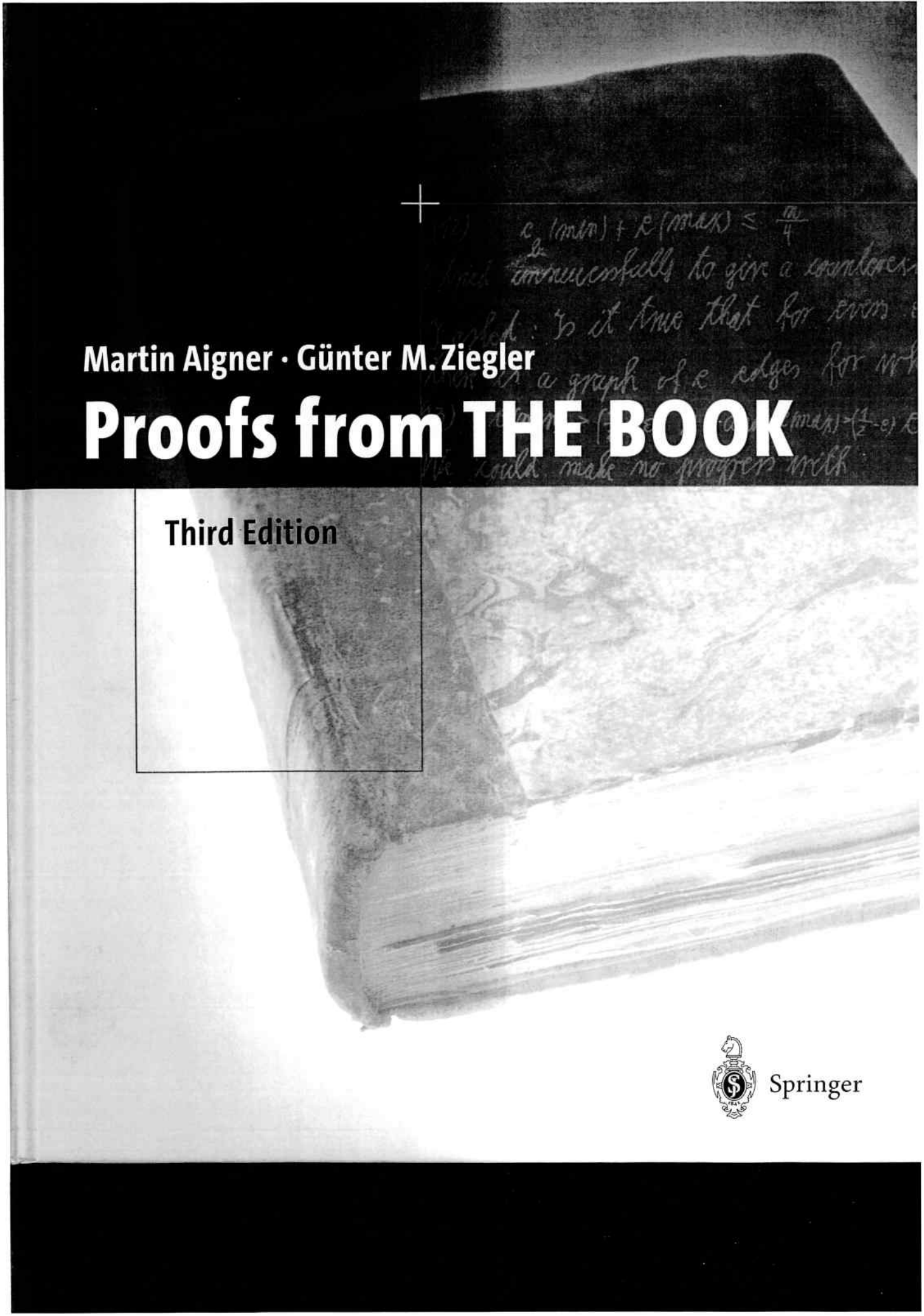

Martin Aigner · Günter M. Ziegler

Proofs from THE BOOK

Third Edition

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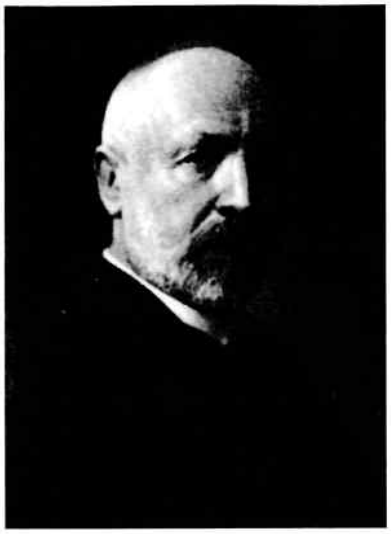
Sets, functions, and the continuum hypothesis

Chapter 16

Set theory, founded by Georg Cantor in the second half of the 19th century, has profoundly transformed mathematics. Modern day mathematics is unthinkable without the concept of a set, or as David Hilbert put it: "Nobody will drive us from the paradise (of set theory) that Cantor has created for us."



