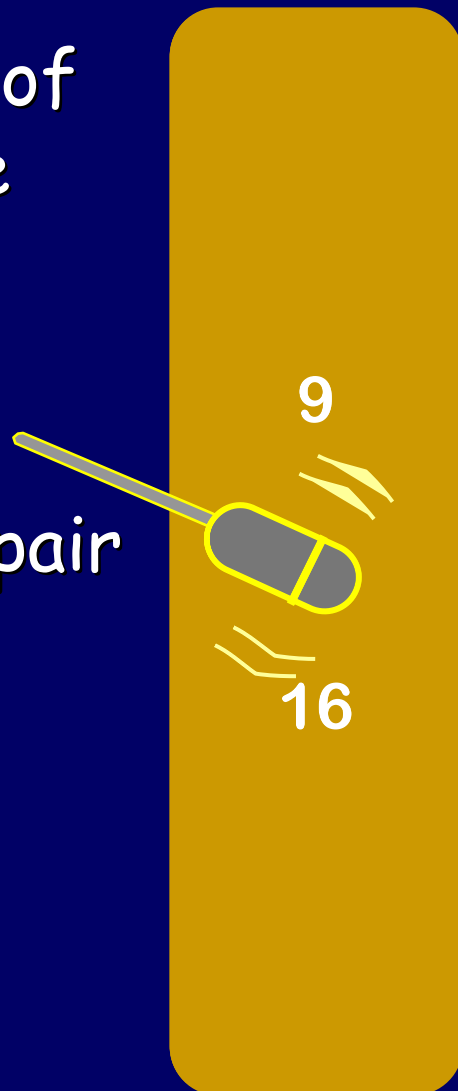


What bounds can you prove on P_n?

Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

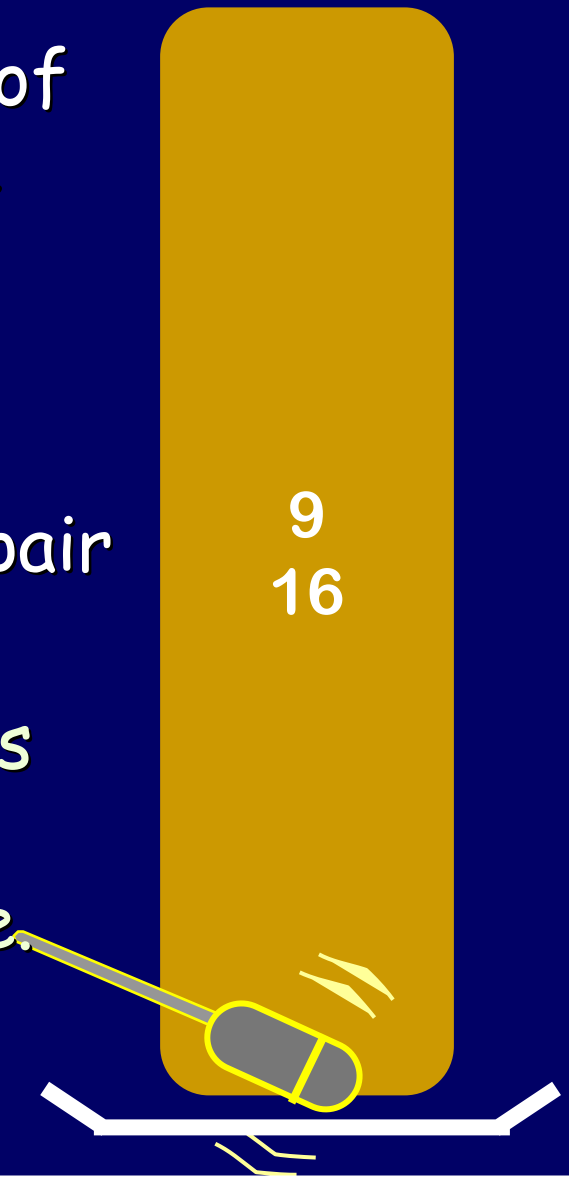


Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

Furthermore, this same principle is true of the “pair” formed by the bottom pancake of S and the plate.



