| Great Theoretical Ideas In Computer Science | | |
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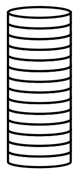
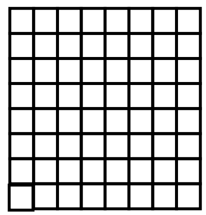
| Lecture 6 | Sep. 9, 2010 | Carnegie Mellon University |

# Mathematical Games II





## Part II - Sums of Games

Consider a game called Boxing Match which was defined in a programming contest
http://potm.tripod.com/BOXINGMATCH/problem.short.html

An n x m rectangular board is initialized with 0 or 1 stone on each cell. Players alternate removing all the stones in any square subarray where all the cells are full. The player taking the last stone wins.

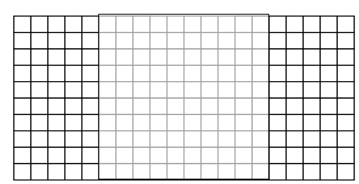
## Boxing Match Example

Suppose we start with a 10 x 20 array that is completely full.

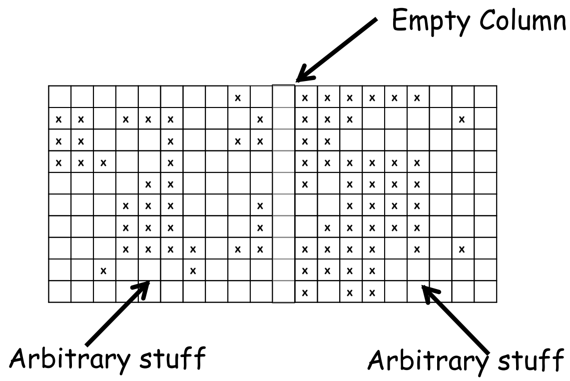
Is this a P or an N-position?

## Example Contd.

The 10 x 20 full board is an N-position. A winning move is to take a 10x10 square in the middle. This leaves a 5x10 rectangle on the left and a 5x10 rectangle on the right. This is a P-position via mirroring. QED.



## Sums of Games




In this kind of situation, the left and right games are completely independent games that don't interact at all. This naturally leads to the notion of the sum of two games.

**A + B**

## A + B

A and B are games. The game A+B is a new game where the allowed moves are to pick one of the two games A or B (that is non-terminal) and make a move in that game. The position is terminal iff both A and B are terminal.

The sum operator is commutative and associative (explain).

## Sums of Games*

We assign a number to any position in any game. This number is called the Nimber of the game.

(It's also called the "Nim Sum" or the "Sprague-Grundy" number of a game. But we'll call it the Nimber.)

*Only applies to normal, impartial games.

## The MEX

The "MEX" of a finite set of natural numbers is the Minimum EXcluded element.

MEX {0, 1, 2, 4, 5, 6} = 3

MEX {1, 3, 5, 7, 9} = 0

MEX {} = 0

## Definition of Nimber

The Nimber of a game $G$ (dnoted $N(G)$) is defined inductively as follows:

$N(G) = 0$ if $G$ is terminal

$N(G) = \mathrm{MEX}\{N(G_1), N(G_2), \ldots N(G_n)\}$

Where $G_1, G_2, \ldots G_n$ are the successor positions of game $G$. (I.e. the positions resulting from all the allowed moves.)

## Another look at Nim

Let $P_k$ denote the game that is a pile of k stones in the game of Nim.

Theorem: $N(P_k) = k$

## Theorem: $N(P_k) = k$

Proof: Use induction.

Base case is when $k=0$. Trivial.
When $k>0$ the set of moves is
$P_{k-1}, P_{k-2}, \ldots P_0$.

By induction these positions have nimbers $k-1, k-2, \ldots 0$.

The MEX of these is k. QED.

Theorem: A game $G$ is a P-position if and only if $N(G)=0$.

(i.e. Nimber = 0 iff P-position)

Proof: Induction.
Trivially true if $G$ is a terminal position.

Suppose $G$ is non-terminal.