Georg Cantor

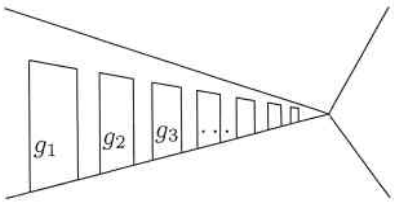
One of Cantor's basic concepts was the notion of the *size* or *cardinality* of a set M, denoted by $|M|$. For finite sets, this presents no difficulties: we just count the number of elements and say that M is an n-set or has size n, if M contains precisely n elements. Thus two finite sets M and N have equal size, $|M| = |N|$, if they contain the same number of elements.

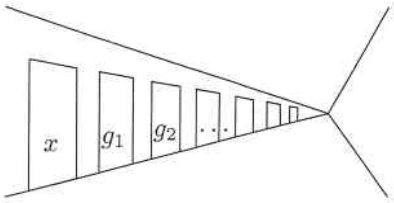
To carry this notion of *equal* size over to infinite sets, we use the following suggestive thought experiment for finite sets. Suppose a number of people board a bus. When will we say that the number of people is the same as the number of available seats? Simple enough, we let all people sit down. If everyone finds a seat, and no seat remains empty, then and only then do the two sets (of the people and of the seats) agree in number. In other words, the two sizes are the same if there is a *bijection* of one set onto the other.

This is then our definition: Two arbitrary sets M and N (finite or infinite) are said to be of *equal size* or *cardinality*, if and only if there exists a bijection from M onto N. Clearly, this notion of equal size is an equivalence relation, and we can thus associate a number, called *cardinal number*, to every class of equal-sized sets. For example, we obtain for finite sets the cardinal numbers $0, 1, 2, \ldots, n, \ldots$ where n stands for the class of n-sets, and, in particular, 0 for the *empty set* $\varnothing$. We further observe the obvious fact that a proper subset of a finite set M invariably has smaller size than M.

The theory becomes very interesting (and highly non-intuitive) when we turn to infinite sets. Consider the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers. We call a set M *countable* if it can be put in one-to-one correspondence with $\mathbb{N}$. In other words, M is countable if we can list the elements of M as $m_1, m_2, m_3, \ldots$. But now a strange phenomenon occurs. Suppose we add to $\mathbb{N}$ a new element x. Then $\mathbb{N} \cup \{x\}$ is still countable, and hence has equal size with $\mathbb{N}$!

This fact is delightfully illustrated by "Hilbert's hotel." Suppose a hotel has countably many rooms, numbered $1, 2, 3, \ldots$ with guest g_i occupying room i; so the hotel is fully booked. Now a new guest x arrives asking for a room, whereupon the hotel manager tells him: Sorry, all rooms are taken. No problem, says the new arrival, just move guest g_1 to room 2, g_2 to room 3, g_3 to room 4, and so on, and I will then take room 1. To the



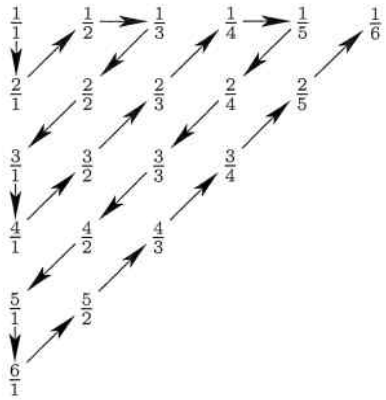


manager's surprise (he is not a mathematician) this works; he can still put up all guests plus the new arrival x!

Now it is clear that he can also put up another guest y, and another one z, and so on. In particular, we note that, in contrast to finite sets, it may well happen that a proper subset of an *infinite* set M has the same size as M. In fact, as we will see, this is a characterization of infinity: A set is infinite if and only if it has the same size as some proper subset.

Let us leave Hilbert's hotel and look at our familiar number sets. The set $\mathbb{Z}$ of integers is again countable, since we may enumerate $\mathbb{Z}$ in the form $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. It may come more as a surprise that the rationals can be enumerated in a similar way.

Theorem 1. *The set $\mathbb{Q}$ of rational numbers is countable.*



■ **Proof.** By listing the set $\mathbb{Q}^+$ of positive rationals as suggested in the figure in the margin, but leaving out numbers already encountered, we see that $\mathbb{Q}^+$ is countable, and hence so is $\mathbb{Q}$ by listing 0 at the beginning and $-\frac{p}{q}$ right after $\frac{p}{q}$. With this listing

$$\mathbb{Q} = \{0, 1, -1, 2, -2, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{3}, 3, -3, 4, -4, \tfrac{3}{2}, -\tfrac{3}{2}, \ldots\}. \qquad \square$$

Another way to interpret the figure is the following statement:

The union of countably many countable sets M_n is again countable.