$n \leq P_n$

Suppose n is even.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

S

S
2
4
6
8
.
.
n
1
3
5
7
.
.
n-1

$n \leq P_n$

S

Suppose n is even.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>2

$n \le P_n$

Suppose n is odd.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

S

1
3
5
7
.
.
n
2
4
6
8
.
.
n-1

$n \leq P_n$

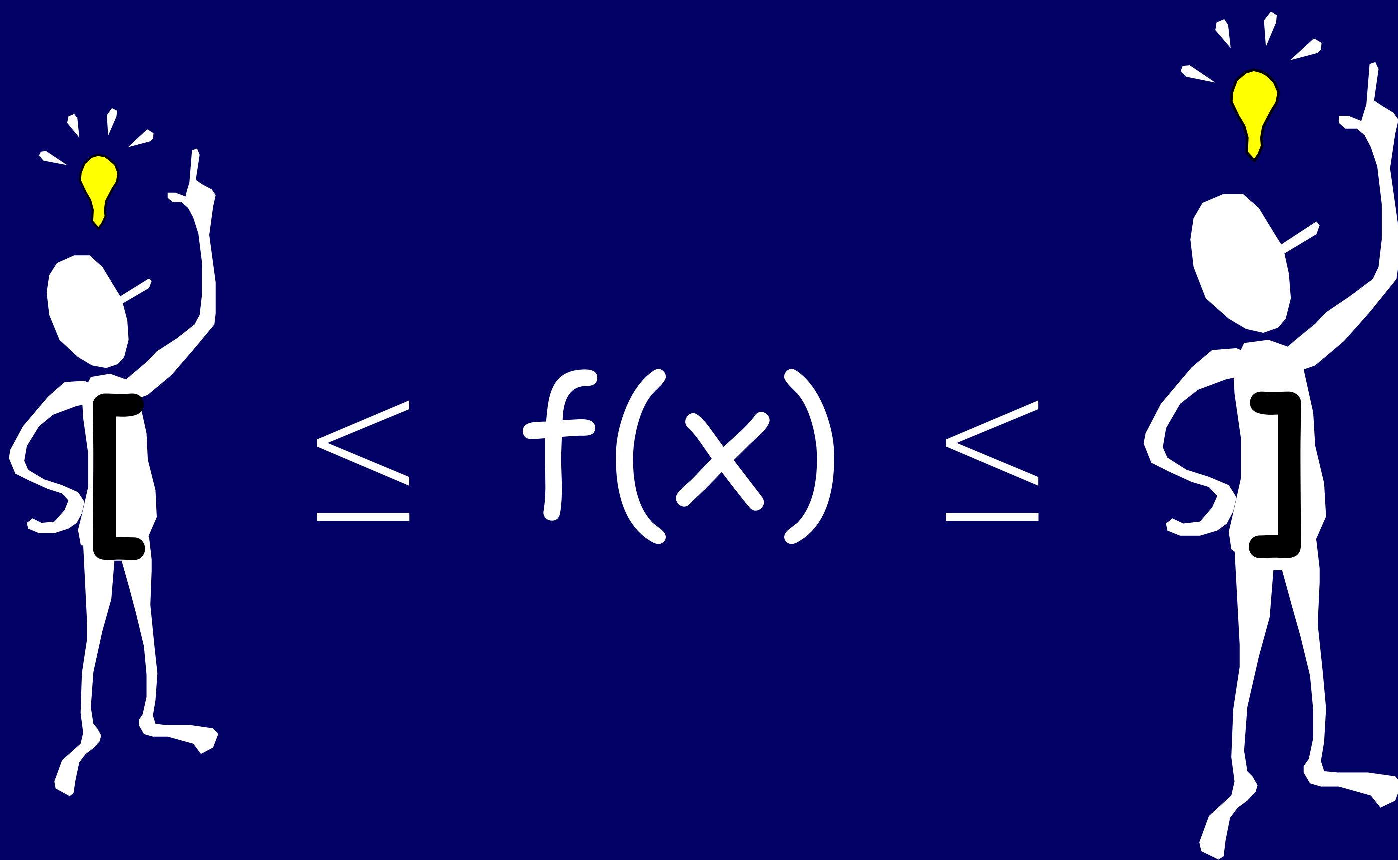
Suppose n is odd.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>3

Bracketing:

What are the best lower and upper bounds that I can prove?



$n \le P_n \le 2n - 3$ (for $n \ge 3$)

Bring To Top is within a factor of two of optimal!