If $N(G)\neq 0$, then by the MEX rule there must be a move $G'$ in $G$ that has $N(G')=0$. By induction this is a P-position. Thus $G$ is an N position.

## Nimber = 0 iff P-position (contd)

If $N(G)=0$, then by the MEX rule none of the successors of $G$ have $N(G')=0$. By induction all of them are N-positions. Therefore $G$ is a P-position.

QED.

## The Nimber Theorem

Theorem: Let $A$ and $B$ be two impartial normal games. Then:

$$N(A+B) = N(A) \oplus N(B)$$

Proof: We'll get to this in a minute.

**The beauty of Nimbers is that they completely capture what you need to know about a game in order to add it to another game. This often allows you to compute winning strategies, and can speed up game search exponentially.**

## Application to Nim

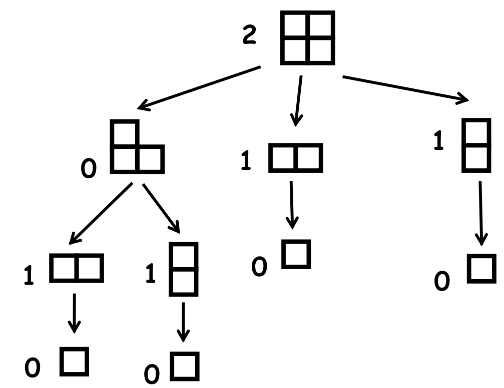
Note that the game of Nim is just the sum of several games. If the piles are of size a, b, and c, then the nim game for these piles is just $P_a + P_b + P_c$.

The nimber of this position, by the Nimber Theorem, is just $a \oplus b \oplus c$.

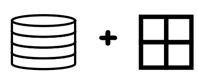
So it's a P-position if and only if $a \oplus b \oplus c = 0$, which is what we proved before.

## Application to Chomp

**What is the nimber of this chomp game?**




**What if we add this to a nim pile of size 4?**



$$4 \oplus 2 = 6$$

**Is this an N-position or a P-position?**

**N() ≠ 0 → it's an N-position. How do you win?**

**If we remove two chips from the nim pile, then the nimber is 0, giving a P-position. This is the unique winning move in this position.**

**Proof of the Nimber Theorem:**

$$N(A+B) = N(A) \oplus N(B)$$

**Let the moves in A be A1, A2, …, An**
**And the moves in B be B1, B2, …, Bm**

**We use induction. If either of these lists is empty the theorem is trivial (base case)**

**The moves in A+B are:**

**A+B1, A+B2, … A+Bm, A1+B, … An+B**

**$N(A+B) = MEX\{N(A+B1),\ldots N(A+Bm),$**
**$N(A1+B),\ldots N(An+B)\}$**

**$N(A+B)$ = (by induction)**
**$MEX\{N(A)\oplus N(B1),\ldots, N(A)\oplus N(Bm),$**
**$N(A1)\oplus N(B),\ldots, N(An)\oplus N(B)\ \}$**

**How do we prove this is $N(A)\oplus N(B)$?**

**We do it by proving two things:**
**(1) $N(A)\oplus N(B)$ is not in the list**
**(2) For all $y < N(A)\oplus N(B)$, y is in the list**

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## (1) $N(A)\oplus N(B)$ is not in the list

**$MEX\{N(A)\oplus N(B1),\ldots, N(A)\oplus N(Bm),$**
**$N(A1)\oplus N(B),\ldots, N(An)\oplus N(B)\ \}$**

**Why is $N(A)\oplus N(B)$ not in this list?**

**Because**
**$N(Bi) \neq N(B) \Rightarrow N(A)\oplus N(Bi) \neq N(A)\oplus N(B)$**

**And**
**$N(Ai) \neq N(A) \Rightarrow N(Ai)\oplus N(B) \neq N(A)\oplus N(B)$**

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## (2) For all $y < N(A)\oplus N(B)$, y is in the list

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N(A)\oplus N(B)$ = | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| y = | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | . | . | . | . | . | . |
| $N(A)$ = | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . |
| $N(B)$ = | . | . | . | . | . | . | . | . | . | 0 | . | . | . | . | . | . |
| $N(Ai)$ = | . | . | . | . | . | . | . | . | . | 0 | x | x | x | x | x | x |

