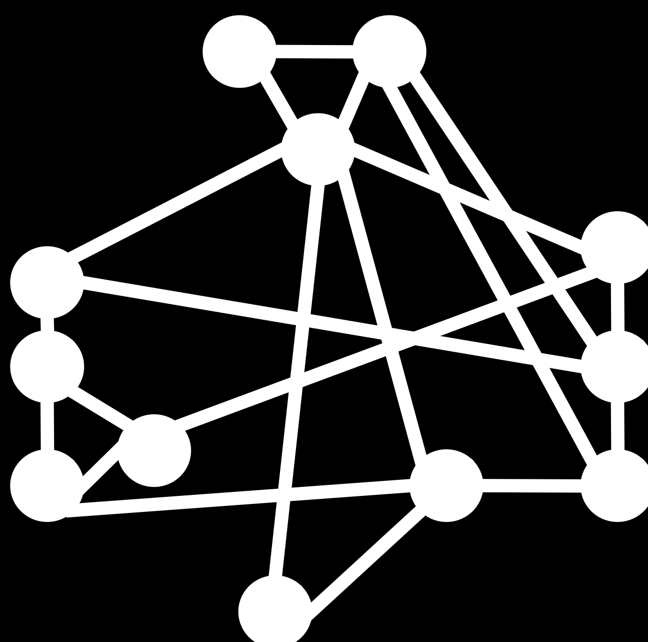
15-251

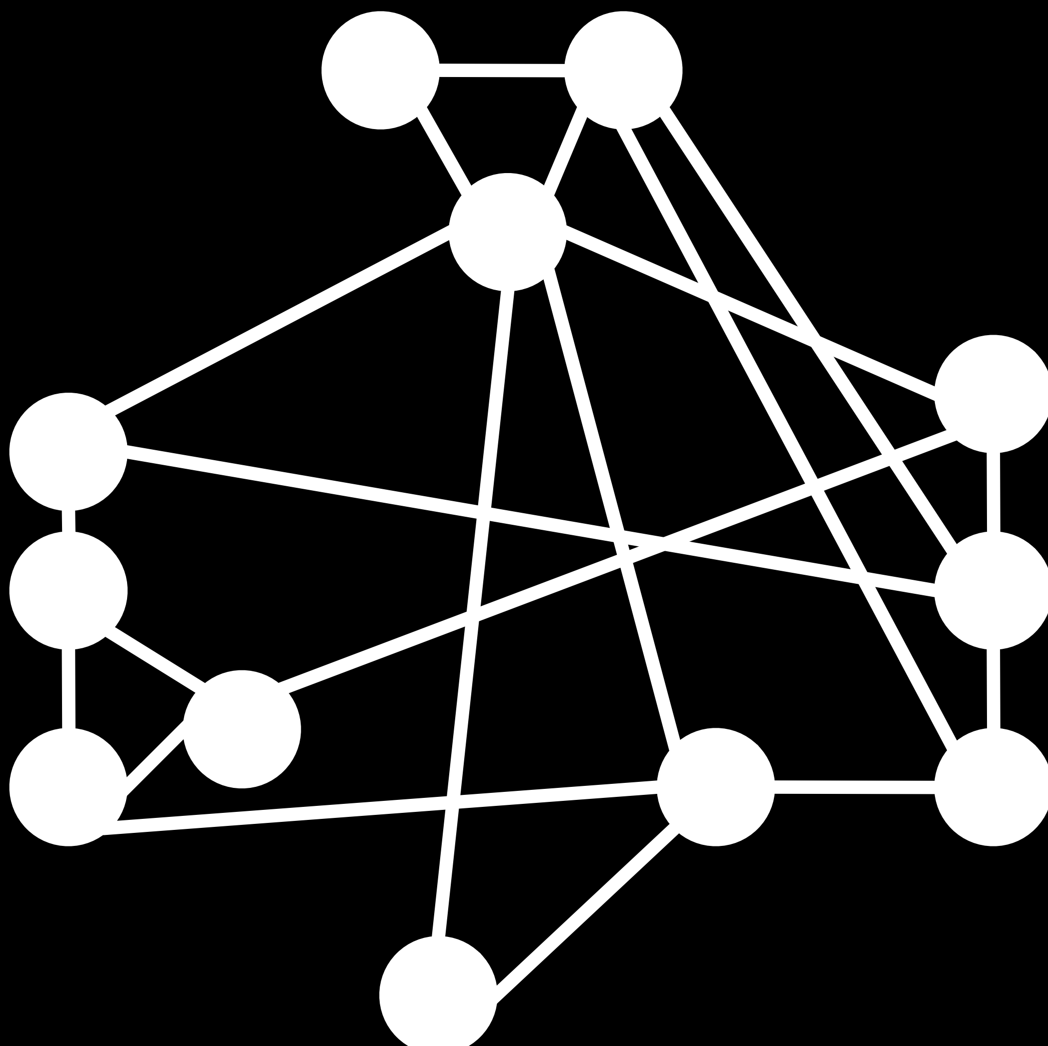
Great Theoretical Ideas in Computer Science

