Great Theoretical Ideas In Computer Science		
Anupam Gupta Danny Sleator		CS 15-251 Fall 2010
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Counting I: Choice Trees and Correspondences

In the next few lectures we will learn some fundamental counting methods.

- •Addition and Product Rules
- •The Inclusion-Exclusion Principal
- •Choice Tree
- •Permutations and Combinations
- •The Binomial Theorem
- •The Pigeonhole Principal
- •Diophantine Equations
- •Generating Functions

If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

Addition Rule

Let A and B be two disjoint finite sets

$$|A \cup B| = |A| + |B|$$

Addition of Multiple Disjoint Sets:

Let A_1, A_2, A_3, …, A_n be disjoint, finite sets:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Addition Rule (2 Possibly Overlapping Sets)

Let A and B be two finite sets:

$$|A \cup B| =$$

$$|A| + |B| - |A \cap B|$$

Inclusion-Exclusion

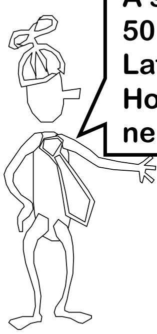
If A, B, C are three finite sets, what is the size of $(A \cup B \cup C)$?

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

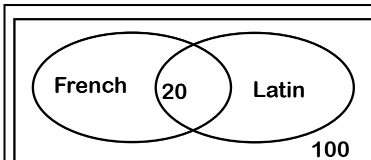
Inclusion-Exclusion

If $A_1, A_2, \ldots, A_n$ are n finite sets, what is the size of $(A_1 \cup A_2 \cup \ldots \cup A_n)$?

$$\Sigma_i |A_i| - \Sigma_{i<j} |A_i \cap A_j| + \Sigma_{i<j<k} |A_i \cap A_j \cap A_k| \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$



A school has 100 students. 50 take French, 40 take Latin, and 20 take both. How many students take neither language?



F ≡ French students
L ≡ Latin students

$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

French OR Latin students:

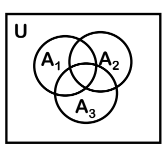
$|F \cup L|=|F|+|L|-|F \cap L|$
$= 50 + 40 – 20 = 70$

Neither language:

$100 – 70 = \boxed{30}$

How many positive integers ≤ 70 are relatively prime to 70?

U = [1..70], 70 = 2x5x7



A_1 ≡ integers in U divisible by 2
A_2 ≡ integers in U divisible by 5
A_3 ≡ integers in U divisible by 7

$|A_1| = 35$ $|A_2| = 14$ $|A_3|=10$

$|A_1 \cap A_2| = 7$
$|A_1 \cap A_3| = 5$
$|A_2 \cap A_3| = 2$
$|A_1 \cap A_2 \cap A_3| = 1$

$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

How many positive integers less than 70 are relatively prime to 70?

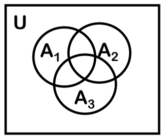
$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$|A_1 \cup A_2 \cup A_3| = 35+14+10-7-5-2+1 = 46$

Thus, the number of relatively prime to 70 is 70 – 46 = 24.

The Principle of Inclusion and Exclusion

Let S_k be the sum of the sizes of
All k-tuple intersections of the A_i's.



$S_1=|A_1|+|A_2|+|A_3|$
$S_2=|A_1\cap A_2|+|A_1\cap A_3|+|A_2\cap A_3|$
$S_3=|A_1\cap A_2\cap A_3|$

$$\boxed{|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3}$$

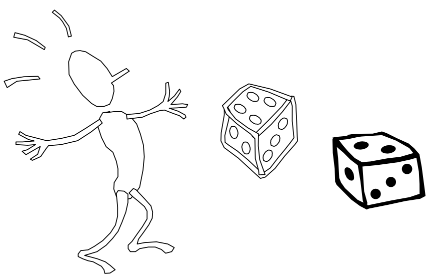
Partition Method

To count the elements of a finite set S, partition the elements into non-overlapping subsets $A_1, A_2, A_3, \ldots, A_n$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

S = all possible outcomes of one white die and one black die.



Partition Method

S = all possible outcomes of one white die and one black die.

