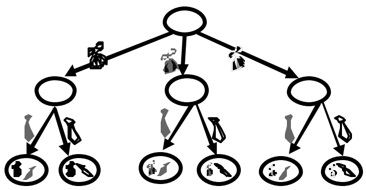
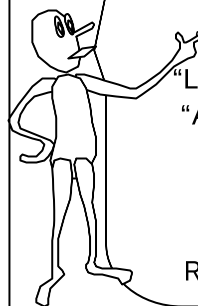
Great Theoretical Ideas In Computer Science		
Steven Rudich		CS 15-251 Spring 2005
Lecture 2	Jan 13, 2005	Carnegie Mellon University

Induction II:
Inductive Pictures

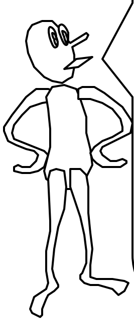




Inductive Proof:
"Standard" Induction
"Least Counter-example"
"All Previous" Induction

Inductive Definition:
Recurrences

Recursive Programming



Theorem? (k ¸ 0)

$1+2+4+8+...+2^k = 2^{k+1} -1$

Try it out on small examples:

$2^0 = 2^1 -1$

$2^0 + 2^1 = 2^2 -1$

$2^0 + 2^1 + 2^2 = 2^3 -1$

S_k ´ "$1+2+4+8+...+2^k = 2^{k+1} -1$"
Use induction to prove ∀k ¸ 0, S_k

<u>Establish "Base Case":</u> S_0. We have already check it.

<u>Establish "Domino Property": ∀k ¸ 0, S_k) S_{k+1}</u>

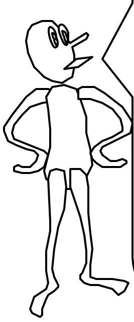
"Inductive Hypothesis" S_k:

$1+2+4+8+...+2^k = 2^{k+1} -1$

Add 2^{k+1} to both sides:

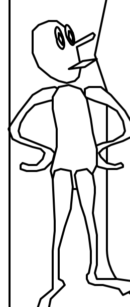
$1+2+4+8+...+2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} -1$

$1+2+4+8+...+2^k + 2^{k+1} = 2^{k+2} -1$



FUNDAMENTAL LEMMA OF THE POWERS OF TWO:

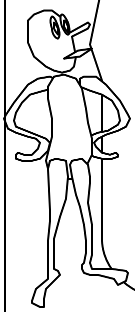
The sum of the first n powers of 2, is one less than the next power of 2.



Yet another way of packaging inductive reasoning is to define an "invariant".

Invariant *(adj.)*
1. Not varying; constant.

2. (mathematics) Unaffected by a designated operation, as a transformation of coordinates.



Yet another way of packaging inductive reasoning is to define an “invariant”.

Invariant *(adj.)*

3. *(programming)* A rule, such as the ordering an ordered list or heap, that applies throughout the life of a data structure or procedure.

Each change to the data structure must maintain the correctness of the invariant.

Invariant Induction

Suppose we have a time varying world state: W_0, W_1, W_2, …

Each state change is assumed to come from a list of permissible operations.

We seek to prove that statement S is true of all future worlds.

Argue that S is true of the initial world.

Show that if S is true of some world – then S remains true after one permissible operation is performed.

Odd/Even Handshaking Theorem:

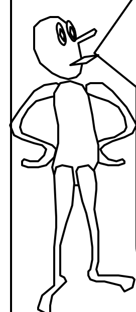
At any party, at any point in time, define a person’s parity as ODD/EVEN according to the number of hands they have shaken.

Statement: The number of people of odd parity must be even.

Initial case: Zero hands have been shaken at the start of a party, so zero people have odd parity.

If 2 people of <u>different parities shake</u>, then they both swap parities and the odd parity count is unchanged.

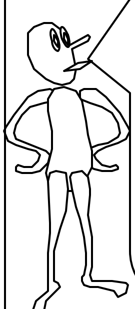
If 2 people of <u>the same parity shake</u>, they both change. But then the odd parity count changes by 2, and remains even.



Inductive Definition of n!

[said n “factorial”]

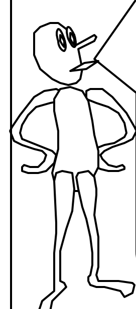
0! = 1; n! = n*(n-1)!