Complexity Theory:
Efficient Reductions Between Computational Problems

Lecture 27 (November 25, 2008)



A Graph Named “Gadget”



K-Coloring

We define a k-coloring of a graph:

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- Each node gets colored with one color

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- Each node gets colored with one color
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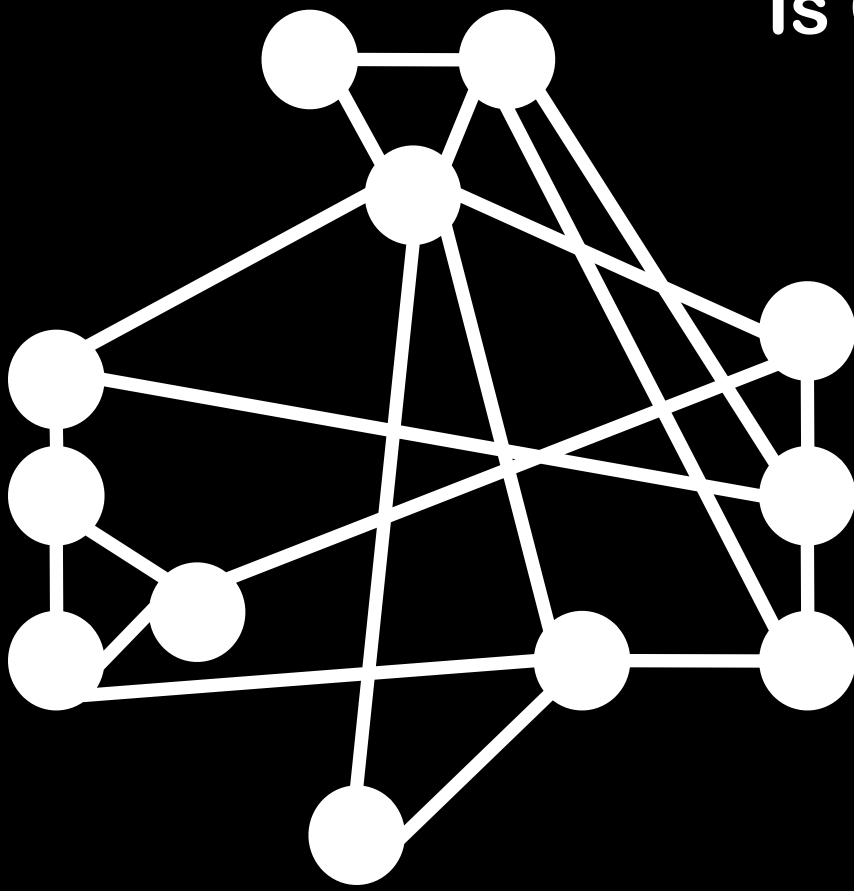
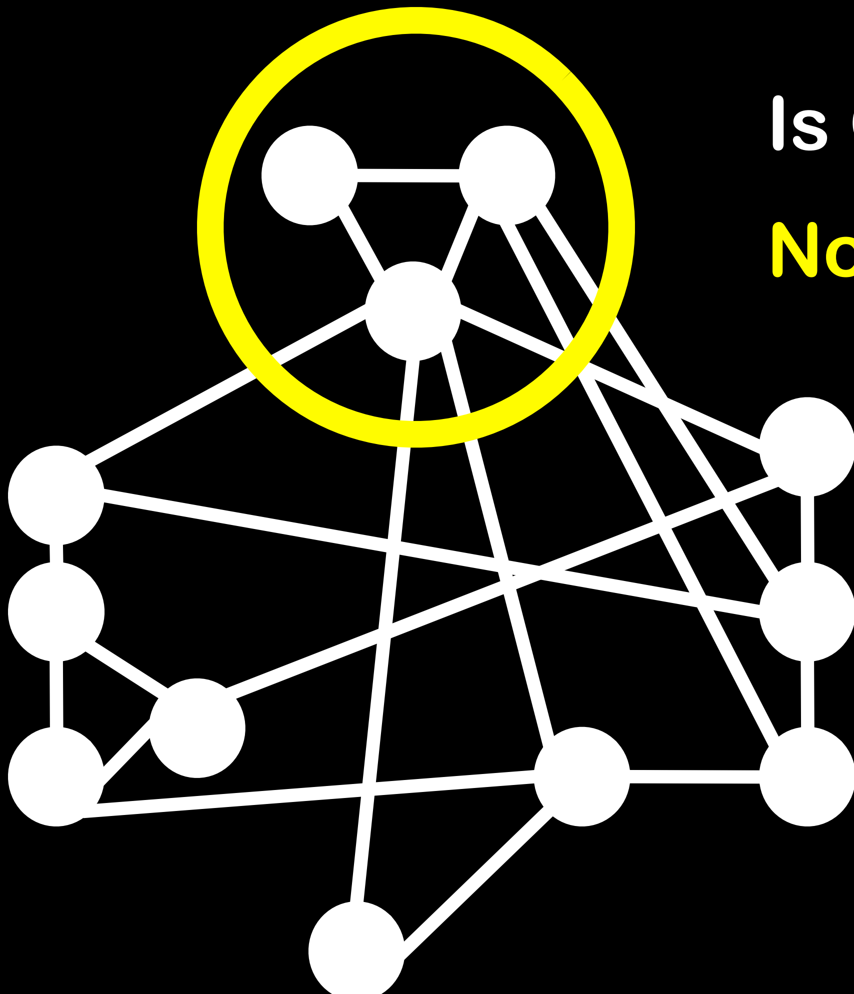
A graph is called k-colorable if and only if it has a k-coloring

