15-251

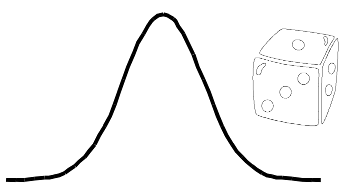
Great Theoretical Ideas in Computer Science

15-251

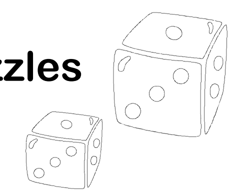
Flipping Coins for Computer Scientists

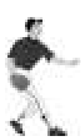
Probability Theory I

Lecture 11 (September 28, 2010)



Some Puzzles





Teams A and B are equally good

In any one game, each is equally likely to win

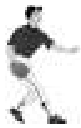
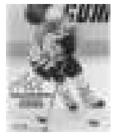
What is most likely length of a "best of 7" series?

Flip coins until either 4 heads or 4 tails
Is this more likely to take 6 or 7 flips?

6 and 7 Are Equally Likely

To reach either one, after 5 games, it must be 3 to 2

½ chance it ends 4 to 2; ½ chance it doesn't

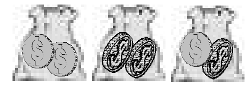
Teams A is now better than team B

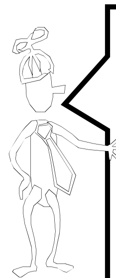
The odds of A winning are 6:5

i.e., in any game, A wins with probability 6/11

What is the chance that A will beat B in the "best of 7" world series?

Silver and Gold



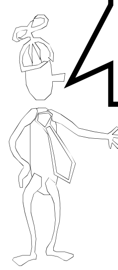


A bag has two silver coins, another has two gold coins, and the third has one of each

One bag is selected at random. One coin from it is selected at random. It turns out to be gold

What is the probability that the other coin is gold?

Let us start simple…



A fair coin is tossed 100 times in a row

What is the probability that we get exactly 50 heads?

The set of all outcomes is $\{H,T\}^{100}$

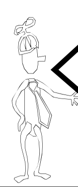
There are 2^{100} outcomes

Out of these, the number of sequences with 50 heads is $\binom{100}{50}$

If we draw a random sequence, the probability of seeing such a sequence:

$$\binom{100}{50} / 2^{100} = 0.07958923739\ldots$$

The Language of Probability

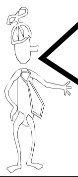


"A fair coin is tossed 100 times in a row"

The sample space S, the set of all outcomes, is $\{H,T\}^{100}$

Each sequence in S is equally likely, and hence has probability $1/|S|=1/2^{100}$

The Language of Probability

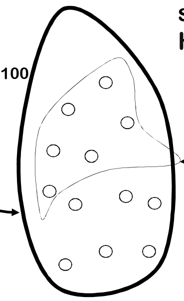


"What is the probability that we get exactly 50 heads?"

Let E = {x in S| x has 50 heads} be the event that we see half heads.

$\Pr(E) = |E|/|S| = |E|/2^{100}$

$\Pr(E) = \sum_{x \text{ in } E} \Pr(x) = |E|/2^{100}$

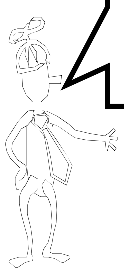


Set S of all 2^{100} sequences $\{H,T\}^{100}$

Event E = Set of sequences with 50 H's and 50 T's

Probability of event E = proportion of E in S

$$\binom{100}{50} / 2^{100}$$



A fair coin is tossed 100 times in a row

What is the probability that we get 50 heads in a row?

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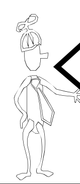
formalizing this problem…



The sample space S, the set of all outcomes, is $\{H,T\}^{100}$ again, each sequence in S equally likely, and hence with probability $1/|S|=1/2^{100}$

Now E = {x in S| x has 50 heads in a row} is the event of interest.

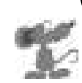
What is |E|?

$\underbrace{HH\cdots\cdots H}_{50}$	anything		2^{50}
T	$HH\cdots\cdots H$	anything	2^{49}
☐ T	$HH\cdots\cdots H$	☐	2^{49}
☐☐ T	$HH\cdots\cdots H$	☐	2^{49}
⋮			
☐ T	$HH\cdots\cdots H$		2^{49}

$50 \bullet 2^{49} + 2^{50} = 52 \bullet 2^{49}$

$$\frac{|E|}{|\Omega|} = \frac{52 \bullet 2^{49}}{2^{100}} = \frac{52}{2^{51}} \approx 0$$

If we roll a fair die, what is the probability that the result is an even number?