0! = 1; n! = n*(n-1)!

```
F:=1;
For x = 1 to n do
        F:=F*x;
Return F
```

Program for n! ?



0! = 1; n! = n*(n-1)!

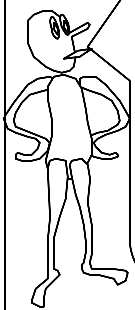
```
F:=1;
For x = 1 to n do
        F:=F*x;
Return F
```

n=0 returns 1

n=1 returns 1

n=2 returns 2

Program for n! ?



0! = 1; n! = n*(n-1)!

```
F:=1;
For x = 1 to n do
        F:=F*x;
Return F
```

Loop Invariant: F=x!

True for x=0. If true after k times through – true after k+1 times through.

Inductive Definition of T(n)

$T(1) = 1$

$T(n) = 4T(n/2) + n$

Notice that T(n) is inductively defined for positive powers of 2, and undefined on other values.

Inductive Definition of T(n)

$T(1) = 1$

$T(n) = 4T(n/2) + n$

Notice that T(n) is inductively defined for positive powers of 2, and undefined on other values.

$T(1)=1$ $T(2)=6$ $T(4)=28$ $T(8)=120$

Guess a closed form formula for T(n). Guess G(n)

$G(n) = 2n^2 - n$

Let the domain of G be the powers of two.

Two equivalent functions?

$G(n) = 2n^2 - n$

Let the domain of G be the powers of two.

$T(1) = 1$

$T(n) = 4\ T(n/2) + n$

Domain of T are the powers of two.

Inductive Proof of Equivalence

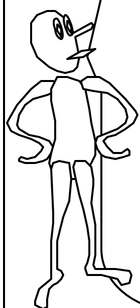
$G(n) = 2n^2 - n$
$T(1) = 1$
$T(n) = 4\ T(n/2) + n$

Base: $G(1) = 1$ and $T(1) = 1$

Induction Hypothesis:
$T(x) = G(x)$ for $x < n$

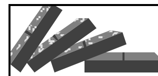
Hence: $T(n/2) = G(n/2) = 2(n/2)^2 - n/2$

$$
\begin{aligned}
T(n) &= 4\ T(n/2) + n \\
&= 4\ G(n/2) + n \\
&= 4\ [2(n/2)^2 - n/2] + n \\
&= 2n^2 - 2n + n \\
&= 2n^2 - n \\
&= G(n)
\end{aligned}
$$



We inductively proved the assertion that G(n) =T(n).

Giving a formula for T with no sums or recurrences is called solving the recurrence T.



Solving Recurrences
Guess and Verify

Guess: $G(n) = 2n^2 - n$

Verify: $G(1) = 1$ and $G(n) = 4\ G(n/2) + n$

Similarly: $T(1) = 1$ and $T(n) = 4\ T(n/2) + n$

So $T(n) = G(n)$

Technique 2
Guess Form and Calculate Coefficients

Guess: $T(n) = an^2 + bn + c$ for some a,b,c

Calculate: $T(1) = 1 \Rightarrow a + b + c = 1$

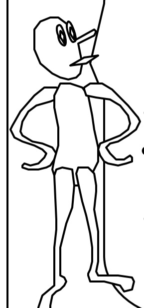
$T(n) = 4\ T(n/2) + n$

$\Rightarrow\ an^2 + bn + c = 4\ [a(n/2)^2 + b(n/2) + c] + n$

$= an^2 + 2bn + 4c + n$

$\Rightarrow\ (b+1)n + 3c = 0$

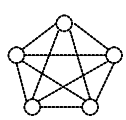
Therefore: $b=-1 \quad c=0 \quad a=2$

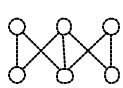


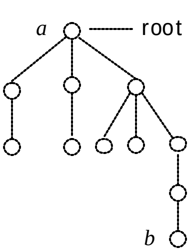
A computer scientist not only deals with numbers, but also with

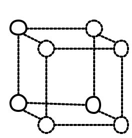
- Finite Strings of symbols
- Very visual objects called graphs
- And especially, especially the special graphs called trees

GRAPHS









Definition: Graphs

A graph G = (V,E) consists of a finite set V of vertices (nodes) and a finite set E of edges. Each edge is a set {a, b} of two different vertices.

A graph may not have self loops or multiple edges.