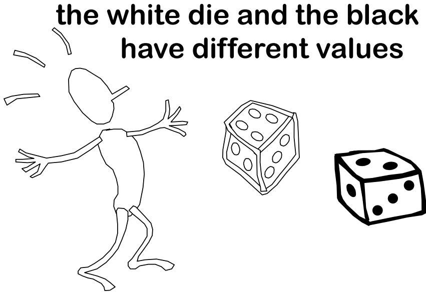
Partition S into 6 sets:

A_1 = the set of outcomes where the white die is 1.
A_2 = the set of outcomes where the white die is 2.
A_3 = the set of outcomes where the white die is 3.
A_4 = the set of outcomes where the white die is 4.
A_5 = the set of outcomes where the white die is 5.
A_6 = the set of outcomes where the white die is 6.

Each of 6 disjoint set have size 6 = 36 outcomes

Partition Method

S = all possible outcomes where the white die and the black die have different values



S ≡ Set of all outcomes where the dice show different values. |S| = ?

A_i ≡ set of outcomes where black die says i and the white die says something else.

$$|S| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

S ≡ Set of all outcomes where the dice show different values. |S| = ?

B ≡ set of outcomes where dice agree.

| S ∪ B | = # of outcomes = 36

|S| + |B| = 36

|B| = 6

|S| = 36 – 6 = 30

Difference Method

To count the elements of a finite set S, find two sets A and B such that

S and B are disjoint

and

S ∪ B = A

then |S| = |A| - |B|

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. |S| = ?

A_i ≡ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. |S| = ?

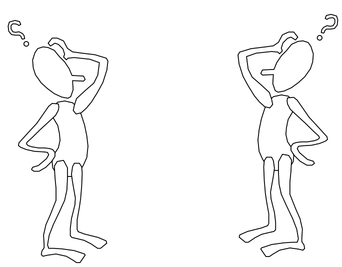
L ≡ set of all outcomes where the black die shows a larger number than the white die.

|S| + |L| = 30

It is clear by symmetry that | S | = | L |.

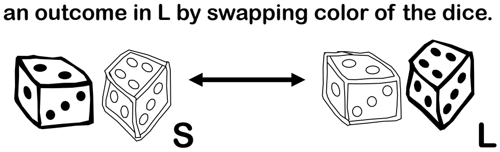
Therefore | S | = 15

"It is clear by symmetry that |S| = |L|?"



Pinning Down the Idea of Symmetry by Exhibiting a Correspondence

Put each outcome in S in correspondence with an outcome in L by swapping color of the dice.



Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: |S| = |L|

Let f : A → B Be a Function From a Set A to a Set B

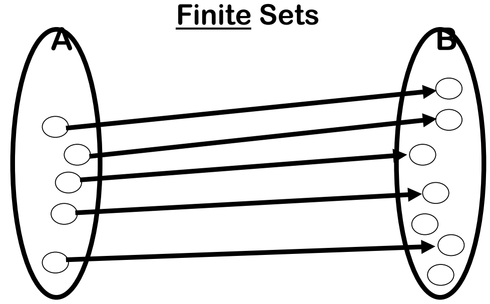
f is injective if and only if
$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$

f is surjective if and only if
$\forall z \in B\ \exists x \in A\ f(x) = z$

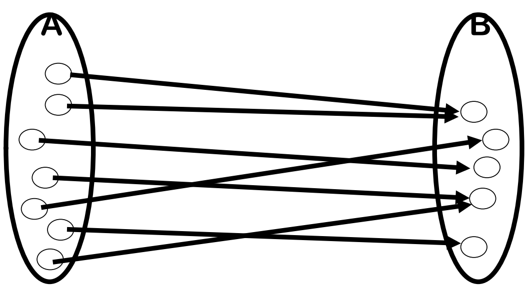
There Exists

For Every

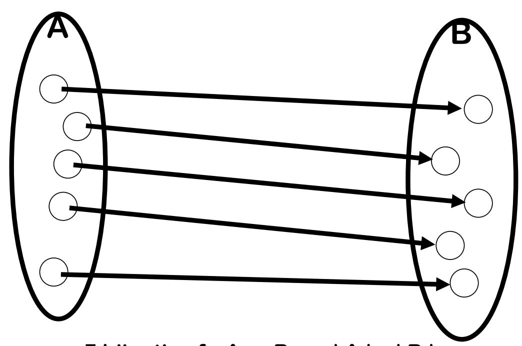
Let's Restrict Our Attention to Finite Sets



