Dating Scenario

There are \( n \) boys and \( n \) girls
Each girl has her own ranked preference list of all the boys
Each boy has his own ranked preference list of the girls
The lists have no ties

Question: How do we pair them off?
More Than One Notion of What Constitutes A “Good” Pairing

Maximizing total satisfaction
Hong Kong and to an extent the USA

Maximizing the minimum satisfaction
Western Europe

Minimizing maximum difference in mate ranks
Sweden

Maximizing people who get their first choice
Barbie and Ken Land

We will ignore the issue of what is “best”!

Rogue Couples

Suppose we pair off all the boys and girls
Now suppose that some boy and some girl prefer each other to the people to whom they are paired
They will be called a rogue couple

Why be with them when we can be with each other?

What use is fairness, if it is not stable?

Any list of criteria for a good pairing must include stability. (A pairing is doomed if it contains a rogue couple.)

Stable Pairings

A pairing of boys and girls is called stable if it contains no rogue couples
A pairing of boys and girls is called stable if it contains no rogue couples.

The study of stability will be the subject of the entire lecture.

We will:
- Analyze various mathematical properties of an algorithm that looks a lot like 1950’s dating
- Discover the naked mathematical truth about which sex has the romantic edge
- Learn how the world’s largest, most successful dating service operates

Given a set of preference lists, how do we find a stable pairing?

Wait! We don’t even know that such a pairing always exists!

Does every set of preference lists have a stable pairing?

Idea: Allow the pairs to keep breaking up and reforming until they become stable.

Can you argue that the couples will not continue breaking up and reforming forever?
An Instructive Variant: Bisexual Dating

 Insight

Any proof that heterosexual couples do not break up and re-form forever must contain a step that fails in the bisexual case.

If you have a proof idea that works equally well in the hetero and bisexual versions, then your idea is not adequate to show the couples eventually stop.

The Traditional Marriage Algorithm

For each day that some boy gets a “No” do:

Morning
- Each girl stands on her balcony
- Each boy proposes to the best girl whom he has not yet crossed off

Afternoon (for girls with at least one suitor)
- To today’s best: “Maybe, return tomorrow”
- To any others: “No, I will never marry you”

Evening
- Any rejected boy crosses the girl off his list
If no boys get a “No”, each girl marries boy to whom she just said “maybe”

Does Traditional Marriage Algorithm always produce a stable pairing?

Wait! There is a more primary question!
Does TMA Always Terminate?

It might encounter a situation where algorithm does not specify what to do next (e.g. “core dump error”)

It might keep on going for an infinite number of days

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Improvement Lemma:
If a girl has a boy on a string, then she will always have someone at least as good on a string (or for a husband)

She would only let go of him in order to “maybe” someone better

She would only let go of that guy for someone even better

She would only let go of that guy for someone even better

AND SO ON...

Corollary: Each girl will marry her absolute favorite of the boys who visit her during the TMA

Lemma: No boy can be rejected by all the girls

Proof (by contradiction):
Suppose boy b is rejected by all the girls
At that point:
Each girl must have a suitor other than b
(By Improvement Lemma, once a girl has a suitor she will always have at least one)

The n girls have n suitors, and b is not among them. Thus, there are at least n+1 boys
Theorem: The TMA always terminates in at most \( n^2 \) days

A “master list” of all \( n \) of the boys lists starts with a total of \( n \times n = n^2 \) girls on it.

Each day that at least one boy gets a “No”, so at least one girl gets crossed off the master list.

Therefore, the number of days is bounded by the original size of the master list.

Great! We know that TMA will terminate and produce a pairing

But is it stable?

Theorem: The pairing \( T \) produced by TMA is stable

Let \( b \) and \( g \) be any couple in \( T \), and \( b \) prefers \( g' \) to \( g \). We claim that \( g' \) prefers her husband to \( b \).

During TMA, \( b \) proposed to \( g' \) before he proposed to \( g \).

Hence, at some point \( g' \) rejected \( b \) for someone she preferred to \( b \).

By the Improvement lemma, the person she married was also preferable to \( b \).

Thus, every boy will be rejected by any girl he prefers to his wife.

\( \Rightarrow \) \( T \) is stable.

Opinion Poll

Forget TMA for a Moment…

How should we define “the optimal girl for boy \( b \)”?

Flawed Attempt: “The girl at the top of \( b \)’s list”
**The Optimal Girl**

A boy’s optimal girl is the highest ranked girl for whom there is some stable pairing in which the boy gets her.

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA.

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**The Pessimal Girl**

A boy’s pessimal girl is the lowest ranked girl for whom there is some stable pairing in which the boy gets her.

She is the worst girl he can conceivably get in a stable world.

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**Dating Heaven and Hell**

A pairing is male-optimal if every boy gets his optimal mate. This is the best of all possible worlds for every boy simultaneously.

A pairing is male-pessimal if every boy gets his pessimal mate. This is the worst of all possible worlds for every boy simultaneously.

BTW, is a male-optimal pairing stable?
How about a male-pessimal one?

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**Dating Heaven and Hell**

A pairing is female-optimal if every girl gets her optimal mate. This is the best of all possible stable worlds for every girl simultaneously.

A pairing is female-pessimal if every girl gets her pessimal mate. This is the worst of all possible stable worlds for every girl simultaneously.

Not clear if either exists!
We’ll show that both exist…

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**The Naked Mathematical Truth!**

The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing.
Theorem: TMA produces a male-optimal pairing

Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA.

Let $t$ be the earliest time at which this happened.

At time $t$, boy $b$ got rejected by his optimal girl $g$ because she said “maybe” to a preferred $b^*$.

By the definition of $t$, $b^*$ had not yet been rejected by his optimal girl.

Therefore, $b^*$ likes $g$ at least as much as his optimal.
Definitions of:
• Stable Pairing
• Traditional Marriage Algorithm
• optimal and pessimal mates

Proof that:
• TMA Produces a Stable Pairing
• TMA Produces a Male-Optimal, Female-Pessimal Pairing

Here’s What You Need to Know…