Definition: Directed Graphs

A graph G = (V,E) consists of a finite set V of vertices (nodes) and a finite set E of edges. Each edge is an ordered pair <a,b> of two different vertices.

Unless we say otherwise, our directed graphs will not have multi-edges, or self loops.

Definition: Tree

A tree is a directed graph with one special node called the root and the property that each node must a unique path from the root to itself.

Child: If <u,v>2E, we sav is a child of u

Parent: If <u,v>2E, we say u is the parent of u

Leaf: If u has no children, we say u is leaf.

Siblings: If u and v have the same parent, they are siblings.

Descendants of u: The set of nodes reachable from u (including u).

Sub-tree rooted at u: Descendants of u and all the edges between them where u has been designated as a root.

Classic Visualization: Tree

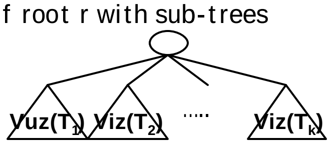
Inductive rule:

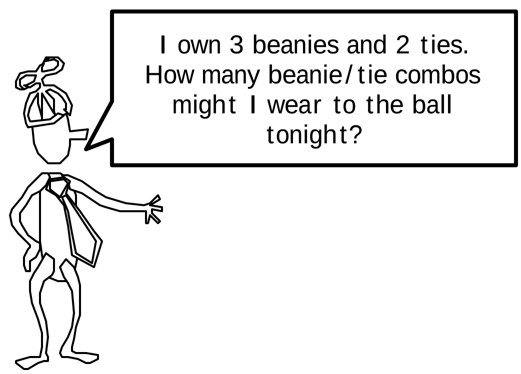
If G is a single node

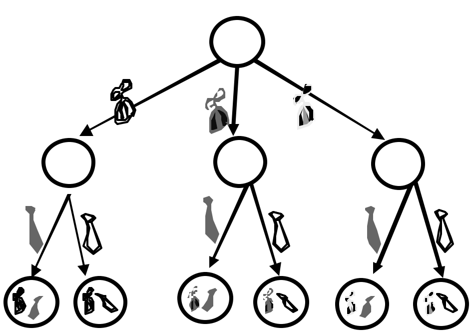
Viz(G) = ◯

If G consists of root r with sub-trees T_1, T_2, …, T_k

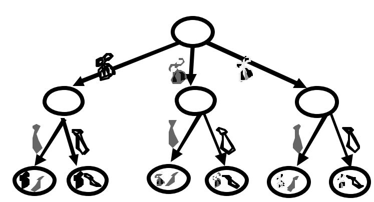
Viz(G) =







Choice Tree

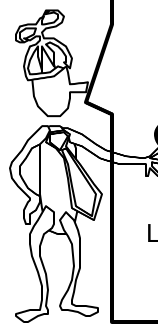
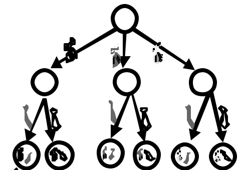


A choice tree is a tree with an object called a "choice" associated with each edge and a label on each leaf.

Definition: Labeled Tree

A tree node labeled by S is a tree T = <V,E> with an associated function $Label_1$: V to S

A tree edge labeled by S is a tree T = <V,E> with an associated function $Label_2$: E to S

was very illuminating.

Let's do something similar to illuminate the nature of T(1)=1; T(n)= 4T(n/2) + n

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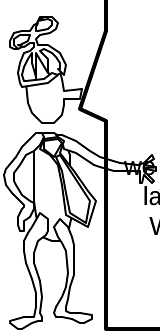


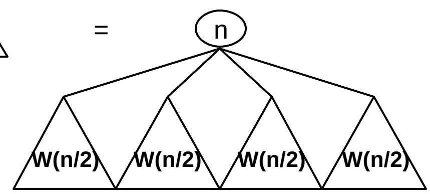
T(1)=1; T(n)= 4T(n/2) + n

For each n (power of 2), we will define a tree W(n) node labeled by Natural numbers. W(n) will give us an incisive picture of T(n).