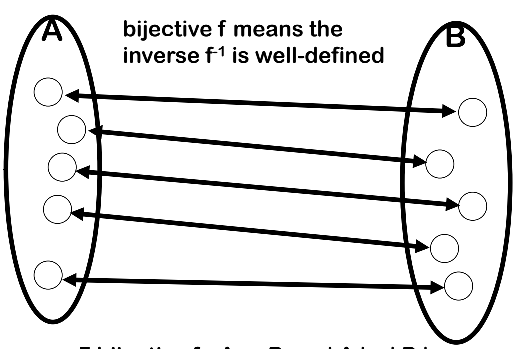
$\exists$ injective (1-1) $f : A \rightarrow B \Rightarrow |A| \leq |B|$



$\exists$ surjective (onto) $f : A \rightarrow B \Rightarrow |A| \geq |B|$



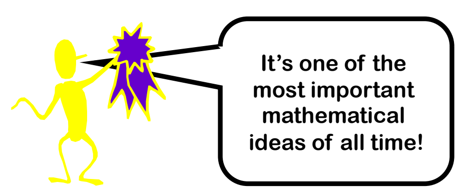
$\exists$ bijective $f : A \rightarrow B \Rightarrow |A| = |B|$



$\exists$ bijective $f : A \rightarrow B \Rightarrow |A| = |B|$

Correspondence Principle

If two finite sets can be placed into bijection, then they have the same size



Question: How many n-bit sequences are there?

000000	↔	0
000001	↔	1
000010	↔	2
000011	↔	3
:	:	:
111111	↔	2^n-1

Each sequence corresponds to a unique number from 0 to 2^n-1. Hence 2^n sequences.

S = { a,b,c,d,e } has Many Subsets

{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, …

The entire set and the empty set are subsets with all the rights and privileges pertaining thereto

Question: How Many Subsets Can Be Made From The Elements of a 5-Element Set?

a	b	c	d	e
0	1	1	0	1

{ b c e }

1 means "TAKE IT"
0 means "LEAVE IT"

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

S = {a_1, a_2, a_3,…, a_n}, T = all subsets of S
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string b = $b_1b_2b_3…b_n$, let f(b) = { a_i | b_i=1}

Claim: f is injective

Any two distinct binary sequences b and b′ have a position i at which they differ

Hence, f(b) is not equal to f(b′) because they disagree on element a_i

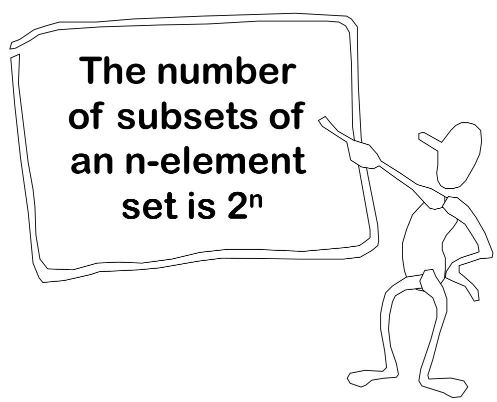
S = {a_1, a_2, a_3,…, a_n}, T = all subsets of S
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string b = $b_1b_2b_3…b_n$, let f(b) = { a_i | b_i=1}

Claim: f is surjective

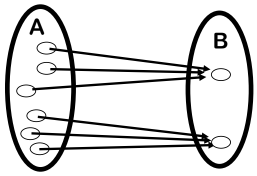
Let X be a subset of {a_1,…,a_n}.
Define b_k = 1 if a_k in X and b_k = 0 otherwise.
Note that f($b_1b_2…b_n$) = X.