$n \leq P_n \leq 2n - 3$ (for $n \geq 3$)

So starting from ANY stack we can get to the sorted stack using no more than P_n flips.

From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$?

Reverse the sequences we use to sort.

From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$.

Hence,

From ANY stack to ANY stack in $\leq 2P_n$.

From ANY stack to ANY stack in $\leq 2P_n$.

Can you find a faster way than $2P_n$ flips to go from ANY to ANY?

From ANY Stack S to ANY stack T in $\leq P_n$

Rename the pancakes in S to be 1,2,3,..,n. Rewrite T using the new naming scheme that you used for S. T will be some list: $\pi(1),\pi(2),..,\pi(n)$. The sequence of flips that brings the sorted stack to $\pi(1),\pi(2),..,\pi(n)$ will bring S to T.

S:	T:
4,3,5,1,2	**5,2,4,3,1**
1,2,3,4,5	**3,5,1,2,4**

The Known Pancake Numbers

n	P_n
1	0
2	1
3	3
4	4
5	5
6	**7**
7	**8**
8	**9**
9	**10**
10	**11**
11	**13**
12	**14**
13	**15**

P_{14} Is Unknown

14! Orderings of 14 pancakes.

14! = 87,178,291,200

Is This Really Computer Science?

Posed in *Amer. Math. Monthly* 82 (1) (1975),
"Harry Dweighter" a.k.a. Jacob Goodman

$(17/16)n \le P_n \le (5n+5)/3$

Bill Gates &
Christos
Papadimitriou:

Bounds For Sorting
By Prefix Reversal.

Discrete Mathematics,
vol 27, pp 47-57,
1979.

$(15/14)n \leq P_n \leq (5n+5)/3$

H. Heydari & Ivan H. Sudborough.

On the Diameter of the Pancake Network.

Journal of Algorithms, vol 25, pp 67-94, 1997.

Permutation

Any particular ordering of all n elements of an n element set S is called a permutation on the set S.

Each different stack of n pancakes is one of the permutations on [1..n].

Permutation

Any particular ordering of all n elements of an n element set S is called a permutation on the set S.

Example: S = {1, 2, 3, 4, 5}

Example permutation: 5 3 2 4 1

120 possible permutations on S

Ultra-Useful Fact

There are $n! = 1*2*3*4*...*n$ permutations on n elements.

Proof by induction on n. IH: There are (n-1)! permutations of n-1 elements.

Let S_i be all permutations on n elements that start with element i. By I,H, each S_i has size (n-1)!

Each permutation on n elements is mentioned exactly once in union of the S_i 's. Hence there are $n*(n-1)! = n!$ permutations on n elements.

Representing A Permutation

We have many choices of how to specify a permutation on S. Here are two methods:

1) List a sequence of all elements of [1..n], each one written exactly once.
   Ex: 6 4 5 2 1 3

2) Give a function π on S s.t. $\pi(1)\ \pi(2)\ \pi(3)\ ..\ \pi(n)$ is a sequence listing [1..n], each one exactly once.
   Ex: $\pi(1)=6$ $\pi(2)=4$ $\pi(3) = 5$ $\pi(4) = 2$ $\pi(4) = 1$ $\pi(6) =3$

A Permutation is a NOUN

An ordering S of a stack of pancakes is a permutation.

A Permutation is a NOUN
A Permutation can also be a VERB

An ordering S of a stack of pancakes is a permutation.

We can permute S to obtain a new stack S'.

<u>Permute</u> also means to rearrange so as to obtain a permutation of the original.

Permute A Permutation.

I start with a permutation S of pancakes.

I continue to use a flip operation to permute my current permutation, so as to obtain the sorted permutation.

FORMALLY

NOUN

A permutation π of a set S is a 1,1 onto function from S to S.