A 2-CRAYOLA Question!

Is Gadget 2-colorable?



A 2-CRAYOLA Question!



Is Gadget 2-colorable?

No, it contains a triangle

A 2-CRAYOLA Question!

Given a graph G, how can we decide if it is 2-colorable?

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Answer: Enumerate all 2^n possible colorings to look for a valid 2-color

A 2-CRAYOLA Question!

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Answer: Enumerate all 2^n possible colorings to look for a valid 2-color

How can we **efficiently** decide if G is 2-colorable?

Theorem: G contains an odd cycle if and only if G is not 2-colorable

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Alternate coloring algorithm:

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If the algorithm terminates without a color conflict, output the 2-coloring

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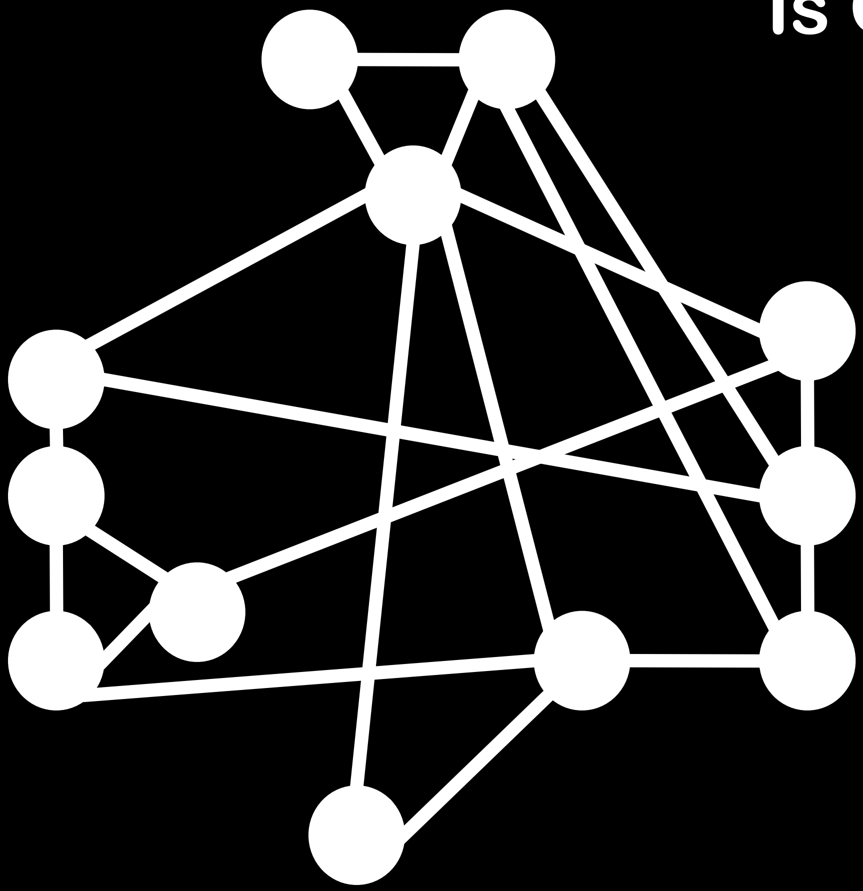
Else, output an odd cycle

A 2-CRAYOLA Question!

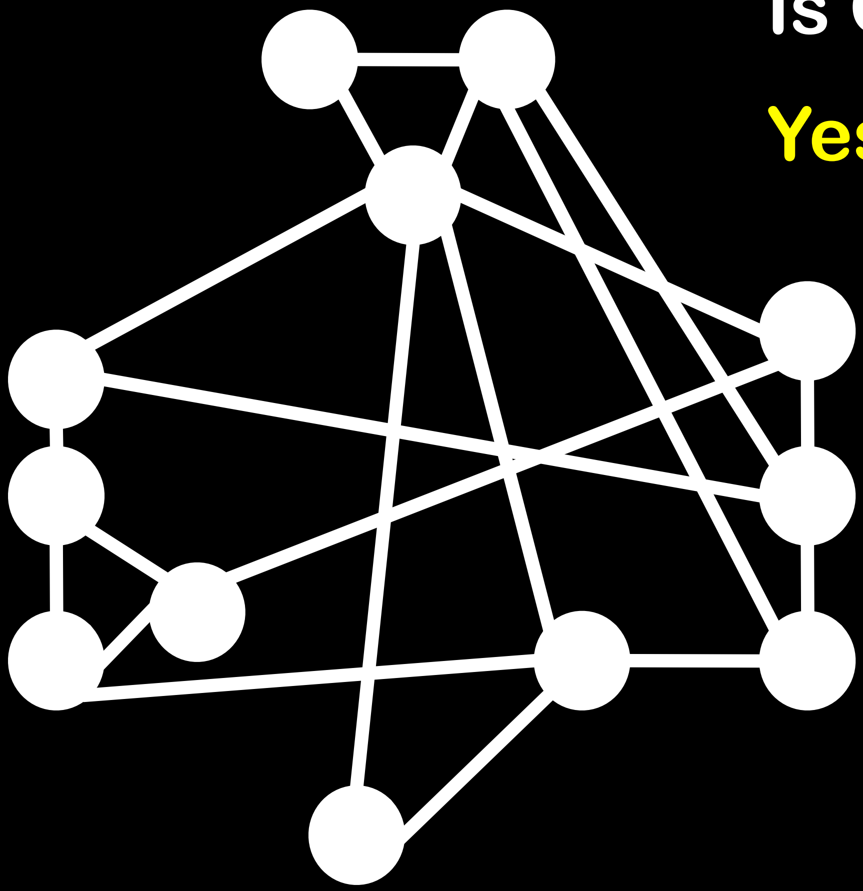
Theorem: G contains an odd cycle if and only if G is not 2-colorable

A 3-CRAYOLA Question!

