# Complexity Theory: The P vs NP question

Lecture 28 (Dec 4, 2007)

## The $1M question

The Clay Mathematics Institute
Millenium Prize Problems

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equations
4. P vs NP
5. Poincaré Conjecture
6. Riemann Hypothesis
7. Yang-Mills Theory

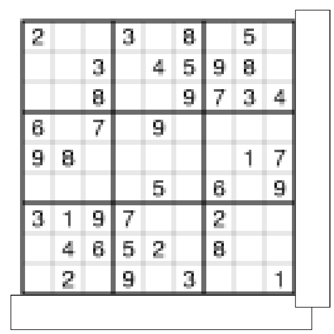
## The P versus NP problem

Is perhaps one of the biggest open problems in computer science (and mathematics!) today.

(Even featured in the TV show NUMB3RS)

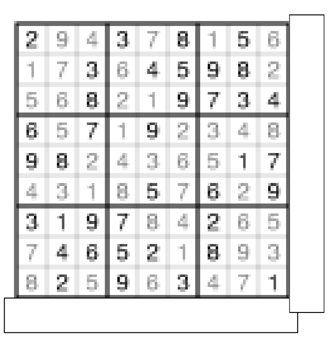
*But what is the P-NP problem?*

## Sudoku



3x3x3

## Sudoku



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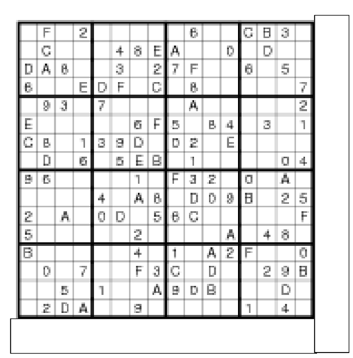
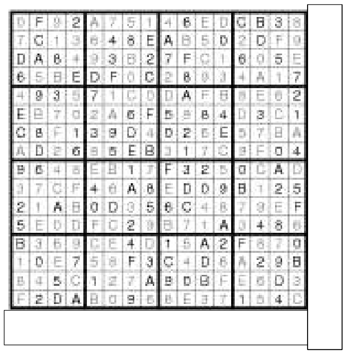
## Sudoku



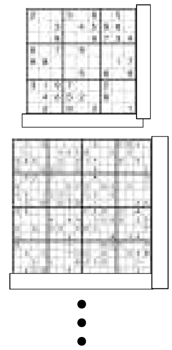
4x4x4

## Sudoku



4x4x4

## Sudoku



n x n x n

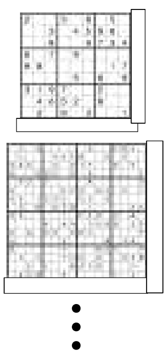
Suppose it takes you S(n) to solve n x n x n

V(n) time to verify the solution

Fact: $V(n) = O(n^2 \times n^2)$

Question: is there some constant such that $S(n) \leq n^{constant}$ ?

## Sudoku



n x n x n

P vs NP problem

=

Does there exist an algorithm for n x n x n Sudoku that runs in time p(n) for some polynomial p( ) ?

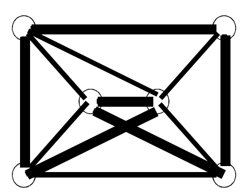
## The P versus NP problem (informally)

Is proving a theorem much more difficult than checking the proof of a theorem?

## Let’s start at the beginning…

## Hamilton Cycle

Given a graph G = (V,E), a cycle that visits all the nodes exactly once



## The Problem "HAM"

Input: Graph G = (V,E)

Output: YES if G has a Hamilton cycle
NO if G has no Hamilton cycle

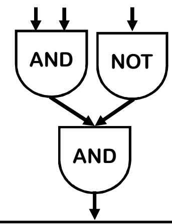
## The Set "HAM"

HAM = { graph G | G has a Hamilton cycle }

## Circuit-Satisfiability

Input: A circuit C with one output

Output: YES if C is satisfiable
NO if C is not satisfiable



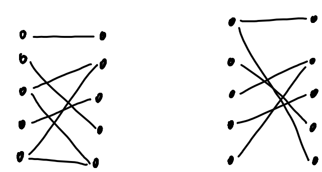

## The Set "SAT"

SAT = { all satisfiable circuits C }

## Bipartite Matching

Input: A bipartite graph G = (U,V,E)