VERB

Let π and π' be permutations. We can compose them to get new ones:

$\pi(\pi'(x))$ and/or

$\pi'(\pi(x))$

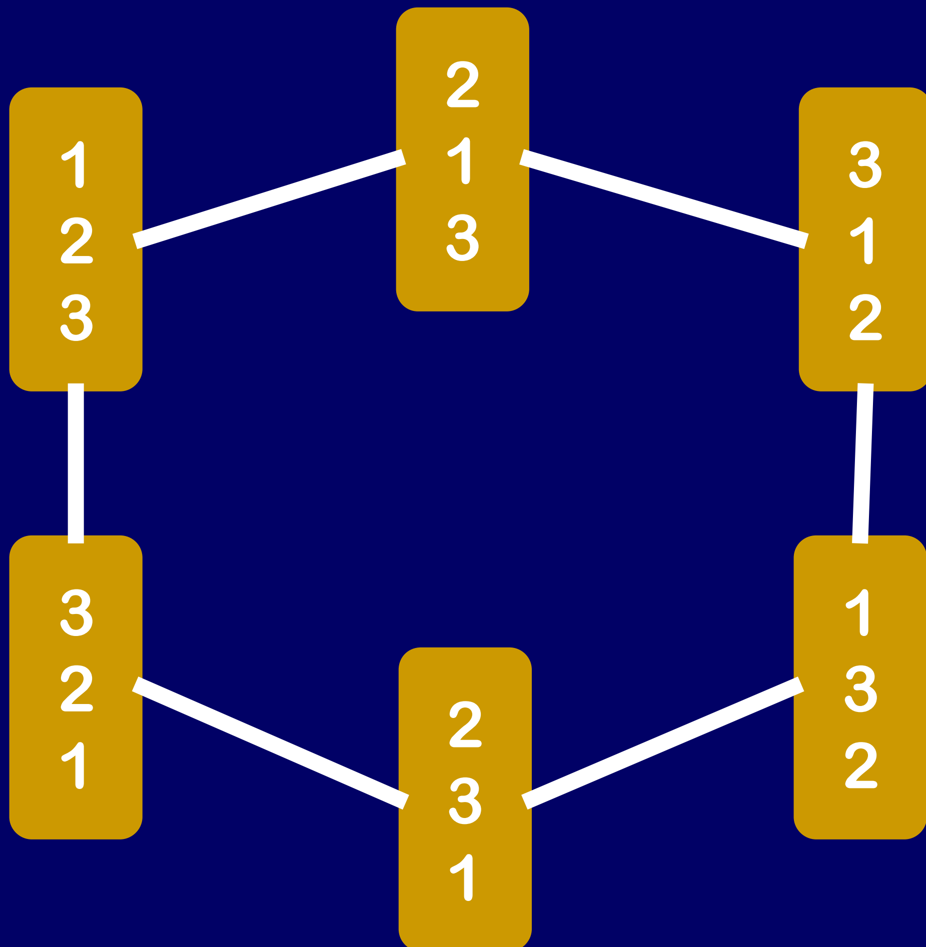
Pancake Network

This network has n! nodes