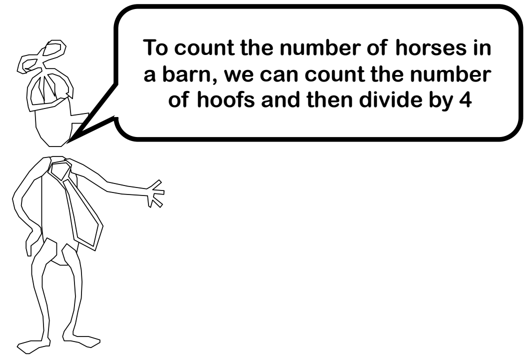
Let f : A → B Be a Function From Set A to Set B

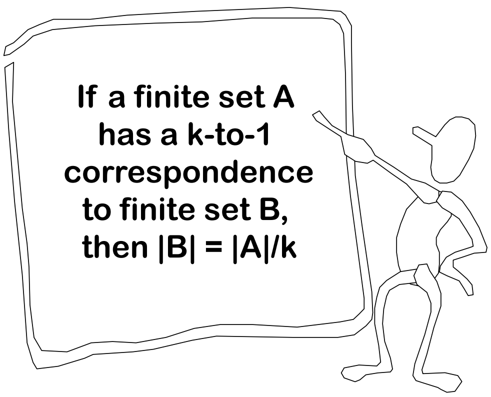
f is a 1 to 1 correspondence (bijection) iff
$\forall z \in B$ $\exists$ exactly one $x \in A$ such that $f(x) = z$

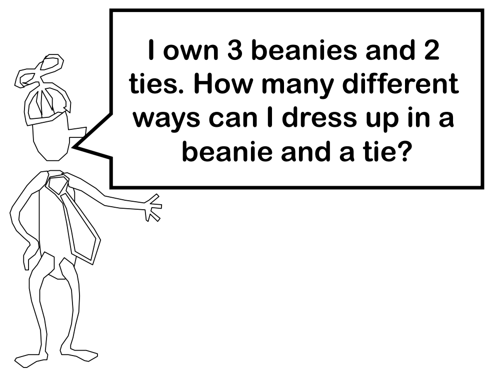
f is a k to 1 correspondence iff
$\forall z \in B$ $\exists$ exactly k $x \in A$ such that $f(x) = z$

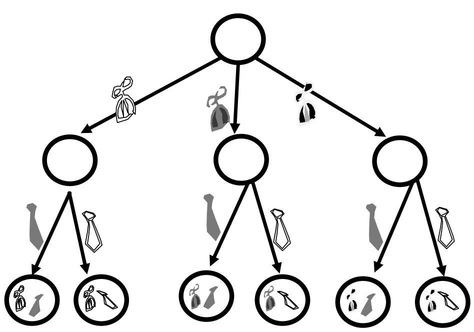


3 to 1 function









A Restaurant Has a Menu With 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts

How many items on the menu?

$5 + 6 + 3 + 7 = 21$

How many ways to choose a complete meal?

$5 \times 6 \times 3 \times 7 = 630$

How many ways to order a meal if I am allowed to skip some (or all) of the courses?

$6 \times 7 \times 4 \times 8 = 1344$

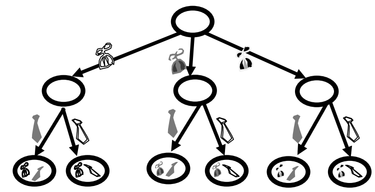
Leaf Counting Lemma

Let T be a depth-n tree when each node at depth $0 \le i \le n-1$ has P_{i+1} children

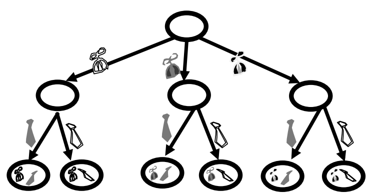
The number of leaves of T is given by:

$P_1 P_2 \ldots P_n$

Choice Tree



A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf



A choice tree provides a "choice tree representation" of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same

We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1 P_2 P_3 \ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

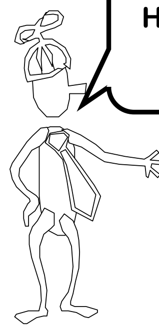
: :

1 possible choice for the 52^{nd} card.