Indeed, set $M_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$ and list

$$\bigcup_{n=1}^{\infty} M_n = \{a_{11}, a_{21}, a_{12}, a_{13}, a_{22}, a_{31}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots\}$$

precisely as before.

Let us contemplate Cantor's enumeration of the positive rationals a bit more. Looking at the figure we obtained the sequence

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{2}, \tfrac{3}{1}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{2}{4}, \tfrac{3}{3}, \tfrac{4}{2}, \tfrac{5}{1}, \ldots$$

and then had to strike out the duplicates such as $\frac{2}{2} = \frac{1}{1}$ or $\frac{2}{4} = \frac{1}{2}$.

But there is a listing that is even more elegant and systematic, and which contains no duplicates — found only quite recently by Neil Calkin and Herbert Wilf. Their new list starts as follows:

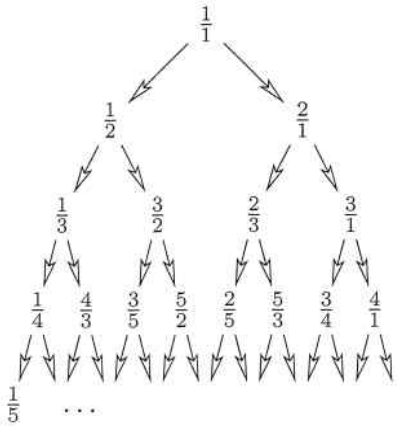
$$\tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{1}, \tfrac{1}{3}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{3}{1}, \tfrac{1}{4}, \tfrac{4}{3}, \tfrac{3}{5}, \tfrac{5}{2}, \tfrac{2}{5}, \tfrac{5}{3}, \tfrac{3}{4}, \tfrac{4}{1}, \ldots .$$

Here the denominator of the n-th rational number equals the numerator of the $(n+1)$-st number. In other words, the n-th fraction is $b(n)/b(n+1)$, where $\big(b(n)\big)_{n\geq 0}$ is a sequence that starts with

$$(1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, \ldots).$$

This sequence has first been studied by a German mathematician, Moritz Abraham Stern, in a paper from 1858, and is has become known as "Stern's diatomic series."

How do we obtain this sequence, and hence the Calkin-Wilf listing of the positive fractions? Consider the infinite binary tree in the margin. We immediately note its recursive rule:



- $\frac{1}{1}$ is on top of the tree, and
- every node $\frac{i}{j}$ has two sons: the left son is $\frac{i}{i+j}$ and the right son is $\frac{i+j}{j}$.

We can easily check the following four properties:

(1) All fractions in the tree are reduced, that is, if $\frac{r}{s}$ appears in the tree, then r and s are relatively prime.

This holds for the top $\frac{1}{1}$, and then we use induction downward. If r and s are relatively prime, then so are r and $r+s$, as well as s and $r+s$.

(2) Every reduced fraction $\frac{r}{s} > 0$ appears in the tree.

We use induction on the sum $r+s$. The smallest value is $r+s=2$, that is $\frac{r}{s}=\frac{1}{1}$, and this appears at the top. If $r>s$, then $\frac{r-s}{s}$ appears in the tree by induction, and so we get $\frac{r}{s}$ as its right son. Similarly, if $r<s$, then $\frac{r}{s-r}$ appears, which has $\frac{r}{s}$ as its left son.

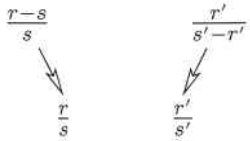
(3) Every reduced fraction appears exactly once.

The argument is similar. If $\frac{r}{s}$ appears more than once, then $r \neq s$, since any node in the tree except the top is of the form $\frac{i}{i+j} < 1$ or $\frac{i+j}{j} > 1$. But if $r>s$ or $r<s$, then we argue by induction as before.

Every positive rational appears therefore exactly once in our tree, and we may write them down listing the numbers level-by-level from left to right. This yields precisely the initial segment shown above.

(4) The denominator of the n-th fraction in our list equals the numerator of the $(n+1)$-st.

This is certainly true for $n=0$, or when the n-th fraction is a left son. Suppose the n-th number $\frac{r}{s}$ is a right son. If $\frac{r}{s}$ is at the right boundary, then $s=1$, and the successor lies at the left boundary and has numerator 1. Finally, if $\frac{r}{s}$ is in the interior, and $\frac{r'}{s'}$ is the next fraction in our sequence, then $\frac{r}{s}$ is the right son of $\frac{r-s}{s}$, $\frac{r'}{s'}$ is the left son of $\frac{r'}{s'-r'}$, and by induction the denominator of $\frac{r-s}{s}$ is the numerator of $\frac{r'}{s'-r'}$, so we get $s=r'$.



