15-251

Some AWESOME

~~Great~~ ~~Theoretical~~ Ideas

~~in~~ ~~Computer~~ ~~Science~~

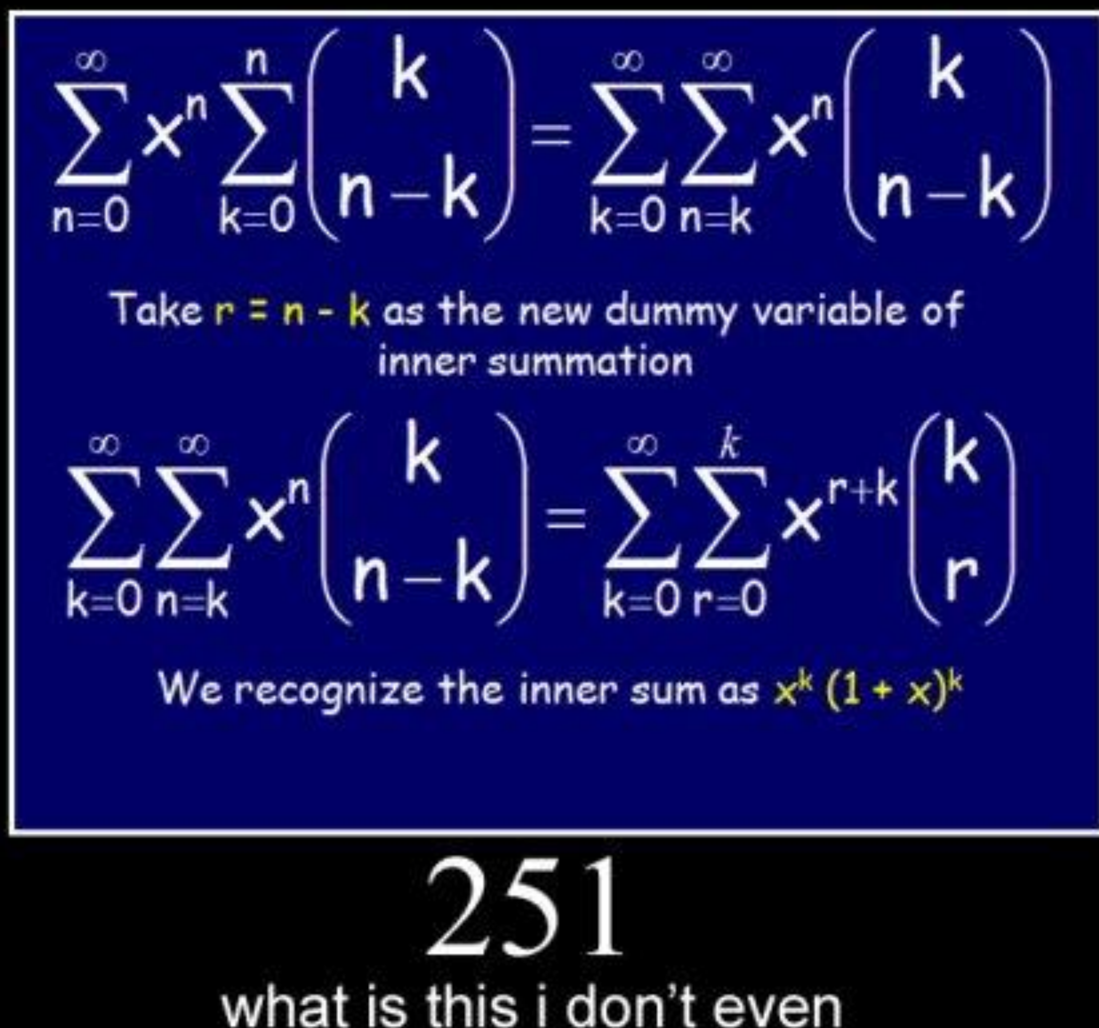
about Generating Functions

Generating Functions

Lecture 9 (September 21, 2010)

$$\sum_{n=0}^{\infty} x^n$$

What is a generating function and why would I use one?