Is Gadget 3-colorable?



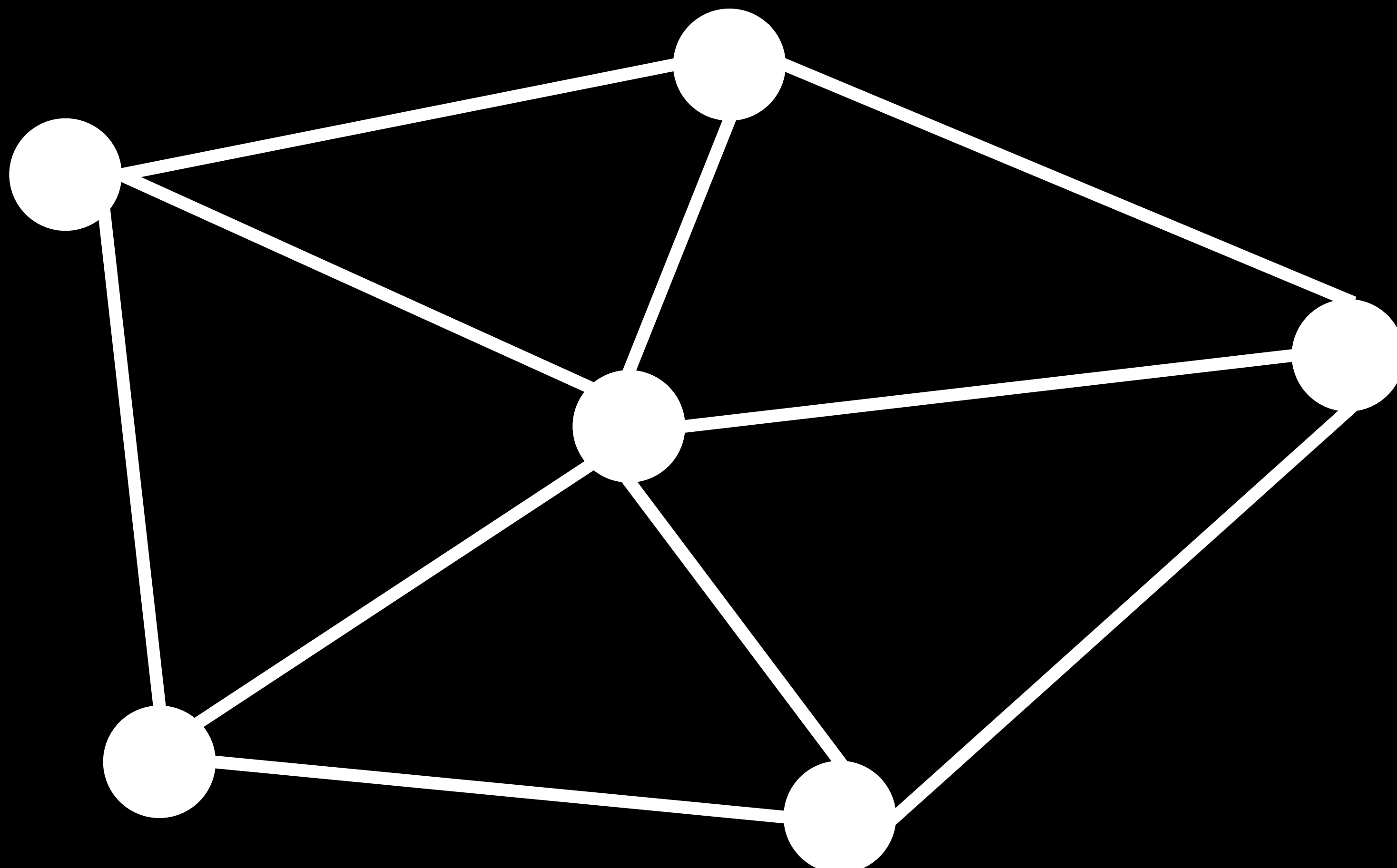
A 3-CRAYOLA Question!