Well, this is nice, but there is even more to come. There are two natural questions:

- Does the sequence $\big(b(n)\big)_{n\geq 0}$ have a "meaning"? That is, does $b(n)$ count anything simple?
- Given $\frac{r}{s}$, is there an easy way to determine the successor in the listing?

To answer the first question, we work out that the node $b(n)/b(n+1)$ has the two sons $b(2n+1)/b(2n+2)$ and $b(2n+2)/b(2n+3)$. By the set-up of the tree we obtain the recursions

$$b(2n+1) = b(n) \quad \text{and} \quad b(2n+2) = b(n) + b(n+1). \tag{1}$$

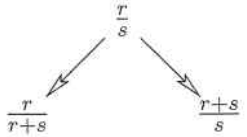
With $b(0) = 1$ the sequence $(b(n))_{n\geq 0}$ is completely determined by (1).

So, is there a "nice" "known" sequence which obeys the same recursion? Yes, there is. We know that any number n can be uniquely written as a sum of distinct powers of 2 — this is the usual binary representation of n. A *hyper-binary* representation of n is a representation of n a sum of powers of 2, where every power 2^k appears at most *twice*. Let $h(n)$ be the number of such representations for n. You are invited to check that the sequence $h(n)$ obeys the recursion (1), and this gives $b(n) = h(n)$ for all n.

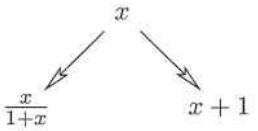
For example, $h(6) = 3$, with the hyper-binary representations
$6 = 4 + 2$
$6 = 4 + 1 + 1$
$6 = 2 + 2 + 1 + 1$.

Incidentally, we have proved a surprising fact: Let $\frac{r}{s}$ be a reduced fraction, there exists precisely one integer n with $r = h(n)$ and $s = h(n+1)$.

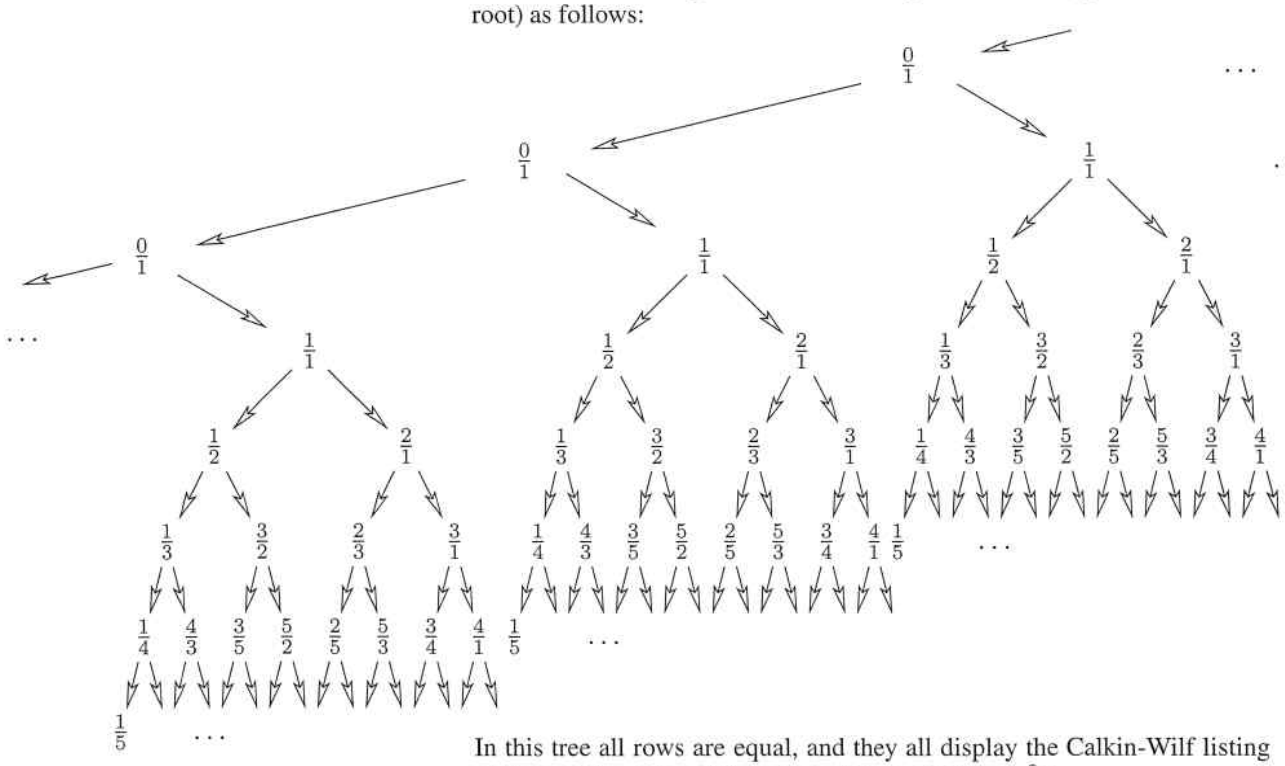
Let us look at the second question. We have in our tree