Representation

$$\langle 1, 1, 1, \ldots \rangle$$

$$a_k = 1$$

$$a_0 = 1$$
$$a_n = a_{n-1}$$

$$1 + 1x + 1x^2 + \cdots = \frac{1}{1-x}$$

What *IS* a Generating Function?

We'll just looking at a particular representation of sequences…

$$1 + 1x + 1x^2 + \ldots \ = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

In general, when a_n is a sequence…

$$\sum_{n=0}^{\infty} a_n x^n$$

Counting 1,2,3…

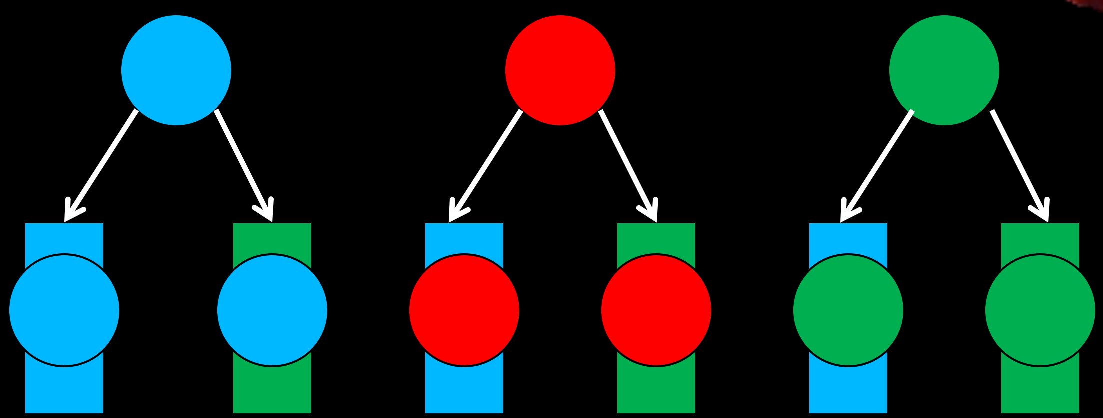
Examples plx…?

Let's talk about a particular counting problem from two lectures ago…

Danny owns 3 beanies and 2 ties. How many ways can he dress up in a beanie and a tie?