Output: YES if G has a perfect matching
NO if G does not



## The Set "BI-MATCH"

BI-MATCH = { all bipartite graphs that have a perfect matching }

## Sudoku

Input: n x n x n sudoku instance

Output: YES if this sudoku has a solution
NO if it does not

## The Set "SUDOKU"

SUDOKU = { All solvable sudoku instances }

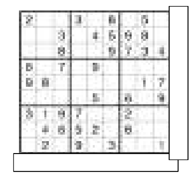
## Decision Versus Search Problems

| Decision Problem | Search Problem |
| --- | --- |
| YES/NO answers | |
| Does G have a Hamilton cycle? | Find a Hamilton cycle in G if one exists, else return NO |
| Can G be 3-colored ? | Find a 3-coloring of G if one exists, else return NO |

## Reducing Search to Decision

Given an algorithm for decision Sudoku, devise an algorithm to find a solution

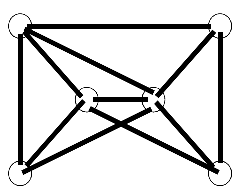
Idea:
Fill in one-by-one and use decision algorithm



## Reducing Search to Decision

Given an algorithm for decision HAM, devise an algorithm to find a solution

Idea:
Find the edges of the cycle one by one



## Decision/Search Problems

We'll study decision problems because they are almost the same (asymptotically) as their search counterparts

# Polynomial Time and The Class "P" of Decision Problems

## What is an efficient algorithm?

Is an O(n) algorithm efficient?
How about O(n log n)?
$O(n^2)$ ?
$O(n^{10})$ ?

} polynomial time — $O(n^c)$ for some constant c

$O(n^{\log n})$ ?
$O(2^n)$ ?
O(n!) ?

} non-polynomial time

Does an algorithm
running in $O(n^{100})$ time
count as efficient?

We consider non-polynomial time algorithms to be inefficient.

And hence a necessary condition for an algorithm to be efficient is that it should run in poly-time.

Asking for a poly-time algorithm for a problem sets a (very) low bar when asking for efficient algorithms.

The question is: can we achieve even this
for 3-coloring?
SAT?
Sudoku?

## The Class P

We say a set $L \subseteq \Sigma^*$ is in **P** if there is

- a program A and
- a polynomial p()

such that for any x in $\Sigma^*$,

A(x) runs for at most p(|x|) time
and answers question “is x in L?” correctly.

## The Class P

The class of all sets L that can be recognized in polynomial time.

The class of all decision problems that can be decided in polynomial time.

Why are we looking only at sets $\subseteq \Sigma^*$?

What if we want to work with graphs or boolean formulas?

## Languages/Functions in P?

Example 1:
CONN = {graph G: G is a connected graph}

Algorithm $A_1$:
If G has n nodes, then run depth first search from any node, and count number of distinct node you see. If you see n nodes, $G \in$ CONN, else not.

Time: $p_1(|x|) = \Theta(|x|)$.

## Languages/Functions in P?

HAM, SUDOKU, SAT are not known to be in P

CO-HAM = { G | G does NOT have a Hamilton cycle}

CO-HAM $\in$ P if and only if HAM $\in$ P

## Onto the new class, NP

## Verifying Membership

Is there a short "proof" I can give you for:

G $\in$ HAM?

G $\in$ BI-MATCH?

G $\in$ SAT?

G $\in$ CO-HAM?

## NP

A set L $\in$ NP

if there exists an algorithm A and a polynomial p( )

| | |
|---|---|
| For all x $\in$ L | For all x′ $\notin$ L |
| there exists y with $\|y\| \le p(\|x\|)$ | For all y′ with $\|y'\| \le p(\|x'\|)$ |
| such that A(x,y) = YES | we have A(x′,y′) = NO |
| in p(\|x\|) time | in p(\|x\|) time |

## Recall the Class P

We say a set L $\subseteq \Sigma^*$ is in **P** if there is
- a program A and
- a polynomial p()

such that for any x in $\Sigma^*$,

{ A(x) runs for at most p(|x|) time and answers question "is x in L?" correctly.

can think of A as "proving" that x is in L

## NP

A set L $\in$ NP

if there exists an algorithm A and a polynomial p( )

| | |
|---|---|
| For all x $\in$ L | For all x′ $\notin$ L |
| there exists a y with $\|y\| \le p(\|x\|)$ | For all y′ with $\|y'\| \le p(\|x'\|)$ |
| such that A(x,y) = YES | Such that A(x′,y′) = NO |
| in p(\|x\|) time | in p(\|x\|) time |

## The Class NP

The class of sets L for which there exist "short" proofs of membership
(of polynomial length)
that can "quickly" verified
(in polynomial time).

