Great Theoretical Ideas in Computer Science

Ancient Wisdom: Unary and Binary
Lecture 3 (August 31, 2010)

9 stones, numbered 1-9
Two players alternate moves.
Each move a player gets to take a new stone
Any subset of 3 stones adding to 15, wins.

How to play the 9 stone game?

Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu in ancient China

Magic Square: Any 3 in a vertical, horizontal, or diagonal line add up to 15.

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>4</td>
<td>9</td>
<td></td>
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<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>6</td>
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Conversely, any 3 that add to 15 must be on a line.

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TIC-TAC-TOE on a Magic Square Represents The Nine Stone Game

Alternate taking squares 1-9. Get 3 in a row to win.

\[
\begin{array}{ccc}
4 & 9 & 2 \\
3 & 5 & 7 \\
8 & 1 & 6 \\
\end{array}
\]

Basic Idea of this Lecture

Don’t stick with the representation in which you encounter problems!

Always seek the more useful one!

This idea requires a lot of practice

Prehistoric Unary

1
2
3
4

Consider the problem of finding a formula for the sum of the first \( n \) numbers

You already used induction to verify that the answer is \( \frac{1}{2}n(n+1) \)

\[
1 + 2 + 3 + \ldots + n-1 + n = S \\
\frac{n + n-1 + n-2 + \ldots + 2 + 1}{2} = S \\
\frac{n+1}{2} + \frac{n+1}{2} + \frac{n+1}{2} + \ldots + \frac{n+1}{2} + \frac{n+1}{2} = 2S \\
\frac{n(n+1)}{2} = 2S
\]
**n^{th} Triangular Number**
\[ \Delta_n = 1 + 2 + 3 + \ldots + n - 1 + n = \frac{n(n+1)}{2} \]

**n^{th} Square Number**
\[ \square_n = n^2 = \Delta_n + \Delta_{n-1} \]
\[ n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} \]

Breaking a square up in a new way
The sum of the first $n$ odd numbers is $n^2$.

Here is an alternative dot proof of the same sum:

$$\square_n = \Delta_n + \Delta_{n-1} = n^2$$
\( n^{th} \text{ Square Number} \)

\[ \square_n = \Delta_n + \Delta_{n-1} \]

\[ = n^2 \]

\[ \square_n = \Delta_n + \Delta_{n-1} \]

\[ = \text{Sum of first } n \text{ odd numbers} \]

\[ \Delta_n = 1 + 3 + 5 + \ldots + (2n-1) \]

\[ \Delta_{n+1} = 1 + 3 + 5 + \ldots + (2n-1) + (2n+1) \]

\[ \Delta_{n+1} \]

\[ \Delta_n \]

Area of square = \((\Delta_n)^2\)

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Area of square = $(\Delta_n)^2$

Can you find a formula for the sum of the first $n$ squares?

Babylonians needed this sum to compute the number of blocks in their pyramids.

$(\Delta_n)^2 = n^3 + (\Delta_{n-1})^2$

$= n^3 + (n-1)^3 + (\Delta_{n-2})^2$

$= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2$

$= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3$

$(\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3$

$= \frac{n(n+1)/2}{2}$

$= \frac{1}{2} n(n+1)(n+2)$
Rhind Papyrus
Scribe Ahmes was Martin Gardner of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7
\[ \sum_{k=0}^{n} 7^k = \frac{7^{n+1}-1}{7-1} \]

A Frequently Arising Calculation
\[(X-1)(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1}) \]
\[= \frac{X^n - 1}{X - 1} - 1 - X - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1} \]
\[= X^n - 1 \]
\[1 + X^1 + X^2 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1} \]
(when \( X \neq 1 \))

Geometric Series for \( X = 2 \)
\[1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1 \]
\[2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2 \]
\[1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1} \]
(when \( X \neq 1 \))

Geometric Series for \( X = \frac{1}{2} \)
\[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}} = \left(\frac{1}{2}\right)^n - 1 \]
\[\leq 2 \left(1 - \left(\frac{1}{2}\right)^n\right) \]
\[1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1} \]
(when \( X \neq 1 \))
A Similar Sum

\[ a^n + a^{n-1}b^1 + a^{n-2}b^2 + \ldots + a^1b^{n-1} + b^n \]

\[ \Rightarrow a^n \left( \frac{1 + (\frac{b}{a}) + \frac{b^2}{a^2} + \ldots + \frac{b^n}{a^n}}{a - b} \right) = \frac{a^n - a^n}{a - b} \]

A slightly different one

\[ 0.2^0 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n = ? \]

\[ S = 0.2^0 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n \]

\[ -2S = \frac{0.2^1 + 1.2^2 + 2.2^3 + 3.2^4 + \ldots + (n-1)2^n + n.2^{n+1}}{2} \]

\[ -S = \frac{0.2^0 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n}{2^{n+1}} - \frac{n2^{n+1}}{2} \]

\[ -S = 2^{n+1} - 2 - n2^{n+1} = -2^{n+1} (n-1) - 2 \]

\[ S = 2^{n+1} (n-1) + 2 \]

Check Your Work!

\[ 0.2^0 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n = S \]

We’re claiming: \( S = 2^{n+1} (n-1) + 2 \)

What is \( S + (n+1)2^{n+1} \)?

\[ 2^{n+1} (n-1) + 2 + (n+1)2^{n+1} \]

= \( 2^{n+1} (2n) + 2 \)

Also, for \( n=0 \) both are 0.