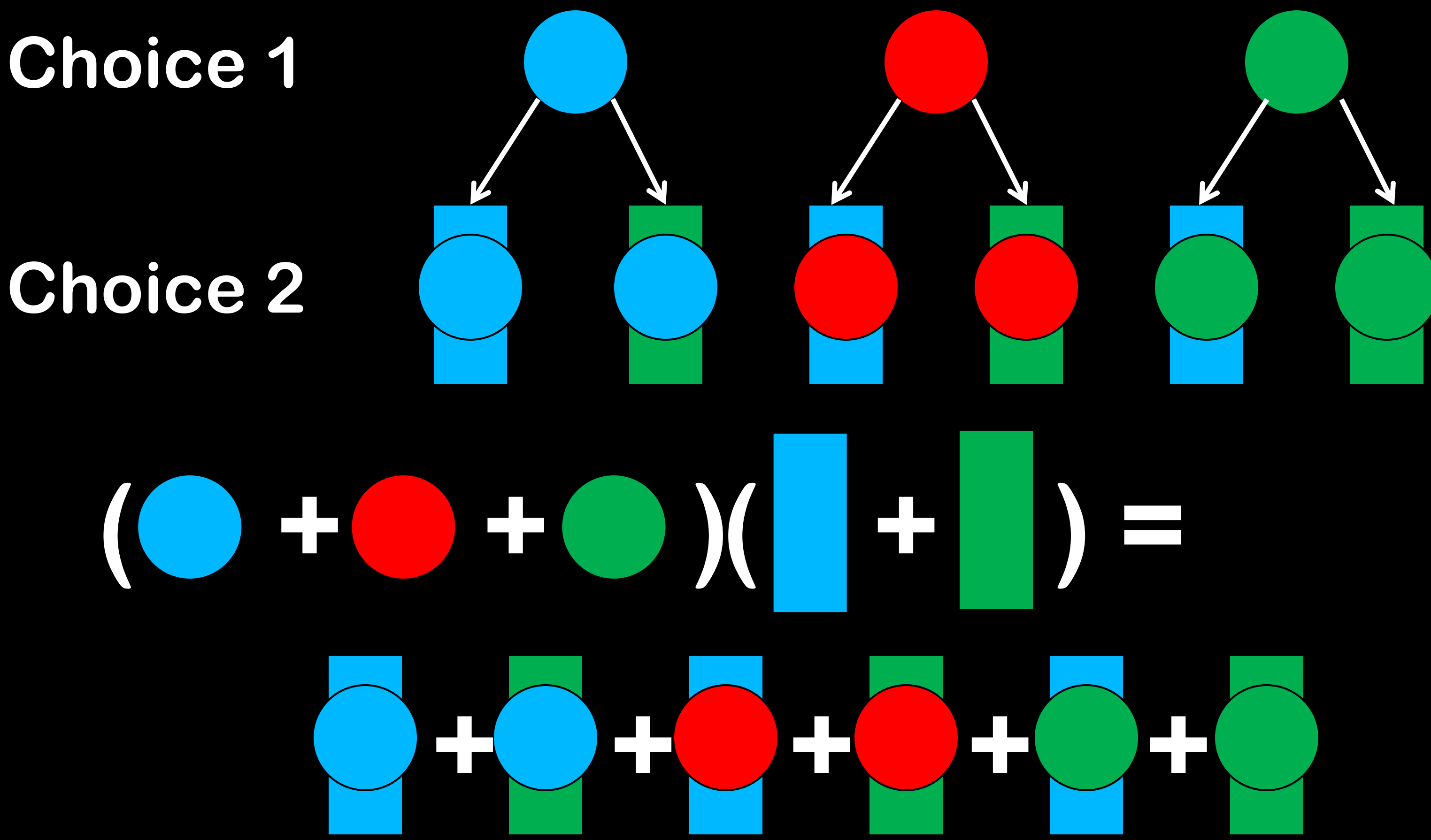
Choice 1

Choice 2



Counting 1,2,3…

Danny owns 3 beanies and 2 ties. How many ways can he dress up in a beanie and a tie?



Counting 1,2,3…

Danny owns 3 beanies and 2 ties. How many ways can he dress up in a beanie and a tie?

How many beanies are we choosing?
How many hats?

Since we only care about the NUMBER,
we can replaces beanies and hats with 'x'

$$(x + x + x)(x + x) =$$

$$x^2 + x^2 + x^2 + x^2 + x^2 + x^2 = 6x^2$$

That is, 6 is the number of ways to choose 2 things

Counting …4,5,6,…

Danny owns 3 beanies and 2 ties. How many ways can he dress up **if he doesn't always wear a beanie or a tie (and wears at most one of each)**?

How many ways for a beanie?

1) 1 way for no beanie
2) 3 ways for one beanie

$$1 + 3x$$

How many ways for a tie?

1) 1 way for no tie
2) 2 ways for one tie

$$1 + 2x$$

$$(1 + 3x)(1 + 2x) = 1 + 5x + 6x^2$$

…And why would I use one?

They're fun!

Solving counting problems

Solving recurrences precisely

Proving identities

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

Let c_n be the number of ways to make change for n cents

For instance, how many ways can we make change for six cents?

1) 6 pennies
2) 1 penny and 1 nickel

So, $c_6 = 2$

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

We want to represent c_n as a generating function

What choices can we make to get n cents?

We choose pennies, nickels, dimes, and quarters separately and then put them together

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

We want to represent c_n as a generating function

Remember that the EXPONENT is the 'n' in c_n and the COEFFICIENT is the number of ways we can make change for n cents

To choose pennies...

$$(1 + x + x^2 + \dots)$$

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

We want to represent c_n as a generating function

Pennies: $(1 + x + x^2 + \dots)$

Nickels: $(1 + x^5 + x^{2\times 5} + \dots)$

Dimes: $(1 + x^{10} + x^{2\times 10} + \dots)$

Quarters: $(1 + x^{25} + x^{2\times 25} + \dots)$

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

Pennies: $(1 + x + x^2 + \dots)$

Nickels: $(1 + x^5 + x^{2\times 5} + \dots)$

Dimes: $(1 + x^{10} + x^{2\times 10} + \dots)$

Quarters: $(1 + x^{25} + x^{2\times 25} + \dots)$

Putting the pieces together…

$$(1 + x + x^2 + \dots)(1 + x^5 + x^{10} + \dots)(1 + x^{10} + x^{20} + \dots)(1 + x^{25} + x^{50} + \dots)$$