½, obviously

True, but let's take the trouble to say this formally.

sample space S = {1,2,3,4,5,6}

Each outcome x in S is equally likely, i.e., ∀x in S, the probability that x occurs is 1/6.

x	$P(x)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$$E = \{2, 4, 6\}$$

$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Suppose that a dice is weighted so that the numbers do not occur with equal frequency.

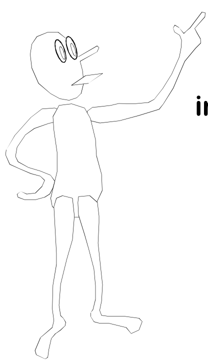
table of frequencies
(proportions)
(probabilities)

x	$P(x)$
1	$1/6$
2	$2/6$
3	$1/12$
4	$1/12$
5	$1/12$
6	$3/12$

$$E = \{2, 4, 6\}$$

$$P(E) = \frac{2}{6} + \frac{1}{12} + \frac{3}{12} = \frac{4}{6} = \frac{2}{3}$$

Language of Probability



The formal language of probability is a crucial tool in describing and analyzing problems involving probabilities…

and in avoiding errors, ambiguities, and fallacious reasoning.

Finite Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element t in S has a non-negative real weight, proportion, or probability p(t)

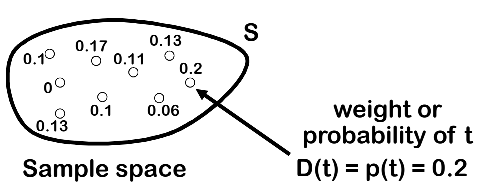
The weights must satisfy:

$$\sum_{t \in S} p(t) = 1$$

For convenience we will define D(t) = p(t)

S is often called the sample space and elements t in S are called samples

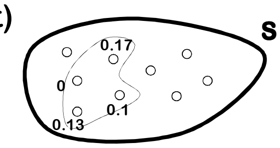
Sample Space



Events

Any set $E \subseteq S$ is called an event

$$Pr_D[E] = \sum_{t \in E} p(t)$$



$$Pr_D[E] = 0.4$$

Uniform Distribution

If each element has equal probability, the distribution is said to be uniform

$$Pr_D[E] = \sum_{t \in E} p(t) = \frac{|E|}{|S|}$$

Using the Language

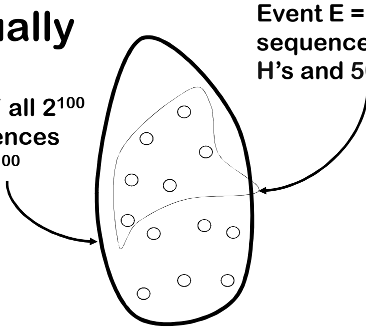
The sample space S is the set of all outcomes $\{H,T\}^{100}$

Each sequence in S is equally likely, and hence has probability $1/|S| = 1/2^{100}$

Visually

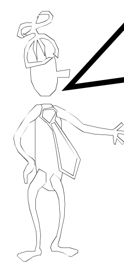
Event E = Set of sequences with 50 H's and 50 T's

Set of all 2^{100} sequences $\{H,T\}^{100}$



Probability of event E = proportion of E in S

$$\binom{100}{50} / 2^{100}$$



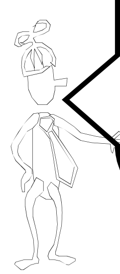
Suppose we roll a white die and a black die

What is the probability that sum is 7 or 11?

Same Methodology!

$$
\begin{array}{lllllll}
S = \{ & (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & \boxed{(1,6)} \\
& (2,1), & (2,2), & (2,3), & (2,4), & \boxed{(2,5)} & (2,6), \\
& (3,1), & (3,2), & (3,3), & \boxed{(3,4)} & (3,5), & (3,6), \\
& (4,1), & (4,2), & \boxed{(4,3)} & (4,4), & (4,5), & (4,6), \\
& (5,1), & \boxed{(5,2)} & (5,3), & (5,4), & (5,5), & \boxed{(5,6)} \\
& \boxed{(6,1)} & (6,2), & (6,3), & (6,4), & \boxed{(6,5)} & (6,6) \}
\end{array}
$$

Pr[E] = |E|/|S| = proportion of E in S = 8/36



23 people are in a room

Suppose that all possible birthdays are equally likely

What is the probability that two people will have the same birthday?