that is, with $x := \frac{r}{s}$,

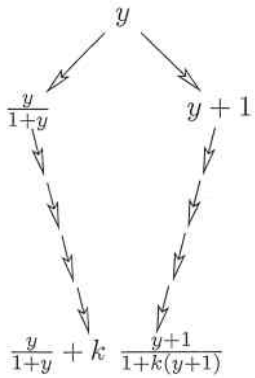


We now use this to generate an even larger infinite binary tree (without a root) as follows:



In this tree all rows are equal, and they all display the Calkin-Wilf listing of the positive rationals (starting with an additional $\frac{0}{1}$).

So how does one get from one rational to the next? To answer this, we first record that for every rational x its right son is $x+1$, the right grand-son is $x+2$, so the k-fold right son is $x+k$. Similarly, the left son of x is $\frac{x}{1+x}$, whose left son is $\frac{x}{1+2x}$, and so on: The k-fold left son of x is $\frac{x}{1+kx}$.



Now to find how to get from $\frac{r}{s} = x$ to the "next" rational $f(x)$ in the listing, we have to analyze the situation depicted in the margin. In fact, if we consider any nonnegative rational number x in our infinite binary tree, then it is the k-fold right son of the left son of some rational $y \geq 0$ (for some $k \geq 0$), while $f(x)$ is given as the k-fold left son of the right son of the same y. Thus with the formulas for k-fold left sons and k-fold right sons, we get

$$x = \frac{y}{1+y} + k,$$

as claimed in the figure in the margin. Here $k = \lfloor x \rfloor$ is the integral part of x, while $\frac{y}{1+y} = \{x\}$ is the fractional part. And from this we obtain

$$f(x) = \frac{y+1}{1+k(y+1)} = \frac{1}{\frac{1}{y+1}+k} = \frac{1}{k+1-\frac{y}{y+1}} = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}.$$

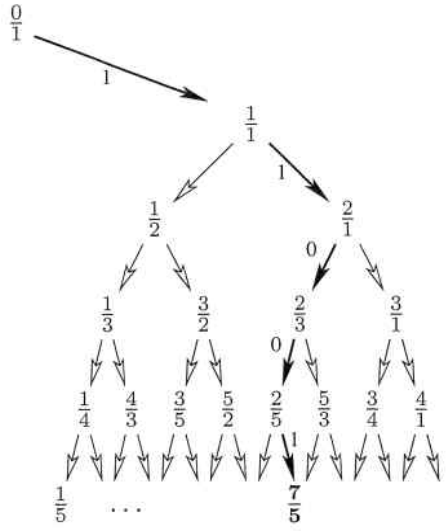
Thus we have obtained a beautiful formula for the successor $f(x)$ of x, found very recently by Moshe Newman:

> *The function*
>
> $$x \longmapsto f(x) = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}$$
>
> *generates the Calkin-Wilf sequence*
>
> $$\tfrac{1}{1} \mapsto \tfrac{1}{2} \mapsto \tfrac{2}{1} \mapsto \tfrac{1}{3} \mapsto \tfrac{3}{2} \mapsto \tfrac{2}{3} \mapsto \tfrac{3}{1} \mapsto \tfrac{1}{4} \mapsto \tfrac{4}{3} \mapsto \dots$$
>
> *which contains every positive rational number exactly once.*

The Calkin-Wilf-Newman way to enumerate the positive rationals has a number of additional remarkable properties. For example, one may ask for a fast way to determine the n-th fraction in the sequence, say for $n = 10^6$. Here it is:

> To find the n-th fraction in the Calkin-Wilf sequence, express n as a binary number $n = (b_k b_{k-1} \dots b_1 b_0)_2$, and then follow the path in the Calkin-Wilf tree that is determined by its digits, starting at $\frac{s}{t} = \frac{0}{1}$. Here $b_i = 1$ means "take the right son," that is, "add the denominator to the numerator," while $b_i = 0$ means "take the left son," that is, "add the numerator to the denominator."

The figure in the margin shows the resulting path for $n = 25 = (11001)_2$: So the 25th number in the Calkin-Wilf sequence is $\frac{7}{5}$. The reader could easily work out a similar scheme that computes for a given fraction $\frac{s}{t}$ (the binary representation of) its position n in the Calkin-Wilf sequence.



Let us move on to the real numbers $\mathbb{R}$. Are they still countable? No, they are not, and the means by which this is shown — Cantor's *diagonalization method* — is not only of fundamental importance for all of set theory, but certainly belongs into The Book as a rare stroke of genius.