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

Pennies **Nickels** **Dimes** **Quarters**

$$(1+x+x^2+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)(1+x^{25}+x^{50}+\ldots)$$

Quick Check…does the GF give the right answer for c_6?

$$(1+x+x^2+x^3+x^4+x^5+x^6)(1+x^5)(1)(1)$$

What is the coefficient of x^6?

$$(x^5 \times x^1 \times 1 \times 1)+(x^6 \times 1 \times 1)$$

Counting Coins

Suppose we have pennies, nickels, dimes, and quarters, and we want to know how many ways we can make change for n cents.

Pennies	Nickels	Dimes	Quarters
$(1+x+x^2+\dots)$	$(1+x^5+x^{10}+\dots)$	$(1+x^{10}+x^{20}+\dots)$	$(1+x^{25}+x^{50}+\dots)$
$\frac{1}{1-x}$	$\frac{1}{1-x^5}$	$\frac{1}{1-x^{10}}$	$\frac{1}{1-x^{25}}$

The infinite sums are clunky, can we find a simpler form?

$$C(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{25}}\right)$$

Technical Terminology

$$(1+x+x^2+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)(1+x^{25}+x^{50}+\ldots)$$

This is the generating function for the change problem

$$C(x)=\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{25}}\right)$$

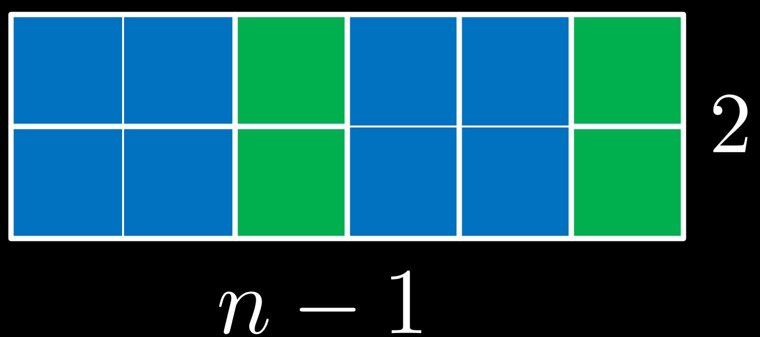
This is the closed form generating function for the change problem

$$[x^n]\,C(x)=c_n$$

c_n **is the coefficient of** x^n **in** $C(x)$

Domino Domination

We have a $2 \times (n-1)$ board, and we would like to fill it with dominos. We have two colors of dominos: green and blue. The green ones are 1×1, and the blue ones are 2×1. How many ways can we tile our board using non-staggered dominos?