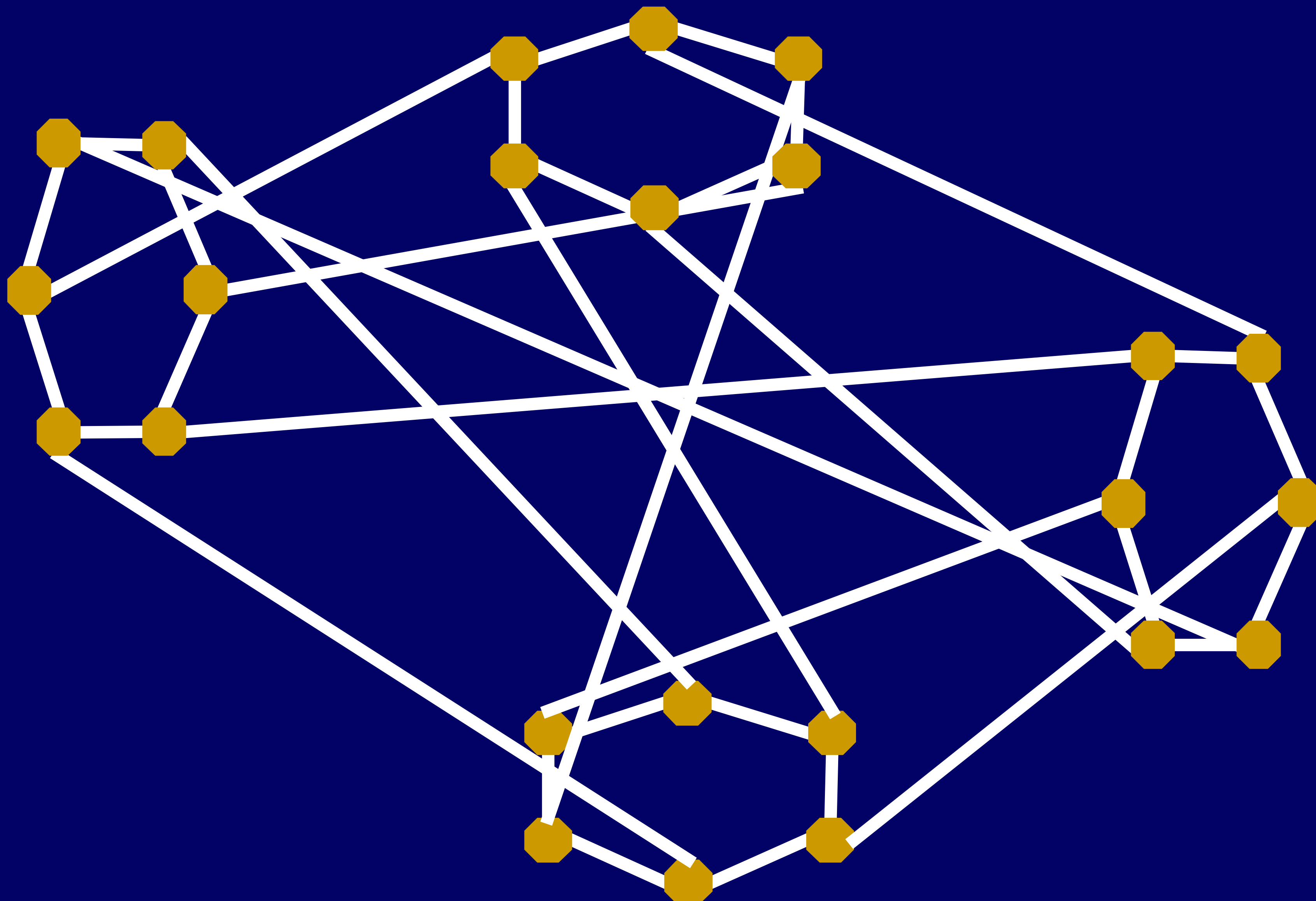
Assign each node the name of one of the possible n! stacks of pancakes.

Put a wire between two nodes
if they are one flip apart.