Is Gadget 3-colorable?

Yes!

A 3-CRAYOLA Question!



3-Coloring Is Decidable by Brute Force

3-Coloring Is Decidable by Brute Force

Try out all 3^n colorings until you determine if G has a 3-coloring

A 3-CRAYOLA Oracle

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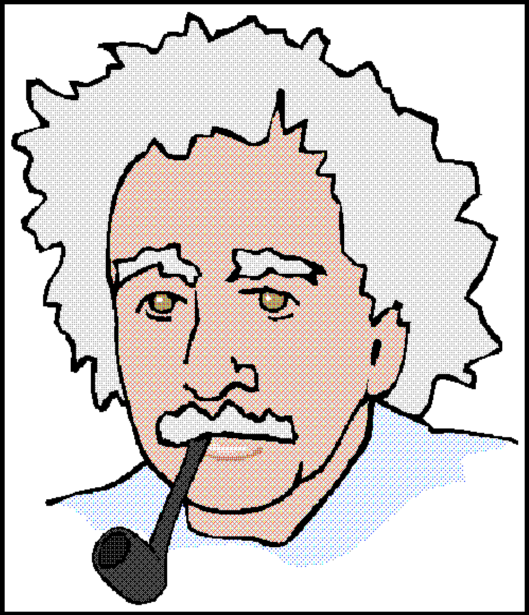
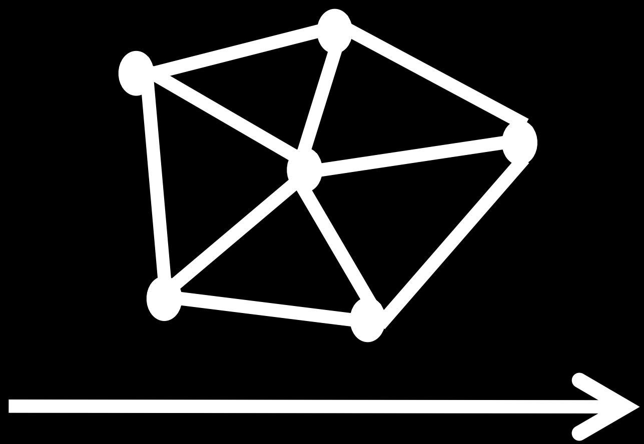
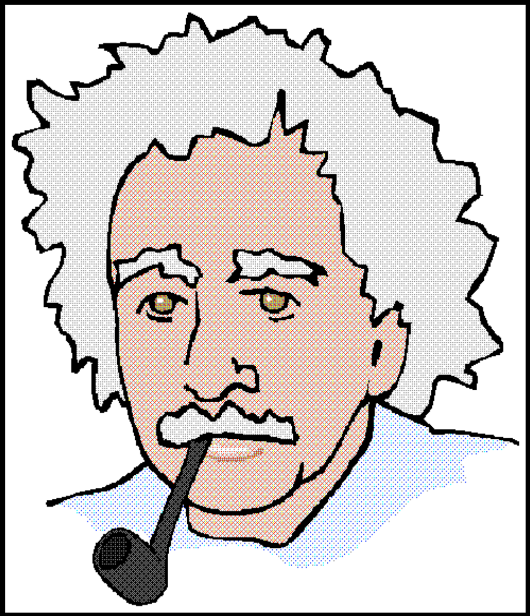