Inductive Definition Of Labeled Tree W(n)

T(n) = n + 4 T(n/2)

W(n) =



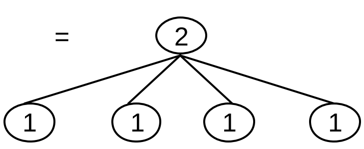
T(1) = 1

W(1) = 1

Inductive Definition Of Labeled Tree W(n)

T(2) = 6

W(2) =



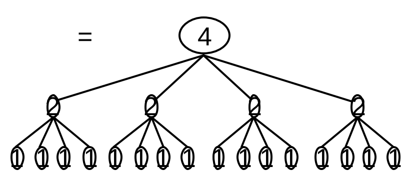
T(1) = 1

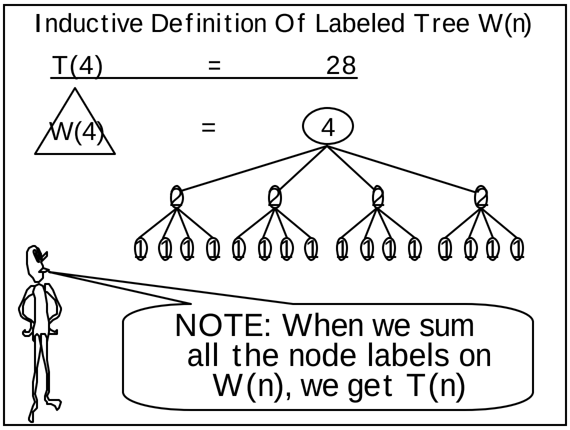
W(1) = 1

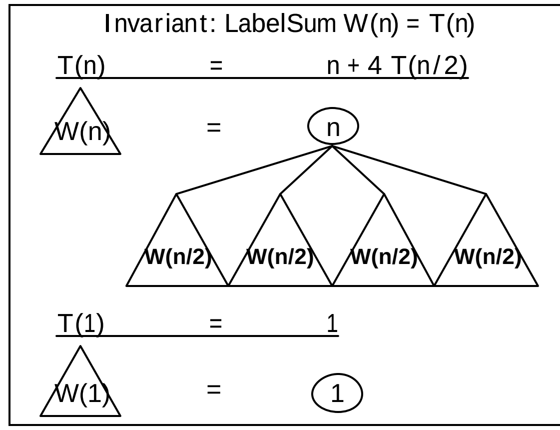
Inductive Definition Of Labeled Tree W(n)