By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

A permutation or arrangement of n objects is an ordering of the objects

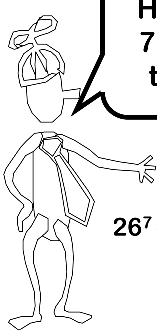
The number of permutations of n distinct objects is n!



How many sequences of 7 letters are there?

26^7

(26 choices for each of the 7 positions)

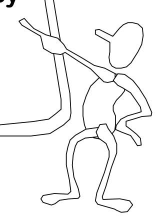


How many sequences of 7 letters contain at least two of the same letter?

$$26^7 - \underbrace{26\times25\times24\times23\times22\times21\times20}_{\text{number of sequences containing all different letters}}$$

The "Difference Principle"

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.



If 10 horses race, how many orderings of the top three finishers are there?

$$10 \times 9 \times 8 = 720$$

Number of ways of ordering, permuting, or arranging r out of n objects

n choices for first place, n-1 choices for second place, . . .

$$n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 × 51

How many unordered pairs?

(52×51) / 2 ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 × 51

How many unordered pairs?

(52×51) / 2 ← divide by overcount

We have a 2-1 map from ordered pairs to unordered pairs.
Hence #unordered pairs = (#ordered pairs)/2

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

52 × 51 × 50 × 49 × 48

How many orderings of 5 cards?

5!

How many unordered 5 card hands?

(52×51×50×49×48)/5! = 2,598,960

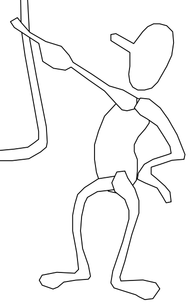
A combination or choice of r out of n objects is an (unordered) set of r of the n objects

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \underbrace{\binom{n}{r}}_{\text{n “choose” r}}$$

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$



How Many 8-Bit Sequences Have 2 0’s and 6 1’s?

Tempting, but incorrect:
8 ways to place first 0, times
7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and then position j for the second 0 gives same sequence as choosing position j for the first 0 and then position i for the second 0 } 2 ways of generating the same object!

How Many 8-Bit Sequences Have 2 0's and 6 1's?

1. Choose the set of 2 positions to put the 0's. The 1's are forced.

$$\binom{8}{2}$$

2. Choose the set of 6 positions to put the 1's. The 0's are forced.

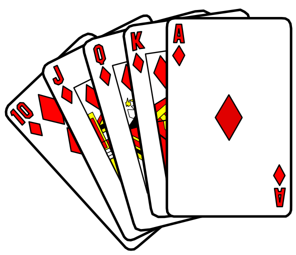
$$\binom{8}{6}$$

Symmetry In The Formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

"# of ways to pick r out of n elements"
=
"# of ways to choose the (n-r) elements to omit"

Counting Cards



How Many 5-card hands Have at Least 3 As?

How Many Hands Have at Least 3 As?

$\binom{4}{3}$ = ways of picking 3 out of 4 aces

$\binom{49}{2}$ = ways of picking 2 cards out of the remaining 49 cards

$$\boxed{4 \times 1176 = 4704}$$

How Many Hands Have at Least 3 As?

How many hands have exactly 3 aces?

$\binom{4}{3}$ = ways of picking 3 out of 4 aces

$\binom{48}{2}$ = ways of picking 2 cards out of the 48 non-ace cards

$$\begin{array}{r} 4 \\ \times\ 1128 \\ \hline 4512 \end{array}$$

How many hands have exactly 4 aces?

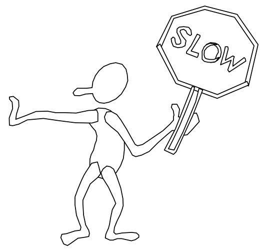
$\binom{4}{4}$ = ways of picking 4 out of 4 aces

$\binom{48}{1}$ = ways of picking 1 cards out of the 48 non-ace cards

$$\begin{array}{r} +\ 48 \\ \hline \boxed{4560} \end{array}$$

4704 ≠ 4560

At least one of the two counting arguments is not correct!