Theorem 2. *The set $\mathbb{R}$ of real numbers is* ***not*** *countable.*

■ **Proof.** Any subset N of a countable set $M = \{m_1, m_2, m_3, \ldots\}$ is *at most countable* (that is, finite or countable). In fact, just list the elements of N as they appear in M. Accordingly, if we can find a subset of $\mathbb{R}$ which is not countable, then a fortiori $\mathbb{R}$ cannot be countable. The subset M of $\mathbb{R}$ we want to look at is the interval $(0, 1]$ of all positive real numbers r with $0 < r \leq 1$. Suppose, to the contrary, that M is countable, and let $M = \{r_1, r_2, r_3, \ldots\}$ be a listing of M. We write r_n as its unique *infinite* decimal expansion without an infinite sequence of zeros at the end:

$$r_n = 0.a_{n1}a_{n2}a_{n3}\ldots$$

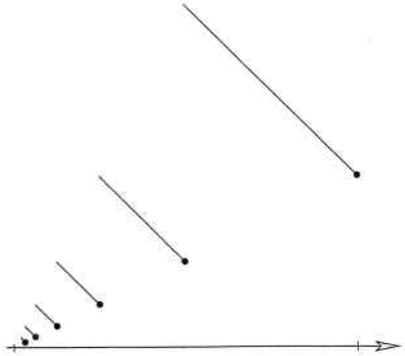
where $a_{ni} \in \{0, 1, \ldots, 9\}$ for all n and i. For example, $0.7 = 0.6999\ldots$ Consider now the doubly infinite array

$$\begin{array}{rcl} r_1 & = & 0.a_{11}a_{12}a_{13}\ldots \\ r_2 & = & 0.a_{21}a_{22}a_{23}\ldots \\ \vdots & & \vdots \\ r_n & = & 0.a_{n1}a_{n2}a_{n3}\ldots \\ \vdots & & \vdots \end{array}$$

For every n, choose $b_n \in \{1, \ldots, 8\}$ different from a_{nn}; clearly this can be done. Then $b = 0.b_1b_2b_3\ldots b_n\ldots$ is a real number in our set M and hence must have an index, say $b = r_k$. But this cannot be, since b_k is different from a_{kk}. And this is the whole proof! □

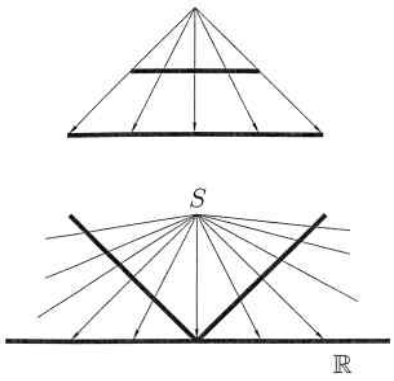
Let us stay with the real numbers for a moment. We note that all four types of intervals $(0, 1), (0, 1], [0, 1)$ and $[0, 1]$ have the same size. As an example, we verify that $(0, 1]$ and $(0, 1)$ have equal cardinality. The map $f : (0, 1] \longrightarrow (0, 1), x \longmapsto y$ defined by

$$y := \begin{cases} \frac{3}{2} - x & \text{for } \frac{1}{2} < x \leq 1, \\ \frac{3}{4} - x & \text{for } \frac{1}{4} < x \leq \frac{1}{2}, \\ \frac{3}{8} - x & \text{for } \frac{1}{8} < x \leq \frac{1}{4}, \\ \quad \vdots \end{cases}$$



A bijective $f : (0, 1] \longrightarrow (0, 1)$

does the job. Indeed, the map is bijective, since the range of y in the first line is $\frac{1}{2} \leq y < 1$, in the second line $\frac{1}{4} \leq y < \frac{1}{2}$, in the third line $\frac{1}{8} \leq y < \frac{1}{4}$, and so on.



Next we find that *any* two intervals (of finite length > 0) have equal size by considering the central projection as in the figure. Even more is true: Every interval (of length > 0) has the same size as the whole real line $\mathbb{R}$. To see this, look at the bent open interval $(0, 1)$ and project it onto $\mathbb{R}$ from the center S.

So, in conclusion, any open, half-open, closed (finite or infinite) interval of length > 0 has the same size, and we denote this size by c, where c stands for *continuum* (a name sometimes used for the interval [0,1]).

That finite and infinite intervals have the same size may come expected on second thought, but here is a fact that is downright counter-intuitive.