Modeling this problem

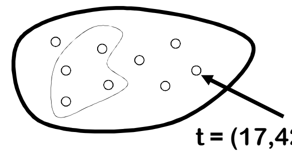
We assume this random experiment:

Each person born on a uniformly random day of the year, independent of the others.

The year has 366 days.

And The Same Methods Again!

Sample space W = $\{1, 2, 3, \ldots, 366\}^{23}$



t = (17,42,363,1,…, 224,177)

23 numbers

Event E = $\{ t \in W \mid$ two numbers in t are same $\}$

What is $|E|$? Count $|\overline{E}|$ instead!

$\overline{E}$ = all sequences in S that have no repeated numbers

$|\overline{E}| = (366)(365)\ldots(344)$

$|W| = 366^{23}$

$$\frac{|\overline{E}|}{|W|} = 0.494\ldots$$

$$\frac{|E|}{|W|} = 0.506\ldots$$

Birthday Paradox

Number of People	Probability of no collisions
21	0.556
22	0.524
23	0.494
24	0.461

Modeling this problem

We assume this random experiment:

Each person born on a uniformly random day of the year, independent of the others.

The year has 366 days.

Accounting for seasonal variations in birthdays would make it more likely to have collisions!

BTW, note that probabilities satisfy the following properties:

$$\left.\begin{array}{l} \text{1) } P(S) = 1 \\ \text{2) } P(E) \geq 0 \text{ for all events } E \\ \text{3) } P(A \cup B) = P(A) + P(B), \\ \qquad \text{for disjoint events A and B} \end{array}\right\} \text{axioms of probability}$$

Hence, $P(\overline{A}) = 1 - P(A)$

BTW, note that probabilities satisfy the following properties:

1) P(S) = 1

2) P(E) $\geq$ 0 for all events E

3) P(A $\cup$ B) = P(A) + P(B), for disjoint events A and B

$\Bigg\}$ axioms of probability

To develop the notion of probability for infinite spaces, we define probabilities as functions satisfying these properties…

Two More Useful Theorems

For any events A and B,
$P(A) = P(A \cap B) + P(A \cap \overline{B})$

For any events A and B,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Inclusion-Exclusion!

Corollary: For any events A and B,
$P(A \cup B) \leq P(A) + P(B)$

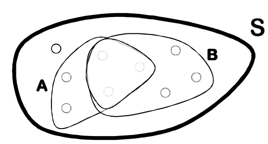
"Union-Bound"
"Boole's inequality"

Conditional Probabilities

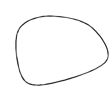
More Language Of Probability

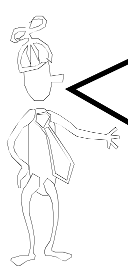
The probability of event A given event B is written Pr[A | B] and is defined to be =

$$\frac{\Pr[\,A \cap B\,]}{\Pr[\,B\,]}$$



proportion of A $\cap$ B 

to B 



Suppose we roll a white die and black die

What is the probability that the white is 1 given that the total is 7?

event A = {white die = 1}

event B = {total = 7}

S = { $\boxed{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),}\ \boxed{(1,6)}$
(2,1), (2,2), (2,3), (2,4), $\boxed{(2,5)}$ (2,6),
(3,1), (3,2), (3,3), $\boxed{(3,4)}$ (3,5), (3,6),
(4,1), (4,2), $\boxed{(4,3)}$ (4,4), (4,5), (4,6),
(5,1), $\boxed{(5,2)}$ (5,3), (5,4), (5,5), (5,6),
$\boxed{(6,1)}$ (6,2), (6,3), (6,4), (6,5), (6,6) }

$$\Pr[\,A \mid B\,] = \frac{\Pr[\,A \cap B\,]}{\Pr[B]} = \frac{|A \cap B|}{|B|} = \frac{1}{6}$$

event A = {white die = 1} event B = {total = 7}

Independence!

A and B are independent events if

Pr[A | B] = Pr[A]

$\Leftrightarrow$

Pr[A $\cap$ B] = Pr[A] Pr[B]

$\Leftrightarrow$

Pr[B | A] = Pr[B]

Two fair coins are flipped
A = {first coin is heads}
B = {second coin is heads}

Are A and B independent?