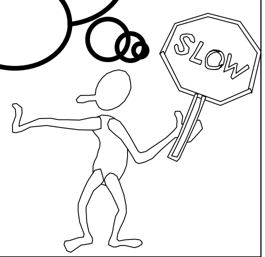
Four Different Sequences of Choices Produce the Same Hand

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

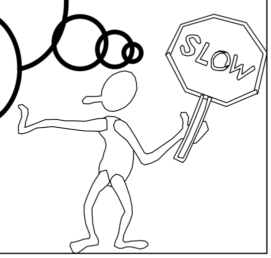
$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Is the other argument correct? How do I avoid fallacious reasoning?



REVERSIBILTY CHECK:
For each object can I reverse engineer the unique sequence of choices that constructed it?



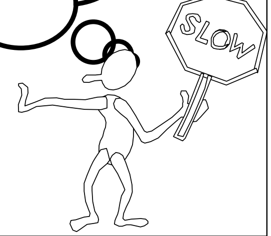
Scheme I
1. Choose 3 of 4 aces
2. Choose 2 of the remaining cards

A♣ A♦ A♥A♠ K♦

For this hand – you can't reverse to a unique choice sequence.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

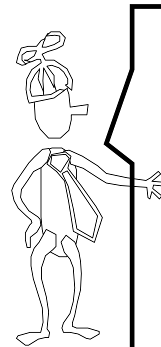
Is the other argument correct? How do I avoid fallacious reasoning?



Scheme II
1. Choose 3 out of 4 aces
2. Choose 2 out of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

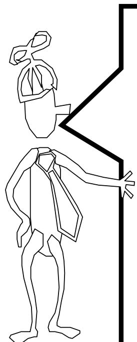


The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S

2. Missing out some objects from the set S

3. Creating the same object two different ways



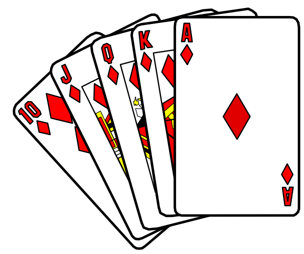
DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I create every object of this type?

Can I reverse engineer my choice sequence from any given object?

Counting Poker Hands



52 Card Deck, 5 card hands

4 possible suits:
♥♦♣♠

13 possible ranks:
2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush: a straight and a flush

4 of a kind: 4 cards of the same rank

Full House: 3 of one kind and 2 of another

Flush: a flush, but not a straight

Straight: a straight, but not a flush

3 of a kind: 3 of the same rank, but not a full house or 4 of a kind

2 Pair: 2 pairs, but not 4 of a kind or a full house

A Pair

Straight Flush

9 choices for rank of lowest card at the start of the straight

4 possible suits for the flush

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2{,}598{,}960} = \text{about 1 in 72193 chance}$$

4 of a Kind

13 choices of rank

48 choices for remaining card

$13 \times 48 = 624$

$$\frac{624}{\binom{52}{5}} = \frac{624}{2{,}598{,}960} = 1 \text{ in } 4165$$

Flush

4 choices of suit

$\binom{13}{5}$ choices of cards

$\left.\right\}$ $4 \times 1287 = 5148$

"but not a straight flush…" - 36 straight flushes

5112 flushes

$$\frac{5{,}112}{\binom{52}{5}} = 1 \text{ in } 508.4\ldots$$

Straight

9 choices of lowest card

4^5 choices of suits for 5 cards

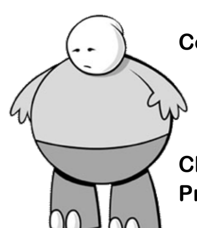
$\left.\right\}$ $9 \times 1024 = 9216$

"but not a straight flush…" - 36 straight flushes

9180 flushes

$$\frac{9{,}180}{\binom{52}{5}} = 1 \text{ in } 208.1\ldots$$

Hand	Number
Straight Flush:	36
Four of a Kind:	624
Full House:	3,744
Flush:	5,112
Straight:	9,180
Three of a Kind:	54,912
Two Pair:	123,552
One Pair:	1,098,240
Nothing:	1,302,540
	2,598,960



Here's What You Need to Know…

Partition and Difference Methods
- Principle of Inclusion and Excl.

Correspondence Principle
- If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

Choice Tree

Product Rule
- Two conditions

Reverse Test

Binomial coefficient

Counting Poker Hands