**15-251**
Great Theoretical Ideas in Computer Science

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**Ancient Wisdom: Unary and Binary**

Lecture 5 (September 2009)

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**How to play the 9 stone game?**

9 stones, numbered 1-9
Two players alternate moves.
Each move a player gets to take a new stone
Any subset of 3 stones adding to 15, wins.

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**Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu in ancient China**

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**Magic Square: Any 3 in a vertical, horizontal, or diagonal line add up to 15.**

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Conversely, any 3 that add to 15 must be on a line.

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TIC-TAC-TOE on a Magic Square
Represents The Nine Stone Game
Alternate taking squares 1-9.
Get 3 in a row to win.

4 9 2
3 5 7
8 1 6

Basic Idea of this Lecture
Don’t stick with the representation in which you encounter problems!
Always seek the more useful one!
This idea requires a lot of practice

Prehistoric Unary
1
2
3
4

Consider the problem of finding a formula for the sum of the first $n$ numbers
You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$

\[ 1 + 2 + 3 + \ldots + n-1 + n = S \]
\[ n + n-1 + n-2 + \ldots + 2 + 1 = S \]
\[ n+1 + n+1 + n+1 + \ldots + n+1 + n+1 = 2S \]
\[ \frac{n(n+1)}{2} = 2S \]
\[ S = \frac{n(n+1)}{2} \]
n^th Triangular Number
\[ \Delta_n = 1 + 2 + 3 + \ldots + n - 1 + n \]
\[ = \frac{n(n+1)}{2} \]

n^th Square Number
\[ \Box_n = n^2 \]
\[ = \Delta_n + \Delta_{n-1} \]

Breaking a square up in a new way

Breaking a square up in a new way

Breaking a square up in a new way

Breaking a square up in a new way

Breaking a square up in a new way
The sum of the first $n$ odd numbers is $n^2$.

Here is an alternative dot proof of the same sum….

$n^{\text{th}}$ Square Number

$$\square_n = \Delta_n + \Delta_{n-1}$$

$$= n^2$$
\[ a_n = \Delta_n + \Delta_{n-1} \]
\[ = n^2 \]

\[ a_n = \Delta_n + \Delta_{n-1} \]

\[ a_n = \Delta_n + \Delta_{n-1} \]

= Sum of first \( n \) odd numbers

Area of square = \((\Delta_n)^2\)
Can you find a formula for the sum of the first $n$ squares?

Babylonians needed this sum to compute the number of blocks in their pyramids.

\[
\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
Rhind Papyrus
Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

A Frequently Arising Calculation

\[
(\frac{1}{x} - 1)(1 + x + x^2 + x^3 + \ldots + x^{n-1})
\]

\[
= x^n - 1 - 1 - x^2 - x^3 - \ldots - x^{n-2} - x^{n-1}
\]

\[
= x^n - 1
\]

\[
1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}
\]

We’ll use this fundamental sum again and again:

The Geometric Series

Geometric Series for \( x = 2 \)

\[
1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1
\]

Geometric Series for \( x = \frac{1}{2} \)

\[
1 + \frac{1}{2} + \frac{1}{2}^2 + \frac{1}{2}^3 + \ldots + \frac{1}{2}^{n-1} = \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1}
\]

\[
= 2 \left(1 - \left(\frac{1}{2}\right)^n\right)
\]

\[
1 + x^1 + x^2 + x^3 + \ldots + x^{n-2} + x^{n-1} = \frac{x^n - 1}{x - 1}
\]

(when \( x = \frac{1}{2} \))
A Similar Sum

\[ a^n + a^{n-1}b^1 + a^{n-2}b^2 + \ldots + a^1b^{n-1} + b^n \]

\[
\left( a^i b^j \right)^2 = a^{i^2} b^{j^2} + \ldots + b^0
\]

\[
a^i \left( \frac{b^{j+1}}{b^{j-1}} \right) = \left( \frac{b^{j+1} - 1}{b^{j-1}} \right)
\]

A slightly different one

\[ S = b_0^2 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n = ? \]

\[
\begin{align*}
S &= \frac{0.2^1 + 1.2^1 + 2.2^2 + \ldots + n2^n}{2S} \\
2S &= 0.2^1 + 1.2^2 + 2.2^3 + \ldots + (n-1)2^n + n2^n \\
S &= -0.2^1 - 1.2^2 - \ldots - 1.2^n - n2^n \\
&= -(2^n - 1) + n.2^n \\
&= n.2^n - 2^n - 2^n + 2 = (n-1)2^n + 2.
\end{align*}
\]