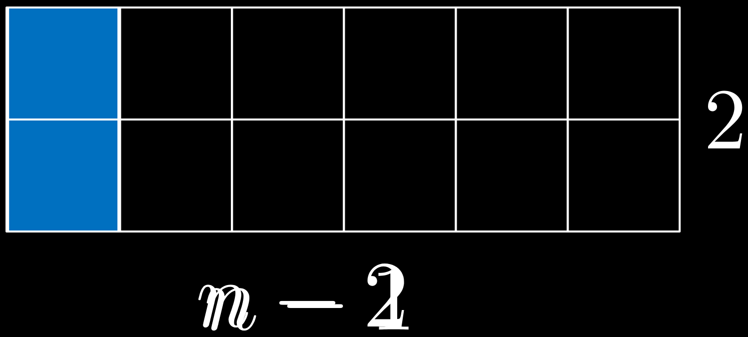


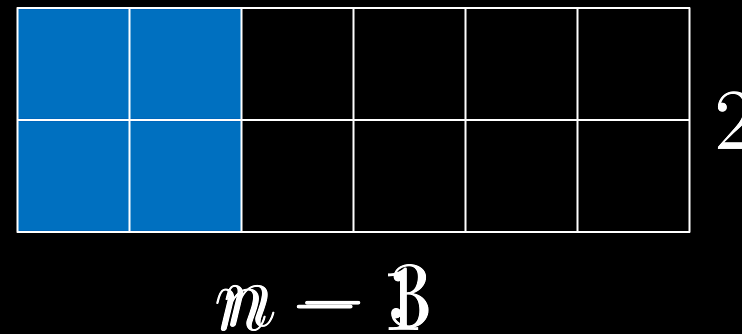
Domino Domination

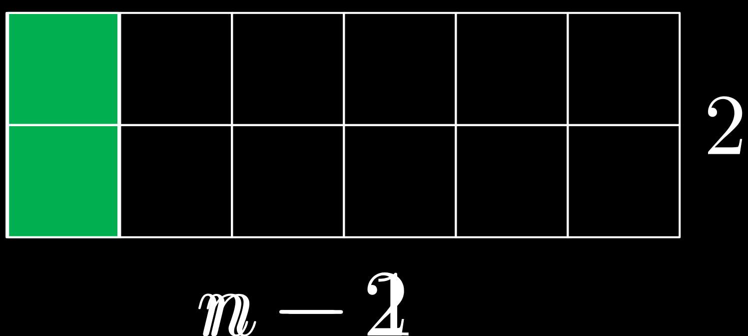
This is a non-obvious combinatorial question!
How should we proceed?!?!

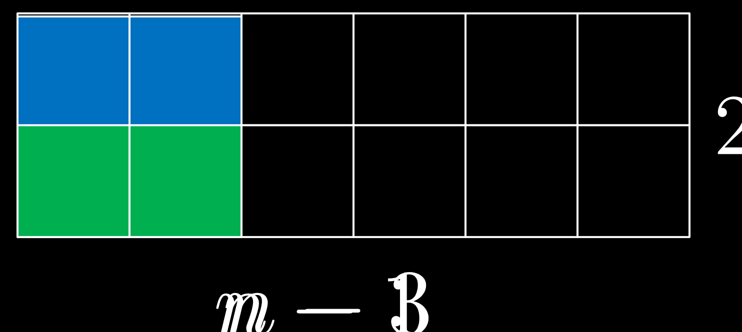
Write a recurrence!

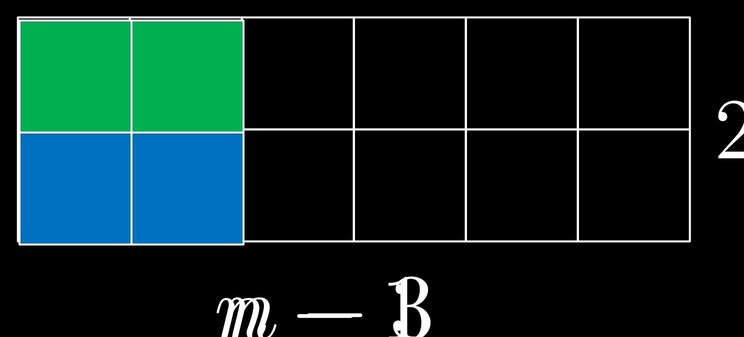
Domino Domination











$$d_n = 2d_{n-1} + 3d_{n-2}$$

So we have a recurrence…but now what?

Domino Domination

$$d_n = 2d_{n-1} + 3d_{n-2}$$

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + \sum_{n=2}^{\infty} (2d_{n-1} + 3d_{n-2}) x^n$

d_n **is the number of ways to tile a** $2 \times (n-1)$ **board.**

We know the base cases:

$$d_0 = 0$$
$$d_1 = 1$$

Domino Domination

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = x + \sum_{n=2}^{\infty} (2d_{n-1} + 3d_{n-2}) x^n$

$$= x + \sum_{n=2}^{\infty} 2d_{n-1} x^n + \sum_{n=2}^{\infty} 3d_{n-2} x^n$$

$$= x + 2x \sum_{n=2}^{\infty} d_{n-1} x^{n-1} + 3x^2 \sum_{n=2}^{\infty} d_{n-2} x^{n-2}$$

$$= x + 2x \sum_{n=1}^{\infty} d_n x^n + 3x^2 \sum_{n=0}^{\infty} d_n x^n$$

$$= x + 2x(D(x) - d_0) + 3x^2 D(x)$$

Domino Domination

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = x + 2x(D(x) - d_0) + 3x^2 D(x)$

$$D(x) = x + 2xD(x) + 3x^2 D(x)$$

$$(1 - 2x - 3x^2)D(x) = x$$

$$D(x) = \frac{x}{1 - 2x - 3x^2}$$

Domino Domination

Why is the closed form of the GF helpful or useful?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1-2x-3x^2} = \frac{-1}{4(1+x)} + \frac{1}{4(1-3x)}$