Recall: A doesn't have to find these proofs y; it just needs to be able to verify that y is a "correct" proof.

## P $\subseteq$ NP

For any L in P, we can just take y to be the empty string and satisfy the requirements.

Hence, every language in P is also in NP.

## Languages/Functions in NP?

G $\in$ HAM?

G $\in$ BI-MATCH?

G $\in$ SAT?

G $\in$ CO-HAM?

## Summary: P versus NP

Set L is in P if membership in L can be decided in poly-time.

Set L is in NP if each x in L has a short "proof of membership" that can be verified in poly-time.

Fact: P $\subseteq$ NP

Question: Does NP $\subseteq$ P ?

## Why Care?

## NP Contains Lots of Problems We Don't Know to be in P

Classroom Scheduling
Packing objects into bins
Scheduling jobs on machines
Finding cheap tours visiting a subset of cities
Allocating variables to registers
Finding good packet routings in networks
Decryption
…

OK, OK, I care…

But where do I begin
if I want to reason about
the P=NP problem?

How can we prove that
$NP \subseteq P$?

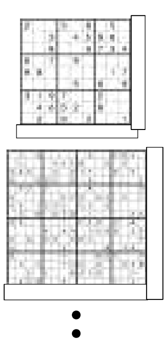
I would have to show that
every set in NP has a
polynomial time algorithm…

How do I do that?
It may take a long time!
Also, what if I forgot one of
the sets in NP?

We can describe
just one problem L in NP,
such that
if this problem L is in P,
then $NP \subseteq P$.

It is a problem that can
capture all other problems
in NP.

The "Hardest" Set in NP



n x n x n

Sudoku

Sudoku has a
polynomial time
algorithm

if and only if

P = NP

The "Hardest" Sets in NP

Sudoku Clique

SAT Independent-Set

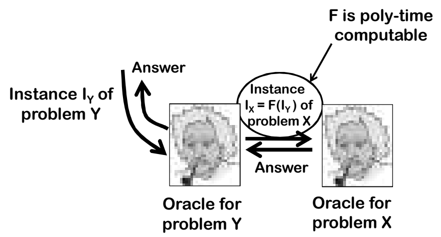
3-Colorability HAM

These problems are all
"polynomial-time equivalent".

I.e., each of these can be reduced to any
of the others in poly-time

## "Poly-time reducible to each other"

**Reducing problem Y to problem X in poly-time**




## How do you prove these are the hardest?

Theorem [Cook/Levin]:

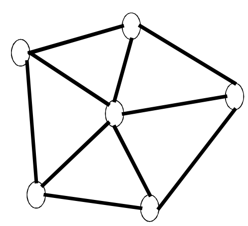
SAT is one language in NP, such that if we can show SAT is in P, then we have shown NP $\subseteq$ P.

SAT is a language in NP that can capture all other languages in NP.

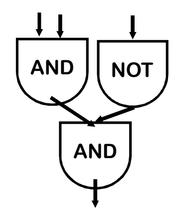
We say SAT is NP-complete.

## Last lecture…

3-colorability



Circuit Satisfiability




## Last lecture…

SAT and 3COLOR: Two problems that seem quite different, but are substantially the same.

Also substantially the same as CLIQUE and INDEPENDENT SET. (Homework)

If you get a polynomial-time algorithm for one, you get a polynomial-time algorithm for ALL.

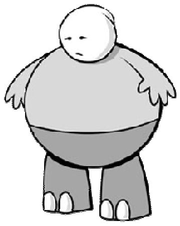
**Any language in NP**

↓ can be reduced (in polytime to) an instance of

**SAT** — hence SAT is NP-complete

↓ can be reduced (in polytime to) an instance of

**3COLOR** — hence 3COLOR is NP-complete



Here's What You Need to Know…

Definition of P and NP

Definition of problems
SAT, 3-COLOR, HAM,
SUDOKU, BI-MATCH

SAT, 3-COLOR, HAM, SUDOKU
all essentially equivalent.

Solve any one in poly-time
$\Rightarrow$ solve all of them in poly-time