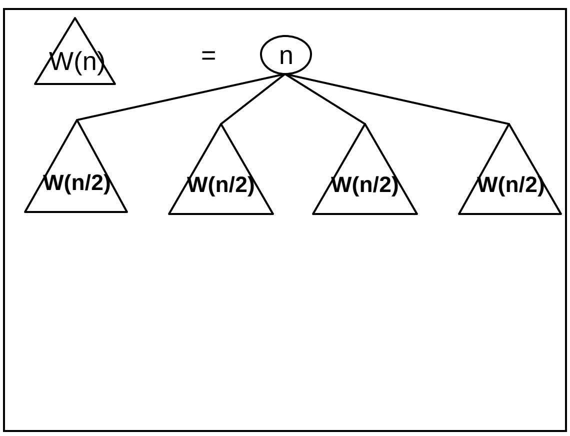
T(4) = 28

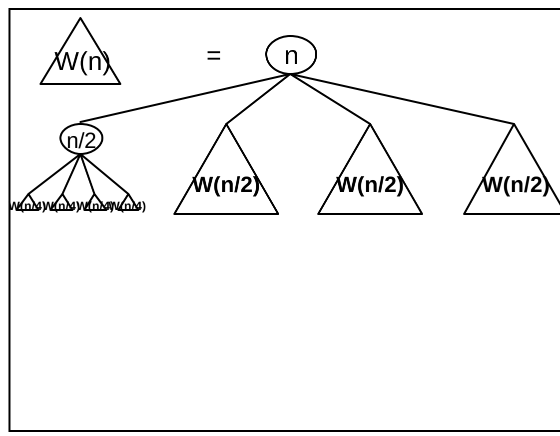
W(4) =

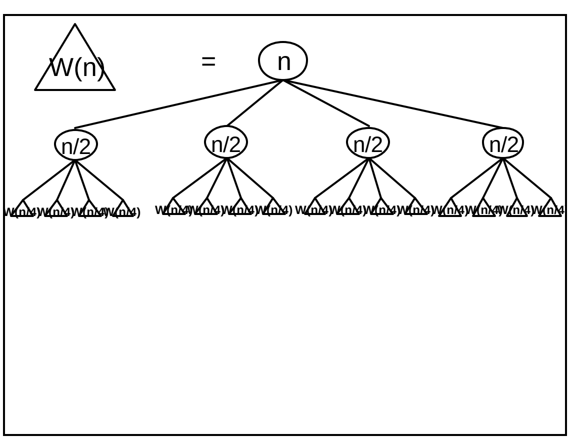


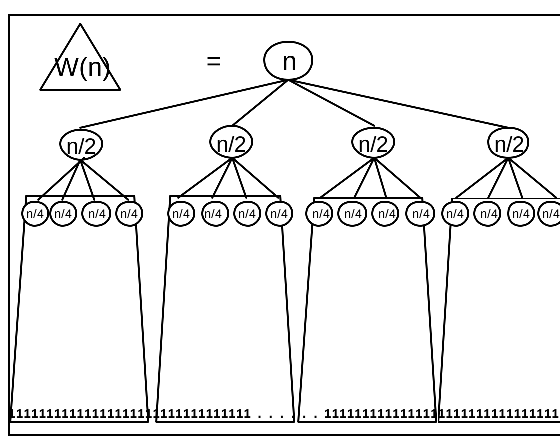

Inductive Definition Of Labeled Tree W(n)
T(4) = 28
W(4) = 4
2 2 2 2
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
NOTE: When we sum all the node labels on W(n), we get T(n)

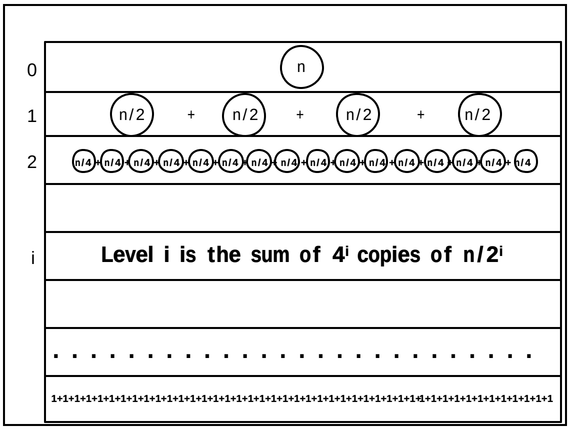

Invariant: LabelSum W(n) = T(n)
T(n) = n + 4 T(n/2)
W(n) = n
W(n/2) W(n/2) W(n/2) W(n/2)
T(1) = 1
W(1) = 1

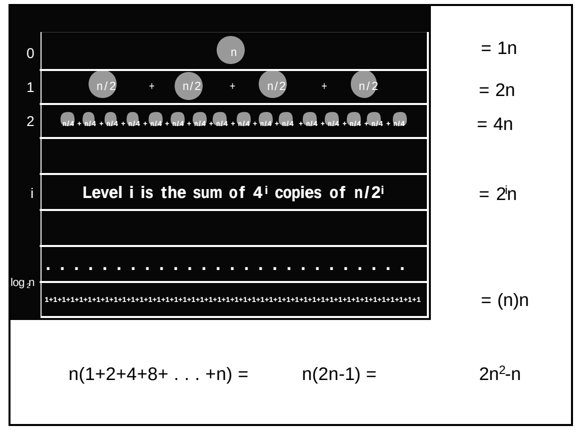

W(n) = n
W(n/2) W(n/2) W(n/2) W(n/2)

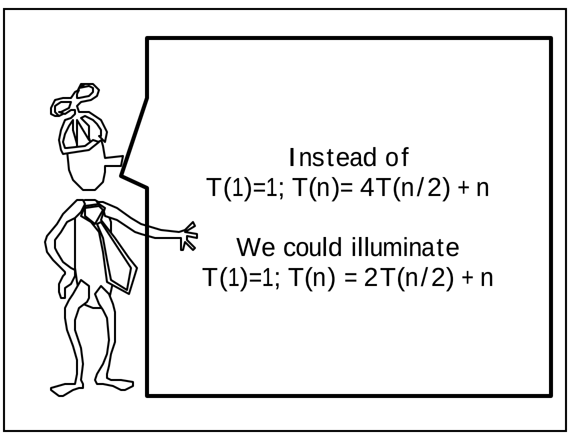

W(n) = n
n/2
W(n/2) W(n/2) W(n/2)