Break it into smaller pieces!

$$\frac{x}{1-2x-3x^2} = \frac{x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}$$

$$x = (1-3x)A + (1+x)B$$

$$1 = -3A + B$$

$$0 = A + B$$

$$A = \frac{-1}{4}$$

$$B = \frac{1}{4}$$

Domino Domination

Why is the closed form of the GF helpful or useful?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \dfrac{x}{1-2x-3x^2} = \dfrac{-1}{4(1+x)} + \dfrac{1}{4(1-3x)}$

$$= \frac{-1}{4}\sum_{n=0}^{\infty}(-x)^n + \frac{1}{4}\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \qquad \frac{1}{1-(3x)} = \sum_{n=0}^{\infty} (3x)^n$$

Domino Domination

Why is the closed form of the GF helpful or useful?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \dfrac{x}{1-2x-3x^2} = \dfrac{-1}{4}\sum_{n=0}^{\infty}(-x)^n + \dfrac{1}{4}\sum_{n=0}^{\infty}(3x)^n$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^{n+1}x^n + \frac{1}{4}\sum_{n=0}^{\infty}3^n x^n$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left((-1)^{n+1}x^n + 3^n x^n\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{4}\left((-1)^{n+1} + 3^n\right)x^n$$

$$d_n = \frac{1}{4}\left((-1)^{n+1} + 3^n\right)$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = x + 4x^2 + \sum_{n=3}^{\infty} (5a_{n-1} - 8a_{n-2} + 4a_{n-3})x^n$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

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Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = x + 4x^2 + \sum_{n=3}^{\infty} (5a_{n-1} - 8a_{n-2} + 4a_{n-3})x^n$

$$= x + 4x^2 + \sum_{n=3}^{\infty} 5a_{n-1}x^n - \sum_{n=3}^{\infty} 8a_{n-2}x^n + \sum_{n=3}^{\infty} 4a_{n-3}x^n$$

$$= x + 4x^2 + 5x\sum_{n=3}^{\infty} a_{n-1}x^{n-1} - 8x^2\sum_{n=3}^{\infty} a_{n-2}x^{n-2} + 4x^3\sum_{n=3}^{\infty} a_{n-3}x^{n-3}$$

$$= x + 4x^2 + 5x\sum_{n=2}^{\infty} a_n x^n - 8x^2\sum_{n=1}^{\infty} a_n x^n + 4x^3\sum_{n=0}^{\infty} a_n x^n$$

$$= x + 4x^2 + 5x(A(x) - a_0 - a_1x^1) - 8x^2(A(x) - a_0) + 4x^3A(x)$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ $= x + 4x^2 + 5x(A(x) - a_0 - a_1 x^1) - 8x^2(A(x) - a_0) + 4x^3 A(x)$

$$A(x) = x + 4x^2 + 5x(A(x) - x^1) - 8x^2(A(x)) + 4x^3 A(x)$$

$$(1 - 5x + 8x^2 - 4x^3)A(x) = x + 4x^2 - 5x^2$$

$$A(x) = \frac{x - x^2}{1 - 5x + 8x^2 - 4x^3}$$

Rogue Recurrence

$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= \frac{x - x^2}{1 - 5x + 8x^2 - 4x^3}$$

$$= \frac{x(1-x)}{(1-2x)^2(1-x)}$$

$$= \frac{x}{(1-2x)^2}$$

What next? Partial fractions?

We could! It would work, but…

Rogue Recurrence

No. Let's be sneaky instead!