Bases In Different Cultures

<table>
<thead>
<tr>
<th>Base</th>
<th>Sumerian-Babylonian</th>
<th>Egyptians</th>
<th>Maya</th>
<th>Africans</th>
<th>French</th>
<th>English</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10, 60, 360</td>
<td>3, 7, 10, 60</td>
<td>20</td>
<td>5, 10</td>
<td>10, 20</td>
<td>10, 12, 20</td>
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</table>
BASE X Representation

S = (a_{n-1}, a_{n-2}, ..., a_1, a_0)_X represents the number:
a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + ... + a_0X^0

Largest number representable in base-X with n “digits”

= (X-1)(X-1)(X-1)(X-1)...

= (X^n - 1)

Fundamental Theorem For Binary

Each of the numbers from 0 to 2^n-1 is
uniquely represented by an n-bit
number in binary

k uses \[ \lceil \log_2 k \rceil + 1 \] digits in base 2

= \lceil \log_2 (x^n) \rceil

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^n-1 is
uniquely represented by an n-"digit"
number in base X

k uses \[ \lceil \log_k k \rceil + 1 \] digits in base X

Other Representations: Egyptian Base 3

Conventional Base 3:
Each digit can be 0, 1, or 2
Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1
Example: \((-1 -1 -1)_{EB3} = 9 - 3 - 1 = 5\)
\((-1 -1 -1)_{EB3} = 9x3 + 3x1 + 1x1\)

We can prove a unique representation theorem

Unary is exponentially longer than binary

How could this be Egyptian?
Historically, negative numbers first appear in the
writings of the Hindu mathematician Brahmagupta (628 AD)
Two Case Studies

**Bases and Representation**

**Solving Recurrences**

using a good representation

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**Example**

<table>
<thead>
<tr>
<th>T(1)</th>
<th>T(2)</th>
<th>T(4)</th>
<th>T(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>28</td>
<td>120</td>
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Notice that \( T(n) \) is inductively defined only for positive powers of 2, and undefined on other values

Give a closed-form formula for \( T(n) \)

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**Technique 1**

**Guess Answer, Verify by Induction**

\[
T(1) = 1, \ T(n) = 4 \ T(n/2) + n
\]

Base Case: \( G(1) = 1 \) and \( T(1) = 1 \)

Induction Hypothesis: \( T(x) = G(x) \) for \( x < n \)

Hence: \( T(n/2) = G(n/2) = 2(n/2)^2 - n/2 \)

\[
T(n) = 4 \ T(n/2) + n
\]

\[
= 4 \ G(n/2) + n
\]

\[
= 4 \ [2(n/2)^2 - n/2] + n
\]

\[
= 2n^2 - 2n + n
\]

\[
= 2n^2 - n = G(n)
\]

---

**Technique 2**

**Guess Form, Calculate Coefficients**

\[
T(1) = 1, \ T(n) = 4 \ T(n/2) + n
\]

Guess: \( T(n) = an^2 + bn + c \) for some \( a, b, c \)

Calculate: \( T(1) = 1, \) so \( a + b + c = 1 \)

\[
T(n) = 4 \ T(n/2) + n
\]

\[
an^2 + bn + c = 4 \ [a(n/2)^2 + b(n/2) + c] + n
\]

\[
= an^2 + 2bn + 4c + n
\]

\[
(b+1)n + 3c = 0
\]

Therefore: \( b = -1 \quad c = 0 \quad a = 2 \)

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**Technique 3**

**The Recursion Tree Approach**

\[
T(1) = 1, \ T(n) = 4 \ T(n/2) + n
\]
A slight variation

\[ T(1) = 1, \quad T(n) = 4T(n/2) + n^2 \]

How about this one?

\[ T(1) = 1, \quad T(n) = 3T(n/2) + n \]

… and this one?

\[ T(1) = 1, \quad T(n) = T(n/4) + T(n/2) + n \]

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Unary and Binary
Triangular Numbers
Dot proofs

\[ (1+x+x^2 + \ldots + x^{n-1}) = \frac{(x^n - 1)}{(x-1)} \]

Base-X representations
\[ k \text{ uses } \left\lfloor \log_k n \right\rfloor + 1 = \left\lfloor \log_2 (k+1) \right\rfloor \]

digits in base 2

Here’s What You Need to Know...

Solving Simple Recurrences

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Bhaskara’s “proof” of Pythagoras' theorem

\[ \sqrt{1 + \sqrt{1 + \sqrt{1 \ldots}}} = \varphi \]

\[ \frac{1}{\varphi} = \varphi \left( 1 + \varphi^2 \ldots \right) \]