Leonardo Fibonacci
In 1202, Fibonacci proposed a problem about the growth of rabbit populations

A rabbit lives forever
The population starts as single newborn pair
Every month, each productive pair begets a new pair which will become productive after 2 months old

\[ F_n = \# \text{ of rabbit pairs at the beginning of the } n^{\text{th}} \text{ month} \]

<table>
<thead>
<tr>
<th>month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>rabbits</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
</tr>
</tbody>
</table>

Fibonacci Numbers
Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Inductive Rule:
For \( n > 3 \), Fib(n) = Fib(n-1) + Fib(n-2)

Sequences That Sum To \( n \)
Let \( f_{n+1} \) be the number of different sequences of 1’s and 2’s that sum to \( n \).

\[ f_1 = 1 \quad 0 = \text{the empty sum} \]
\[ f_2 = 1 \quad 1 = 1 \]
\[ f_3 = 2 \quad 2 = 1 + 1 \]
\[ 2 \]
Sequences That Sum To n
Let $f_{n+1}$ be the number of different sequences of 1's and 2's that sum to $n$.

$$
4 = 2 + 2 \\
= 2 + 1 + 1 \\
1 + 2 + 1 \\
1 + 1 + 2 \\
1 + 1 + 1 + 1
$$

Fibonacci Numbers Again
Let $f_{n+1}$ be the number of different sequences of 1's and 2's that sum to $n$.

$$
f_{n+1} = f_n + f_{n-1}
$$

$f_1 = 1$  \quad $f_2 = 1$

Visual Representation: Tiling
Let $f_{n+1}$ be the number of different ways to tile a $1 \times n$ strip with squares and dominoes.

$f_n$ tilings that start with a square.

$f_{n-1}$ tilings that start with a domino.

$f_{n+1}$ is number of ways to tile length $n$. 

1 way to tile a strip of length 0

1 way to tile a strip of length 1:

2 ways to tile a strip of length 2:
Fibonacci Identities

Some examples:

\[ F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1} \]

\[ F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n \]

\[ (F_n)^2 = F_{n-1} F_{n+1} + (-1)^n \]

\[ (F_n)^2 = F_{n-1} F_{n+1} + (-1)^n \]

\[ (F_n)^2 = F_{n-1} F_{n+1} + (-1)^n \]

Draw a vertical “fault line” at the rightmost position (<n) possible without cutting any dominoes

Swap the tails at the fault line to map to a tiling of 2 (n-1)’s to a tiling of an n-2 and an n.
(F_n)^2 = F_{n-1}F_{n+1} + (-1)^{n-1}

n even

n odd

Counting Petals
5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)
8 petals: delphiniums
13 petals: ragwort, corn marigold, cineraria, some daisies
21 petals: aster, black-eyed susan, chicory
34 petals: plantain, pyrethrum
55, 89 petals: michaelmas daisies, the asteraceae family.

Definition of ϕ (Euclid)
Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

ϕ = \frac{AC}{AB} = \frac{AB}{BC}

ϕ^2 = \frac{AC}{BC}

ϕ^2 - ϕ - 1 = 0

ϕ = \frac{1 + \sqrt{5}}{2}

Sneezwort (Achilleaptarmica)
Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

The Fibonacci Quarterly

ϕ^2 - ϕ - 1 = 0

ϕ = \frac{1 + \sqrt{5}}{2}
Golden ratio supposed to arise in...

Parthenon, Athens (400 B.C.)

The great pyramid at Gizeh

Ratio of a person’s height to the height of his/her navel

Mostly circumstantial evidence...

Expanding Recursively

Continued Fraction Representation

A (Simple) Continued Fraction Is Any Expression Of The Form:

\[ \frac{a}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \ddots}}}} \]

where \( a, b, c, \ldots \) are whole numbers.
A Continued Fraction can have a finite or infinite number of terms.

\[ a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \ldots}}}}} \]

We also denote this fraction by \([a,b,c,d,e,f,\ldots]\)

A Finite Continued Fraction

\[ 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}} \]

Denoted by \([2,3,4,2,0,0,0,\ldots]\)

An Infinite Continued Fraction

\[ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}} \]

Denoted by \([1,2,2,2,\ldots]\)

Recursively Defined Form For CF

\[ \text{CF} = \text{whole number, or} = \text{whole number} + \frac{1}{\text{CF}} \]

Continued fraction representation of a standard fraction

\[ \frac{67}{29} = 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}} \]

\[ \frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \frac{1}{3 + \frac{2}{9}} = 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}} \]

e.g., \(67/29 = 2\) with remainder \(9/29\)
\[ = 2 + \frac{1}{(29/9)} \]
Ancient Greek Representation: Continued Fraction Representation

\[
\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}} = [1, 1, 1, 0, 0, 0, \ldots]
\]

Ancient Greek Representation: Continued Fraction Representation

\[
\frac{8}{5} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = [1, 1, 1, 1, 0, 0, 0, \ldots]
\]

A Pattern?

Let \( r_1 = [1, 0, 0, 0, \ldots] = 1 \)
\( r_2 = [1, 1, 0, 0, 0, \ldots] = 2/1 \)
\( r_3 = [1, 1, 1, 0, 0, 0, \ldots] = 3/2 \)
\( r_4 = [1, 1, 1, 1, 0, 0, 0, \ldots] = 5/3 \)
and so on.