Pr[A] =
Pr[B] =
Pr[A | B] =

H,H	H,T
T,H	T,T

Two fair coins are flipped
A = {first coin is heads}
C = {two coins have different outcomes}

Are A and C independent?

Pr[A] =
Pr[C] =
Pr[A | C] =

H,H	H,T
T,H	T,T

Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring

E.g., $\{A_1, A_2, A_3\}$ are independent events if:

$Pr[A_1 \mid A_2 \cap A_3] = Pr[A_1]$
$Pr[A_2 \mid A_1 \cap A_3] = Pr[A_2]$
$Pr[A_3 \mid A_1 \cap A_2] = Pr[A_3]$

$Pr[A_1 \mid A_2] = Pr[A_1]$ $\quad$ $Pr[A_1 \mid A_3] = Pr[A_1]$
$Pr[A_2 \mid A_1] = Pr[A_2]$ $\quad$ $Pr[A_2 \mid A_3] = Pr[A_2]$
$Pr[A_3 \mid A_1] = Pr[A_3]$ $\quad$ $Pr[A_3 \mid A_2] = Pr[A_3]$

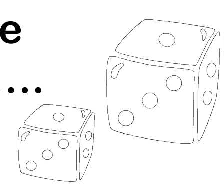
Two fair coins are flipped
A = {first coin is heads}
B = {second coin is heads}
C = {two coins have different outcomes}

A&B independent?
A&C independent?
B&C independent?

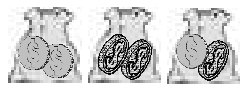
A&B&C independent?

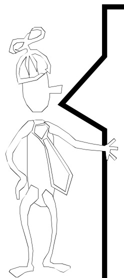
H,H	H,T
T,H	T,T

Let's solve some more problems….



Silver and Gold

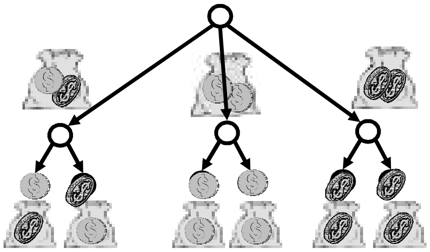


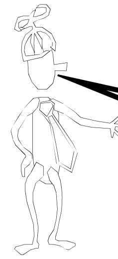


One bag has two silver coins, another has two gold coins, and the third has one of each

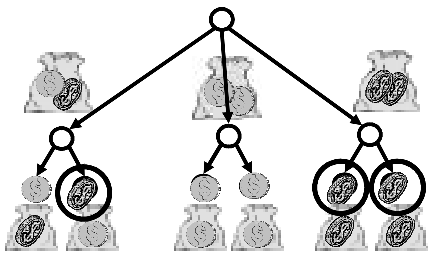
One bag is selected at random. One coin from it is selected at random. It turns out to be gold

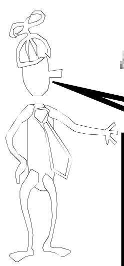
What is the probability that the other coin is gold?





3 choices of bag
2 ways to order bag contents
6 equally likely paths





Given that we see a gold, 2/3 of remaining paths have gold in them!

Formally…

Let G_1 be the event that the first coin is gold

$Pr[G_1] = 1/2$

Let G_2 be the event that the second coin is gold

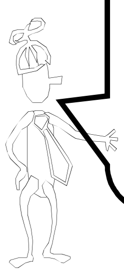
$Pr[G_2 \mid G_1] = Pr[G_1 \text{ and } G_2] / Pr[G_1]$