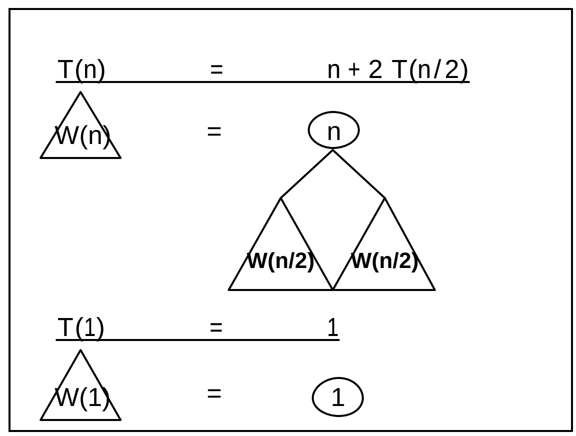

W(n) = n
n/2 n/2 n/2 n/2

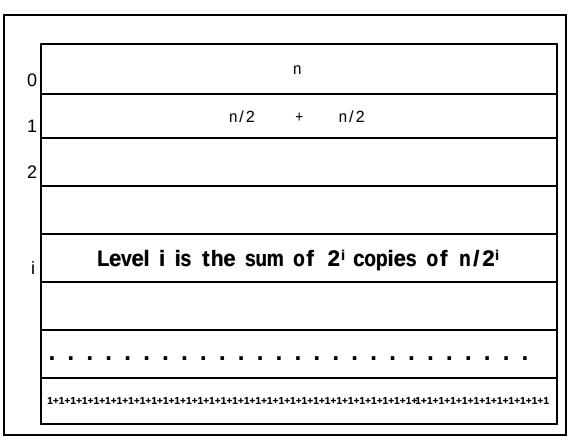

W(n) = n
n/2 n/2 n/2 n/2
n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4 n/4

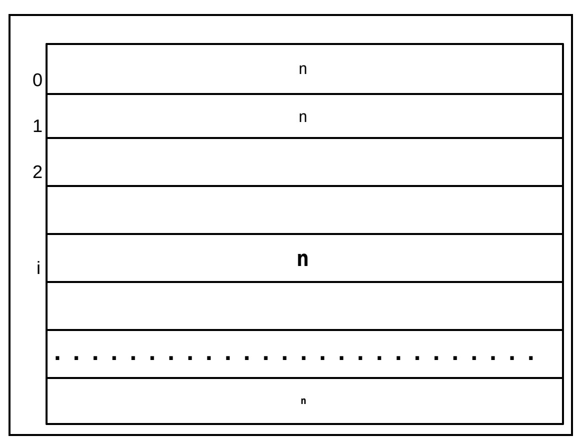

0
1
2
i
n
n/2 + n/2 + n/2 + n/2
n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4 + n/4
Level i is the sum of 4^i copies of n/2^i
. . . .


0
1
2
i
log2n
n
n/2 + n/2 + n/2 + n/2
Level i is the sum of 4^i copies of n/2^i
= 1n
= 2n
= 4n
= 2^i n
= (n)n
n(1+2+4+8+ . . . +n) =
n(2n-1) =
2n^2-n


Instead of
T(1)=1; T(n)= 4T(n/2) + n
We could illuminate
T(1)=1; T(n) = 2T(n/2) + n


T(n) = n + 2 T(n/2)
W(n)
=
n
W(n/2)
W(n/2)
T(1) = 1
W(1)
=
1


0
1
2
i
n
n/2 + n/2
Level i is the sum of 2^i copies of n/2^i
. . . .


0
1
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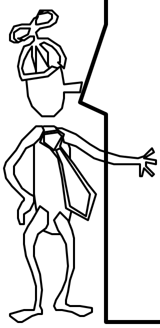
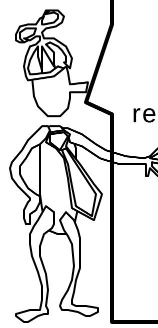
i
n
n
n
. . . .
n



$T(1)=1$; $T(n) = 2T(n/2) + n$

Has closed form: $n\log_2(n)$ where n is a power of 2



Representing a recurrence relation as a labeled tree is one of the basics tools you will have to put recurrence relations in closed form.