Network For n=3

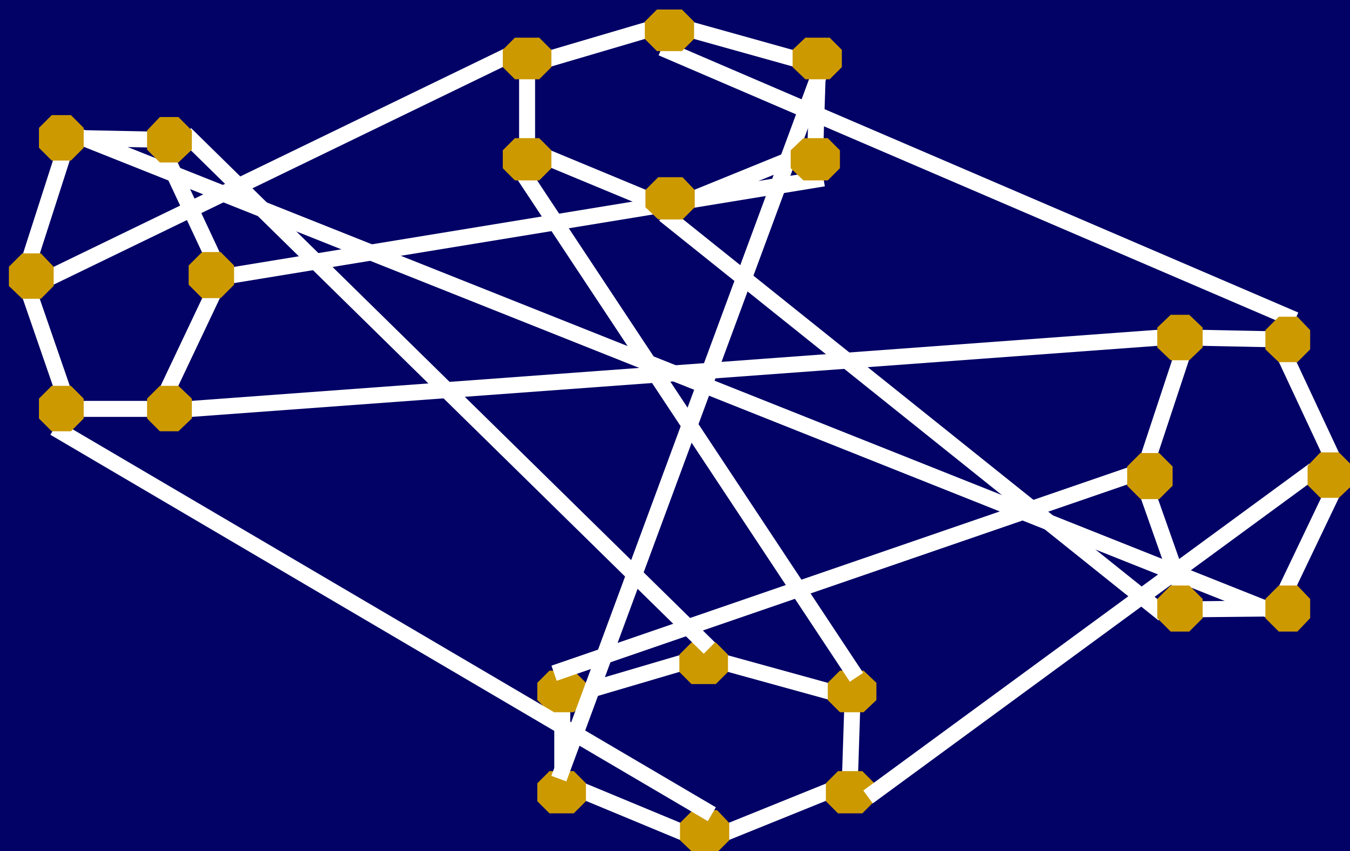


Network For n=4



Pancake Network: Routing Delay

What is the maximum distance between two nodes in the pancake network?