$= (1/3) / (1/2)$

$= 2/3$

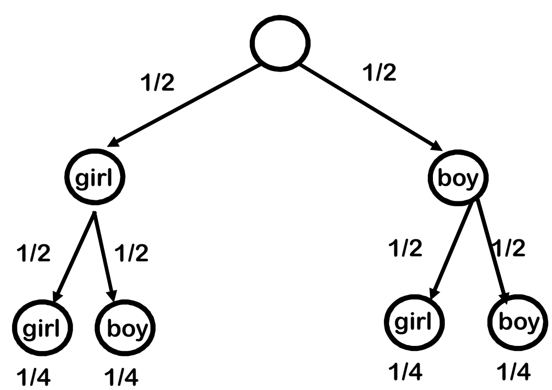
Note: G_1 and G_2 are not independent

Boys and Girls

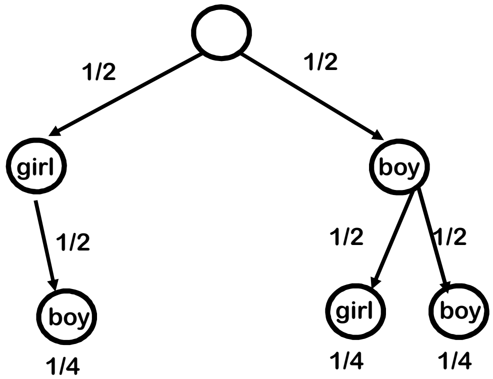


Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

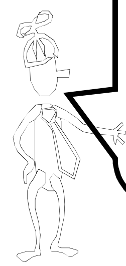
1/3



conditioning on at least one boy…



Boys and Girls



Consider a family with two children. Given that <u>the first</u> child is a boy, what is the probability that both children are boys?

1/2

Monty Hall Problem

Announcer hides prize behind one of 3 doors at random

You select some door

Announcer opens one of others with no prize

You can decide to keep or switch

What to do?

Monty Hall Problem

Sample space = { prize behind door 1, prize behind door 2, prize behind door 3 }

Each has probability 1/3

Staying	Switching
we win if we choose the correct door	we win if we choose the incorrect door
Pr[choosing correct door] = 1/3	Pr[choosing incorrect door] = 2/3

Monty Hall Problem

Let the doors be called X, Y and Z.

Let Cx, Cy, Cz be events that car is behind door X, etc

Let Hx, Hy, Hz be events that host opens door X, etc.

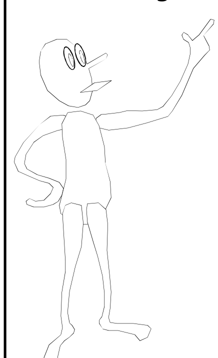
Supposing that you choose door X, the possibility that you win a car if you switch is

$P(Hz \cap Cy) + P(Hy \cap Cz) =$

$P(Hz \mid Cy)\, P(Cy) + P(Hy \mid Cz)\, P(Cz) =$

$1 \times 1/3 + 1 \times 1/3 = 2/3$

Why Was This Tricky?

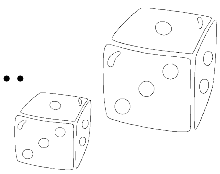


We are inclined to think:

"After one door is opened, others are equally likely…"

But his action is not independent of yours!

Some useful sample spaces…



1) A fair coin

sample space S = {H, T}

Pr(H) = ½ , Pr(T) = ½.

2) A “bias-p” coin

sample space S = {H, T}

Pr(H) = p, Pr(T) = 1-p.

3) Two bias-p coins

sample space S = {HH, HT, TH, TT}

x	$\Pr[x]$
$\langle T,T\rangle$	$(1-p)^2$
$\langle T,H\rangle$	$(1-p)p$
$\langle H,T\rangle$	$(1-p)p$
$\langle H,H\rangle$	p^2

“Binomial Distribution B(n,p)”

3) n bias-p coins

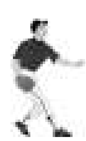
sample space S = $\{H,T\}^n$

if outcome x in S has k heads and n-k tails

$\Pr(x) = p^k (1-p)^{n-k}$

Event E = {x in S | x has k heads}

$\Pr(x) = \sum_{x \text{ in } E} \Pr(x) = \binom{n}{k} p^k (1-p)^{n-k}$



Teams A is better than team B

The odds of A winning are 6:5

i.e., in any game, A wins with probability 6/11

What is the chance that A will beat B in the “best of 7” world series?

Team A beats B with probability 6/11 in each game

(implicit assumption: true for each game, independent of past.)

Sample space S = $\{W, L\}^7$

$\Pr(x) = p^k(1-p)^{7-k}$ if there are k W’s in x

Want event E = “team A wins at least 4 games”

E = {x in S | x has at least 4 W’s}

$\Pr(E) = \sum_{x \text{ in } E} \Pr(x) = \sum_{k=4}^{7} \binom{7}{k} p^k (1-p)^{7-k}$

= 0.5986…

Question:

Why is it permissible to assume that the two teams play a full seven-game series even if one team wins four games before seven have been played?

Given a fair coin ($p = \frac{1}{2}$)

$$\Pr(\text{see } k \text{ heads in } n \text{ flips}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} 2^{-n}$$

As $n \to \infty$, the plot for Pr(k heads) tends to "bell curve" or "Gaussian/normal distribution"