Theorem 3. *The set $\mathbb{R}^2$ of all ordered pairs of real numbers (that is, the real plane) has the same size as $\mathbb{R}$.*

■ **Proof.** To see this, it suffices to prove that the set of all pairs (x, y), $0 < x, y \leq 1$, can be mapped bijectively onto $(0, 1]$. The proof is again from The Book. Consider the pair (x, y) and write x, y in their unique non-terminating decimal expansion as in the following example:

$$\begin{array}{rcrrrrrr} x & = & 0.3 & 01 & 2 & 007 & 08 & \ldots \\ y & = & 0.009 & 2 & 05 & 1 & 0008 & \ldots \end{array}$$

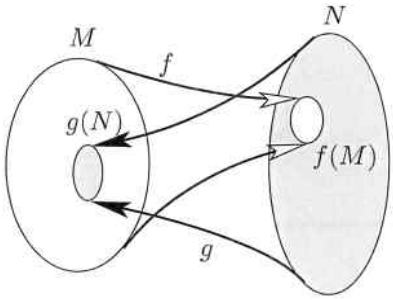
Note that we have separated the digits of x and y into groups by always going to the next nonzero digit, inclusive. Now we associate to (x, y) the number $z \in (0, 1]$ by writing down the first x-group, after that the first y-group, then the second x-group, and so on. Thus, in our example, we obtain

$$z = 0.3\,009\,01\,2\,2\,05\,007\,1\,08\,0008\ldots$$

Since neither x nor y exhibits only zeros from a certain point on, we find that the expression for z is again a non-terminating decimal expansion. Conversely, from the expansion of z we can immediately read off the preimage (x, y), and the map is bijective — end of proof. □

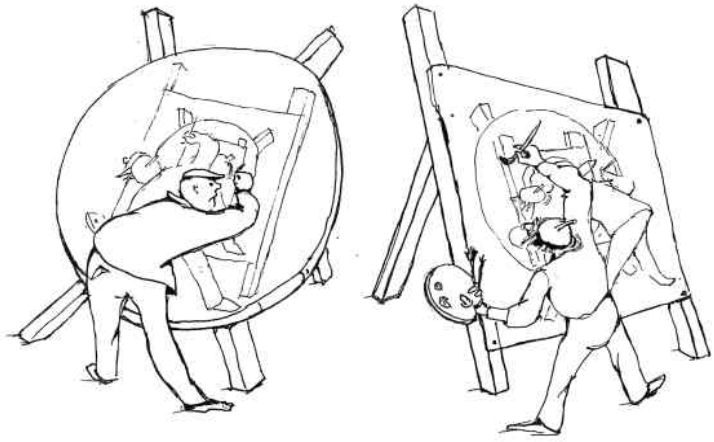
As $(x, y) \longmapsto x + iy$ is a bijection from $\mathbb{R}^2$ onto the complex numbers $\mathbb{C}$, we conclude that $|\mathbb{C}| = |\mathbb{R}| = c$. Why is the result $|\mathbb{R}^2| = |\mathbb{R}|$ so unexpected? Because it goes against our intuition of *dimension*. It says that the 2-dimensional plane $\mathbb{R}^2$ (and, in general, by induction, the n-dimensional space $\mathbb{R}^n$) can be mapped bijectively onto the 1-dimensional line $\mathbb{R}$. Thus dimension is not generally preserved by bijective maps. If, however, we require the map and its inverse to be continuous, then the dimension is preserved, as was first shown by Luitzen Brouwer.

Let us go a little further. So far, we have the notion of equal size. When will we say that M is at most as large as N? Mappings provide again the key. We say that the cardinal number $\mathbf{m}$ is *less than or equal to* $\mathbf{n}$, if for sets M and N with $|M| = \mathbf{m}$, $|N| = \mathbf{n}$, there exists an *injection* from M into N. Clearly, the relation $\mathbf{m} \leq \mathbf{n}$ is independent of the representative sets M and N chosen. For finite sets this corresponds again to our intuitive notion: An m-set is at most as large as an n-set if and only if $m \leq n$.



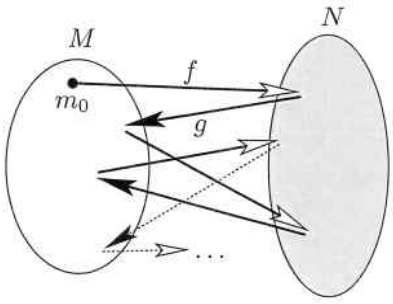
Now we are faced with a basic problem. We would certainly like to have that the usual laws concerning inequalities also hold for cardinal numbers. But is this true for infinite cardinals? In particular, is it true that $\mathbf{m} \leq \mathbf{n}$, $\mathbf{n} \leq \mathbf{m}$ imply $\mathbf{m} = \mathbf{n}$? This is not at all obvious: We are given infinite sets M and N as well as maps $f : M \longrightarrow N$ and $g : N \longrightarrow M$ that are injective but not necessarily surjective. This suggests to construct a bijection by relating some elements $m \in M$ to $f(m) \in N$, and some elements $n \in N$ to $g(n) \in M$. But it is not clear whether the many possible choices can be made to "fit together."