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (2x)^n\right)$$

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^n x^{n-1}$$

$$\frac{x}{2}\frac{2}{(1-2x)^2} = \frac{x}{2}\sum_{n=0}^{\infty} n2^n x^{n-1}$$

$$\frac{x}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^{n-1} x^{n}$$

Rogue Recurrence

$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Now back to the recurrence…

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n \quad = \frac{x}{(1-2x)^2}$

$$\frac{x}{(1-2x)^2} = \sum_{n=0}^{\infty} n 2^{n-1} x^n$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n 2^{n-1} x^n$$

$$a_n = n 2^{n-1}$$

Some Common GFs

Sequence	Generating Function
$\langle 1, 1, 1, \ldots \rangle$	$\dfrac{1}{1-x}$
$\langle 1, 2, 4, \ldots \rangle$	$\dfrac{1}{1-2x}$
$\langle 1, 2, 3, \ldots \rangle$	$\dfrac{1}{(1-x)^2}$
$\langle 0, 1, 1, 2, 3, \ldots \rangle$	$\dfrac{x}{1-x-x^2}$

Double Sums OMGWTFBBQ!

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k}$$

Our first step is to swap the summations. Let's try a small example…

$$\sum_{n=0}^{10} \sum_{k=0}^{n} x^n \binom{k}{n-k} =$$

$$= \left(x^0 \binom{0}{n}\right) + \left(x^0 \binom{0}{n} + x^1 \binom{1}{n-1}\right) + \cdots + \left(x^0 \binom{0}{n-0} + x^1 \binom{1}{n-1} + \cdots + x^{10} \binom{10}{n-10}\right)$$

$$= \sum_{n=0}^{10} x^n \binom{0}{n} + \sum_{n=0}^{10} x^n \binom{1}{n-1} + \cdots + \sum_{n=0}^{10} x^n \binom{10}{n-10} \quad = \sum_{k=0}^{10} \sum_{n=0}^{10} x^n \binom{k}{n-k}$$

Double Sums OMGWTFBBQ!

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k}$$

Our first step is to swap the summations.

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} x^n \binom{k}{n-k}$$

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{n+k} \binom{k}{n}$$

Double Sums OMGWTFBBQ!

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} x^{n+k}\binom{k}{n}$$

$$\sum_{k=0}^{\infty} x^k \sum_{n=0}^{\infty} x^n \binom{k}{n}$$

$$\sum_{k=0}^{\infty} x^k \left(\sum_{n=0}^{k} x^n \binom{k}{n} + \sum_{n=k+1}^{\infty} x^n \binom{k}{n}\right)$$

$$\sum_{k=0}^{\infty} x^k \sum_{n=0}^{k} x^n \binom{k}{n}$$

We know that…

$$\sum_{i=0}^{p} \binom{p}{i} x^i = (1+x)^p$$

$$\sum_{k=0}^{\infty} x^k (1+x)^k = \frac{1}{1-x(1+x)} = \frac{1}{1-x-x^2}$$

Double Sums OMGWTFBBQ!

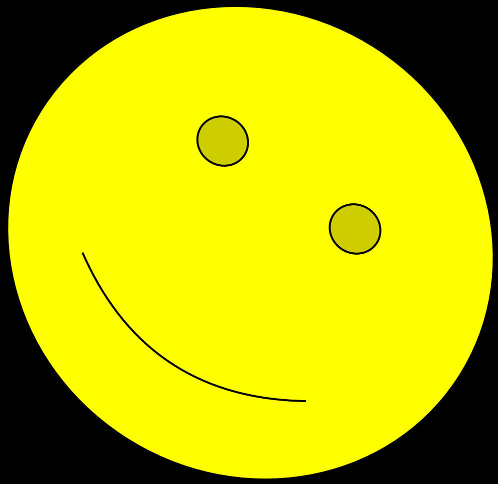
$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k} = \sum_{k=0}^{\infty} x^k (1+x)^k = \frac{1}{1-x-x^2}$$

From the table... $$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

So... $$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{1}{1-x-x^2}$$

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n$$

$$\sum_{k=0}^{n} \binom{k}{n-k} = F_{n+1}$$



Generating Functions

- Counting with GFs
- Solving recurrences with GFs
- How to derive base cases of recurrences
- Basic partial fractions

Here's What You Need to Know…