Theorem:

\( r_n = \frac{\text{Fib}(n+1)}{\text{Fib}(n)} \)
1,1,2,3,5,8,13,21,34,55,…

\[
\begin{align*}
2/1 &= 2 \\
3/2 &= 1.5 \\
5/3 &= 1.666… \\
8/5 &= 1.6 \\
13/8 &= 1.625 \\
21/13 &= 1.6153846… \\
34/21 &= 1.61904…
\end{align*}
\]

\[\varphi = 1.6180339887498948482045\]

Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral.

The ratio of consecutive ring lengths approaches the Golden Ratio.

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The ratio of consecutive ring lengths approaches the Golden Ratio.

Proposition:
Any finite continued fraction evaluates to a rational.

Theorem
Any rational has a finite continued fraction representation.

An infinite continued fraction

\[\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots}}}}\]

Hmm.
Finite CFs = Rationals.

Then what do infinite continued fractions represent?

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Finite CFs = Rationals.

Then what do infinite continued fractions represent?
Quadratic Equations

- \( x^2 - 3x - 1 = 0 \)
  \[ x = \frac{3 + \sqrt{13}}{2} \]
- \( x^2 = 3x + 1 \)
- \( x = 3 + \frac{1}{x} \)
- \( x = 3 + \frac{1}{3 + \frac{1}{x}} = \ldots \)

A Periodic CF

Theorem:
Any solution to a quadratic equation has a periodic continued fraction.

Converse:
Any periodic continued fraction is the solution of a quadratic equation.
(try to prove this!)

Non-periodic CFs

\[ e - 1 = \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{6 + \ldots}}}}} \]

What is the pattern?

\[ \pi = \frac{1}{2 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ldots}}}}} \]

So they express more than just the rationals...

What about those non-recurring infinite continued fractions?

No one knows!
What a cool representation!

Finite CF: Rationals
Periodic CF: Quadratic roots
And some numbers reveal hidden regularity.

More good news: Convergents

Let \( \alpha = [a_1, a_2, a_3, \ldots] \) be a CF.

Define:
\[
C_1 = [a_1,0,0,0,0,\ldots] \\
C_2 = [a_1,a_2,0,0,0,\ldots] \\
C_3 = [a_1,a_2,a_3,0,0,\ldots] \text{ and so on.}
\]

\( C_k \) is called the \( k \)-th convergent of \( \alpha \)

\( \alpha \) is the limit of the sequence \( C_1, C_2, C_3, \ldots \)

Best Approximator Theorem

- A rational \( p/q \) is the best approximator to a real \( \alpha \) if no rational number of denominator smaller than \( q \) comes closer to \( \alpha \).

BEST APPROXIMATOR THEOREM:
Given any CF representation of \( \alpha \), each convergent of the CF is a best approximator for \( \alpha \)!

Best Approximators of \( \phi \)

\[
\begin{align*}
C_1 &= 3 \\
C_2 &= \frac{22}{7} \\
C_3 &= \frac{333}{106} \\
C_4 &= \frac{355}{113} \\
C_5 &= \frac{103993}{33102} \\
C_6 &= \frac{104348}{33215}
\end{align*}
\]

Continued Fraction Representation
Remember?

We already saw the convergents of this CF 
\[ [1,1,1,1,1,1,1,1,1,1,1,\ldots] \]
are of the form \( \text{Fib}(n+1)/\text{Fib}(n) \)

Hence:
\[ \frac{1 + \sqrt{5}}{2} \]

1,1,2,3,5,8,13,21,34,55,\ldots

- \( 2/1 = 2 \)
- \( 3/2 = 1.5 \)
- \( 5/3 = 1.666\ldots \)
- \( 8/5 = 1.6 \)
- \( 13/8 = 1.625 \)
- \( 21/13 = 1.6153846\ldots \)
- \( 34/21 = 1.61904\ldots \)
- \( \varphi = 1.6180339887498948482045\ldots \)

As we’ve seen...

\[ \frac{x}{1-x-x^2} = 0 + x + x^3 + 2x^3 + 3x^4 + 5x^5 + \ldots \]
\[ = F_0 + F_1x + F_2x^2 + F_3x^3 + F_4x^4 + F_5x^5 + \ldots \]

Going the Other Way

\[
\begin{align*}
(1 - z - z^2)(F_0 + F_1z + F_2z^2 + F_3z^3 + \cdots) \\
= F_0 + F_1z + F_2z^2 + F_3z^3 + \cdots \\
- F_0z - F_1z^2 - F_2z^3 - \cdots \\
- F_0z^2 - F_1z^3 - \cdots \\
= F_0 + (F_1 - F_0)z \\
= z
\end{align*}
\]

\[
F(z) = F_0 + F_1z + F_2z^2 + \cdots = \frac{z}{1 - z - z^2} \\
\frac{z}{1 - z - z^2} = \sum_{n \geq 0} \frac{1}{\sqrt{5}} (\varphi^n - \bar{\varphi}^n) z^n.
\]

\[
F_n = \frac{\varphi^n - (-\frac{1}{\varphi})^n}{\sqrt{5}} \approx \frac{\varphi^n}{\sqrt{5}}
\]
Recurrences and generating functions

Golden ratio

Continued fractions

Convergents

Closed form for Fibonacci

Here's What You Need to Know...