The Lindenmayer Game

$\Sigma = \{a,b\}$
Start word: a

SUB(a) = ab SUB(b) = a
For each $w = w_1 w_2 \dots w_n$
$NEXT(w) = SUB(w_1)SUB(w_2)..SUB(w_n)$

The Lindenmayer Game

SUB(a) = ab SUB(b) = a
For each $w = w_1 w_2 \dots w_n$
$NEXT(w) = SUB(w_1)SUB(w_2)..SUB(w_n)$

Time 1: a
Time 2: ab
Time 3: aba
Time 4: abaab
Time 5: abaababa

The Lindenmayer Game

SUB(a) = ab SUB(b) = a
For each $w = w_1 w_2 \dots w_n$
$NEXT(w) = SUB(w_1)SUB(w_2)..SUB(w_n)$

Time 1: a
Time 2: ab
Time 3: aba
Time 4: abaab
Time 5: abaababa

How long are the strings as a function of time?

Aristid Lindenmayer (1925-1989)

1968 Invents L-systems in Theoretical Botany

Time 1: a
Time 2: ab
Time 3: aba
Time 4: abaab
Time 5: abaababa

FIBONACCI(n) cells at time n

The Koch Game

$\Sigma = \{F,+,-\}$
Start word: F

SUB(F) = F+F--F+F SUB(+)=+ SUB(-)=-
For each $w = w_1 w_2 \ldots w_n$
NEXT(w) = SUB(w_1)SUB(w_2)..SUB(w_n)

The Koch Game

Gen 0:F
Gen 1: F+F--F+F
Gen 2: F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F

The Koch Game

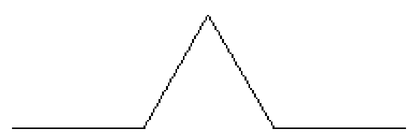
Picture representation:

F draw forward one unit
+ turn 60 degree left
- turn 60 degrees right.

Gen 0: F
Gen 1: F+F--F+F
Gen 2: F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F

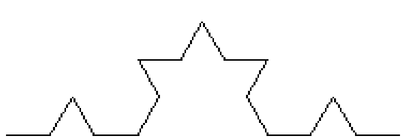
The Koch Game

F+F--F+F

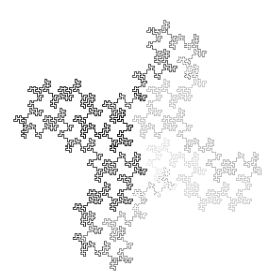


The Koch Game

F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F



Koch Curve



Dragon Game

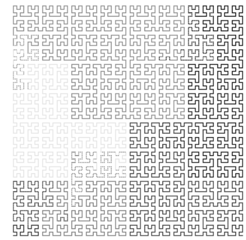
SUB(X) =X+YF+
SUB(Y) = -FX-Y

Dragon Curve:

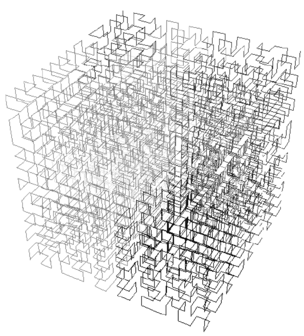
Hilbert Game
SUB(L)= +RF-LFL-FR+
SUB(R)= -LF+RFR+FL-

Hilbert Curve:

Note: Make 90 degree turns instead of 60 degrees.

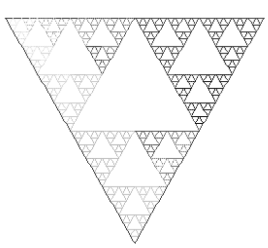


Hilbert's Space Filling Curve



Peano-Gossamer Curve

Sierpinski Triangle

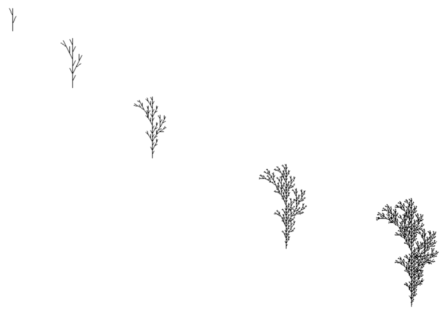


Lindenmayer 1968

SUB(F) = F[-F]F[+F][F]

Interpret the stuff inside brackets as a branch.

Lindenmayer 1968



Inductive Leaf