Two Case Studies

Bases and Representation

\[
\begin{align*}
S &= \left( a^i, 2^1, (a-2)^1, 2^2, \ldots, (a^n-1)^1 \right) \\
S &= n2^n - \sum_{i=1}^{n} (2^i - 1) \cdot 2^i
\end{align*}
\]

BASE X Representation

\[ S = a_{n-1}a_{n-2} \ldots a_1a_0 \]

represents the number:

\[ a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots + a_0X^0 \]

Base 2  Binary Notation

101 represents: 1 (2^2) + 0 (2^1) + 1 (2^0)

= 3

Base 7

1015 represents: 0 (7^2) + 1 (7^1) + 5 (7^0)

= 22

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE X Representation

\[ S = (a_{n-1}a_{n-2} \ldots a_1a_0)_X \]

represents the number:

\[ a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots + a_0X^0 \]

Largest number representable in base-X with n “digits”

\[ = (X-1)X^{n-1}X^{n-1}X^{n-1}X^{n-1} \ldots X^{n-1} \]

\[ = (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0) \]

\[ = (X^n - 1) \]
**Fundamental Theorem For Binary**

Each of the numbers from 0 to $2^n-1$ is uniquely represented by an n-bit number in binary

$$k \text{ uses } \lceil \log_2 k \rceil + 1 \text{ digits in base } 2$$

**Fundamental Theorem For Base-$X$**

Each of the numbers from 0 to $X^n-1$ is uniquely represented by an n-"digit" number in base $X$

$$k \text{ uses } \lceil \log_X k \rceil + 1 \text{ digits in base } X$$

**Other Representations: Egyptian Base 3**

Conventional Base 3:
Each digit can be 0, 1, or 2
Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1
Example: $(1 -1 -1)_{EB3} = 9 - 3 - 1 = 5$

We can prove a unique representation theorem

How could this be Egyptian?
Historically, negative numbers first appear in the writings of the Hindu mathematician Brahmagupta (628 AD)

One weight for each power of 3
Left = "negative". Right = "positive"
Two Case Studies

Bases and Representation

Solving Recurrences using a good representation

Example

T(1) = 1
T(n) = 4T(n/2) + n

Notice that T(n) is inductively defined only for positive powers of 2, and undefined on other values

T(1) = 1  T(2) = 6  T(4) = 28  T(8) = 120

Give a closed-form formula for T(n)

Guess: G(n) = 2n

Base Case: G(1) = 1 and T(1) = 1
Induction Hypothesis: T(x) = G(x) for x < n
Hence: T(n/2) = G(n/2) = 2(n/2) – n

T(n) = 4T(n/2) + n
= 4G(n/2) + n
= 4[2(n/2) – n/2] + n
= 2n – 2n + n
= 2n – n = G(n)

Technique 1

Guess Answer, Verify by Induction

T(1) = 1, T(n) = 4T(n/2) + n

Technique 2

Guess Form, Calculate Coefficients

T(1) = 1, T(n) = 4T(n/2) + n

Guess: T(n) = an^2 + bn + c
for some a, b, c

Calculate: T(1) = 1, so a + b + c = 1
T(n) = 4T(n/2) + n
an^2 + bn + c = 4[a(n/2)^2 + b(n/2) + c] + n
= an^2 + 2bn + 4c + n
(b+1)n + 3c = 0
Therefore: b = -1  c = 0  a = 2

Technique 3

The Recursion Tree Approach

T(1) = 1, T(n) = 4T(n/2) + n

A slight variation

T(1) = 1, T(n) = 4T(n/2) + n^2
How about this one?

\[ T(1) = 1, \ T(n) = 3 \ T(n/2) + n \]

... and this one?

\[ T(1) = 1, \ T(n) = T(n/4) + T(n/2) + n \]

Unary and Binary

Triangular Numbers

Dot proofs

\[ (1+x+x^2+\ldots+x^{n-1}) = \frac{(x^n - 1)}{(x-1)} \]

Base-X representations

\[ k \text{ uses } \left\lfloor \log_k k \right\rfloor + 1 = \left\lfloor \log_2 (k+1) \right\rfloor \text{ digits in base 2} \]

Here's What You Need to Know...

Solving Simple Recurrences

Bhaskara’s “proof” of Pythagoras’ theorem