**The highlighted column is the 1st where y and $N(A)\oplus N(B)$ differ.** At that bit position $N(A)\oplus N(B)$ is 1 and y is 0. Therefore one of $N(A)$ and $N(B)$ =1. WLOG assume $N(A)$=1

**Because $N(A)=MEX\{N(A1),\ldots N(An)\}$ there is a move in A such that the bits after the 1 form any desired pattern.**

**Therefore we can produce the desired y. QED.**

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## Application to Breaking Chocolates

**http://www.spoj.pl/problems/BCHOCO/**

**Bored of setting problems for Bytecode, Venkatesh and Akhil decided to take some time off and started to play a game. The game is played on an nxn bar of chocolate consisting of Black and White chocolate cells.**

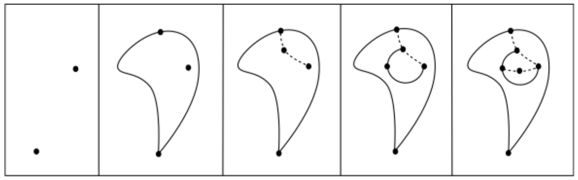
Both of them do not like black chocolate, so if the bar consists only of black chocolate cells, it is discarded (Discarding the bar is not considered as a move). If the bar consists only of white chocolate cells, they do not break it further and the bar can be consumed at any time (Eating the bar is considered as a move). If the bar consists of both black and white chocolate cells, it must be broken down into two smaller pieces by breaking the bar along any horizontal or vertical line (Breaking the bar is considered as a move). The player who cannot make a move on any of the remaining bars loses.

Assuming Venkatesh starts the game and both players are infinitely intelligent, determine who wins the game.

Example --- See document camera.

If the dimension of the original chocolate bar is nxn, give an algorithm to compute the who can win, and analyze its running time as a function of n.

## Sprouts



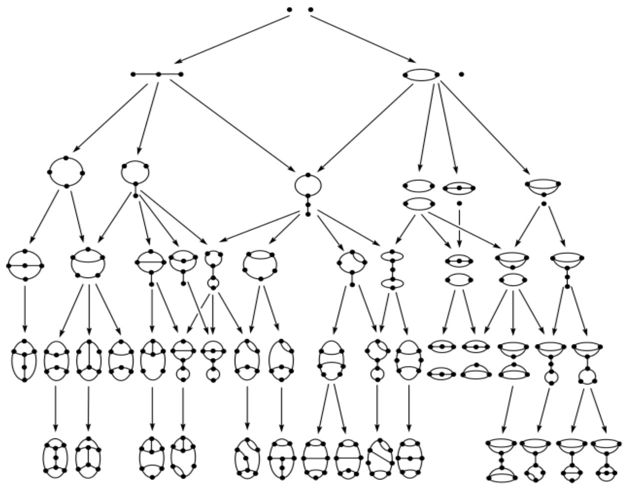


Figure 2: The complete game tree for two-spot Sprouts

| number of spots | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| normal play | L | L | W | W | W | L | L* | L* | W* | W* | W* |
| misère play | W | L | L | L | W* | W* | L* | L* | L* | | |

W denotes that the game is a win for the first player;
L denotes a loss for the first player.
A "*" indicates new results obtained by our program.

See "Computer Analysis of Sprouts" by Applegate, Jacobson, and Sleator http://www.cs.cmu.edu/~sleator/papers/Sprouts.htm

## Application to Boxing Match

The beauty of Nimbers is that they completely capture what you need to know about a game in order to add it to another game. This can speed up game search exponentially.

How would you use this to win in Boxing Match against an opponent who did not know about Nimbers?

(My friends Guy Jacobson and David Applegate used this to cream all the other players in the Boxing Match contest.)

Study Bee

- Sums of games
- Definition of Nimbers
- The MEX operator
- Sum of Games
- The Nimber Theorem
- Applications of the theorem:
  - Boxing Match
  - Breaking Chocolates
  - Sprouts