The affirmative answer is provided by the famous Schröder-Bernstein theorem, which Cantor announced in 1883. The first proofs were given by Friedrich Schröder and Felix Bernstein quite some time later. The following proof appears in a little book by one of the twentieth century giants of set theory, Paul Cohen, who is famous for resolving the continuum hypothesis (which we will discuss below).



"Schröder and Bernstein painting"

Theorem 4. *If each of two sets M and N can be mapped injectively into the other, then there is a bijection from M to N, that is, $|M| = |N|$.*



■ **Proof.** We may certainly assume that M and N are disjoint — if not, then we just replace N by a new copy.

Now f and g map back and forth between the elements of M and those of N. One way to bring this potentially confusing situation into perfect clarity and order is to align $M \cup N$ into chains of elements: Take an arbitrary element $m_0 \in M$, say, and from this generate a chain of elements by applying f, then g, then f again, then g, and so on. The chain may close up (this is Case 1) if we reach m_0 again in this process, or it may continue with distinct elements indefinitely. (The first "duplicate" in the chain cannot be an element different from m_0, by injectivity.)

If the chain continues indefinitely, then we try to follow it backwards: From m_0 to $g^{-1}(m_0)$ if m_0 is in the image of g, then to $f^{-1}(g^{-1}(m_0))$ if $g^{-1}(m_0)$ is in the image of f, and so on. Three more cases may arise here: The process of following the chain backwards may go on indefinitely

(Case 2), it may stop in an element of M that does not lie in the image of g (Case 3), or it may stop in an element of N that does not lie in the image of f (Case 4).

Thus $M \cup N$ splits perfectly into four types of chains, whose elements we may label in such a way that a bijection is simply given by putting $F : m_i \longmapsto n_i$. We verify this in the four cases separately:

Case 1. Finite cycles on $2k+2$ distinct elements ($k \geq 0$)

$$m_0 \overset{f}{\twoheadrightarrow} n_0 \overset{g}{\rightarrow} m_1 \overset{f}{\twoheadrightarrow} \cdots \quad m_k \overset{f}{\twoheadrightarrow} n_k \quad (g: n_k \to m_0)$$

Case 2. Two-way infinite chains of distinct elements

$$\cdots \rightarrow m_0 \overset{f}{\twoheadrightarrow} n_0 \overset{g}{\rightarrow} m_1 \overset{f}{\twoheadrightarrow} n_1 \overset{g}{\rightarrow} m_2 \overset{f}{\twoheadrightarrow} \cdots$$

Case 3. The one-way infinite chains of distinct elements that start at the elements $m_0 \in M \backslash g(N)$

$$m_0 \overset{f}{\twoheadrightarrow} n_0 \overset{g}{\rightarrow} m_1 \overset{f}{\twoheadrightarrow} n_1 \overset{g}{\rightarrow} m_2 \overset{f}{\twoheadrightarrow} \cdots$$

Case 4. The one-way infinite chains of distinct elements that start at the elements $n_0 \in N \backslash f(M)$

$$n_0 \overset{g}{\rightarrow} m_0 \overset{f}{\twoheadrightarrow} n_1 \overset{g}{\rightarrow} m_1 \overset{f}{\twoheadrightarrow} \cdots \quad \square$$

What about the other relations governing inequalities? As usual, we set $\mathfrak{m} < \mathfrak{n}$ if $\mathfrak{m} \leq \mathfrak{n}$, but $\mathfrak{m} \neq \mathfrak{n}$. We have just seen that for any two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ at most one of the three possibilities

$$\mathfrak{m} < \mathfrak{n}, \ \mathfrak{m} = \mathfrak{n}, \ \mathfrak{m} > \mathfrak{n}$$

holds, and it follows from the theory of cardinal numbers that, in fact, precisely one relation is true. (See the appendix to this chapter, Proposition 2.)

Furthermore, the Schröder-Bernstein Theorem tells us that the relation $<$ is transitive, that is, $\mathfrak{m} < \mathfrak{n}$ and $\mathfrak{n} < \mathfrak{p}$ imply $\mathfrak{m} < \mathfrak{p}$. Thus the cardinalities are arranged in linear order starting with the finite cardinals $0, 1, 2, 3, \ldots$. Invoking the usual Zermelo-Fraenkel axiom system (in particular, the axiom of choice) we easily find that any infinite set M contains a countable subset. In fact, M contains an element, say m_1. The set $M \setminus \{m_1\}$ is not empty (since it is infinite) and hence contains an element m_2. Considering $M \setminus \{m_1, m_2\}$ we infer the existence of m_3, and so on. So, the size of a countable set is the *smallest infinite* cardinal, usually denoted by $\aleph_0$ (pronounced "aleph zero").

"The smallest infinite cardinal"