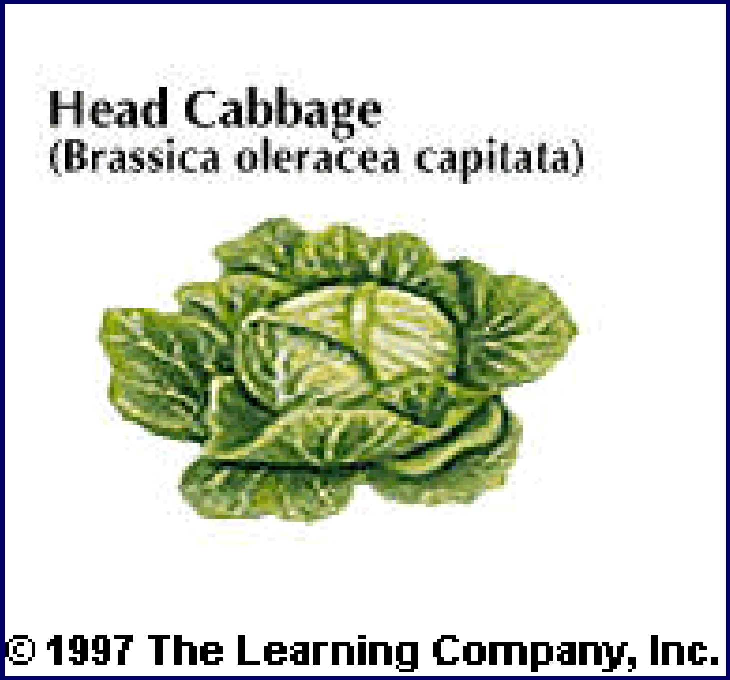
P_n

Pancake Network: Reliability

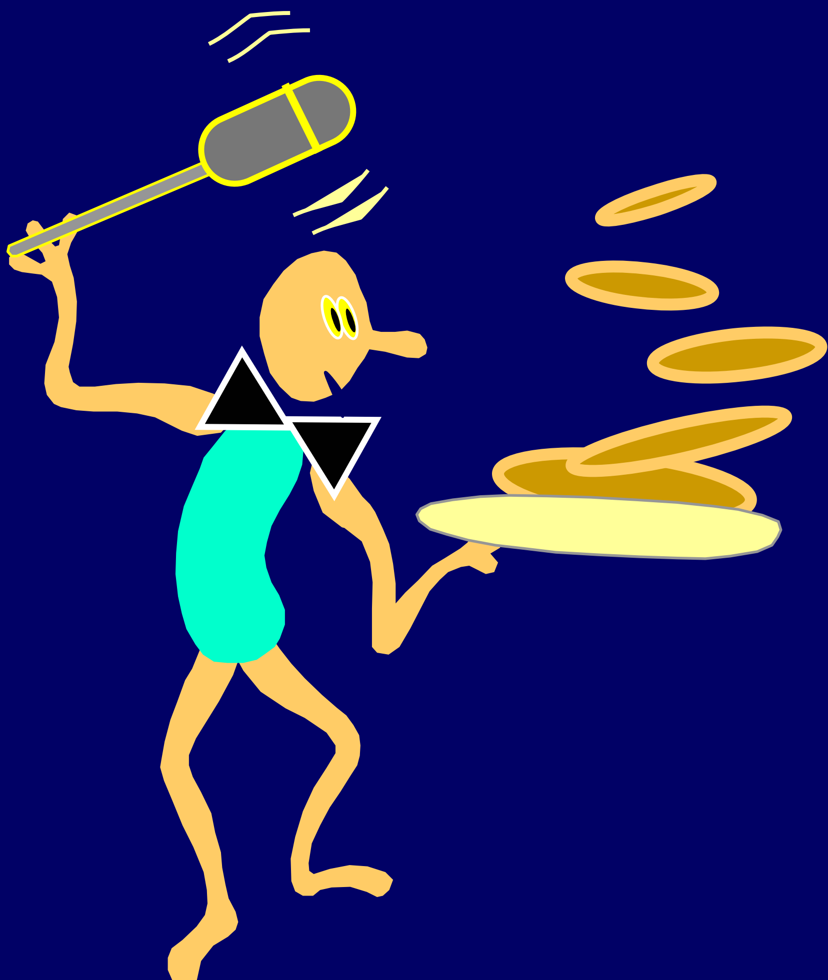
If up to n-2 nodes get hit by lightning the network remains connected, even though each node is connected to only n-1 other nodes.

The Pancake Network is optimally reliable for its number of edges and nodes.

Mutation Distance



One "Simple" Problem



A host of problems and applications at the frontiers of science

Definitions of:

nth pancake number

lower bound

upper bound

permutation

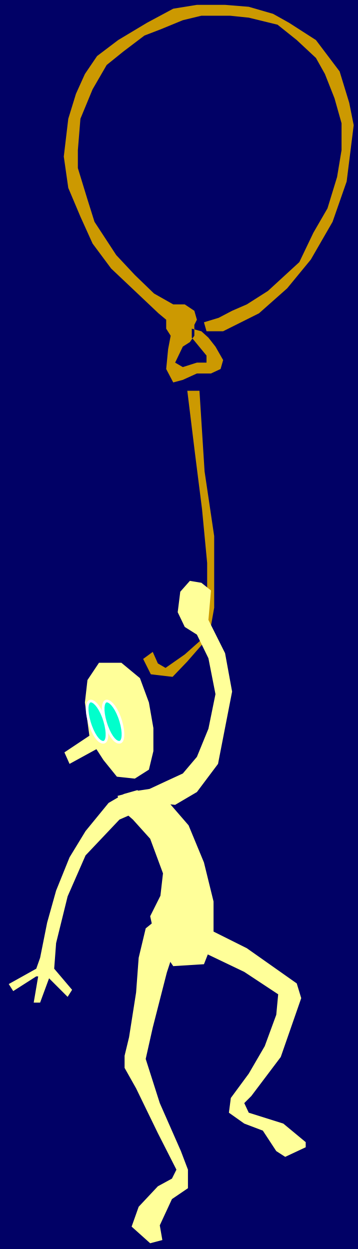
Proof of:

ANY to ANY in $\leq P_n$

Technique:

BRACKETING

High Level Point



This lecture is a microcosm of mathematical modeling and optimization.

References

Bill Gates & Christos Papadimitriou: Bounds For Sorting By Prefix Reversal. *Discrete Mathematics,* vol 27, pp 47-57, 1979.

H. Heydari & H. I. Sudborough: On the Diameter of the Pancake Network. *Journal of Algorithms,* vol 25, pp 67-94, 1997