3-Colorability
Oracle

A 3-CRAYOLA Oracle





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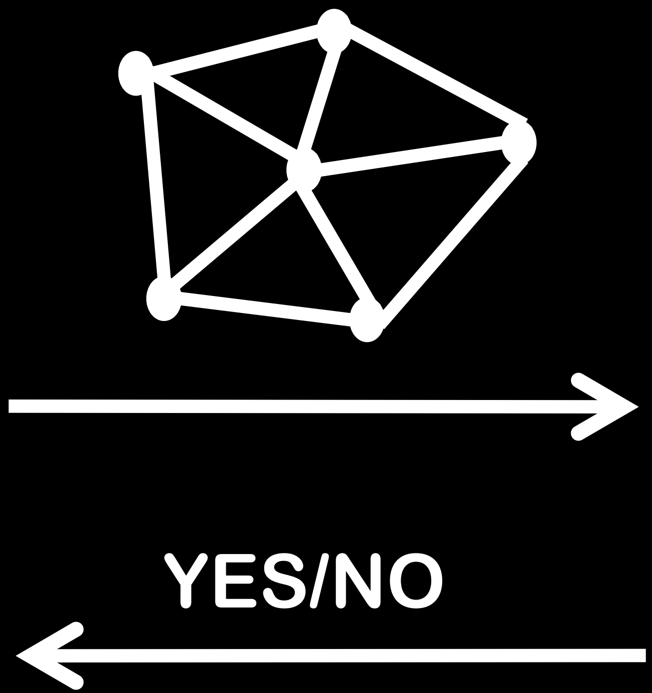

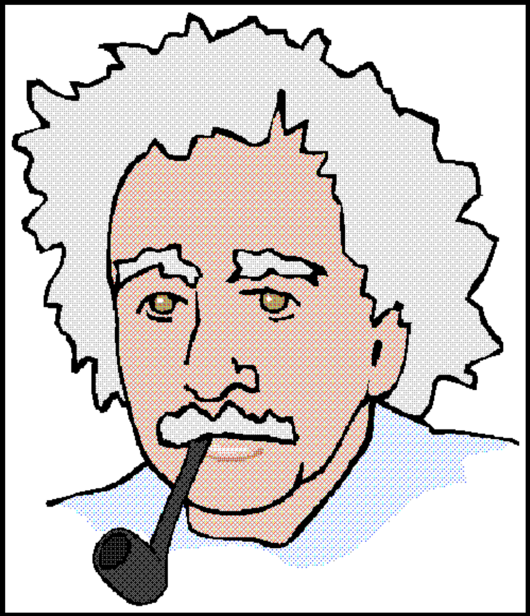

3-Colorability Oracle

Better 3-CRAYOLA Oracle

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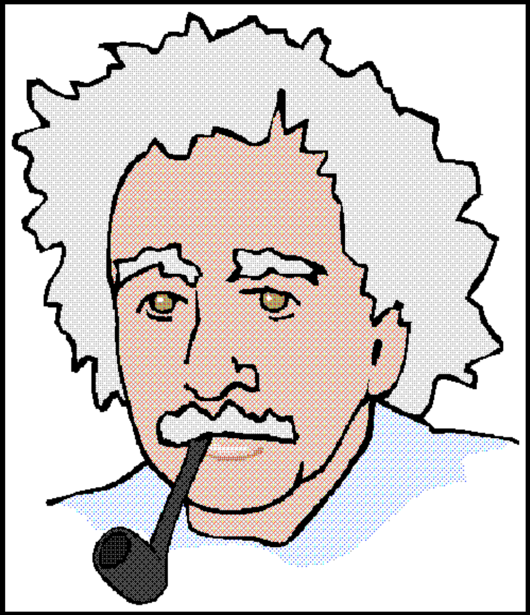
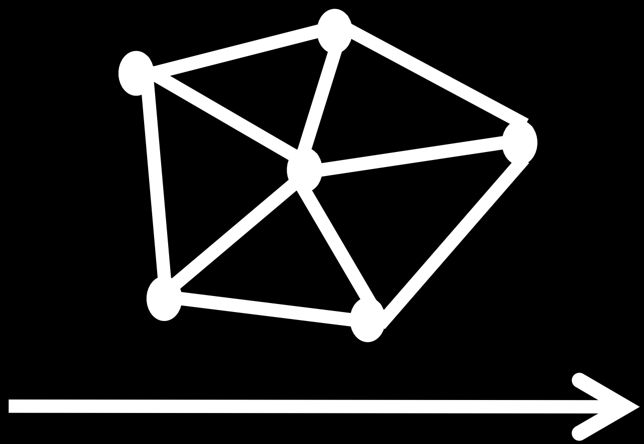
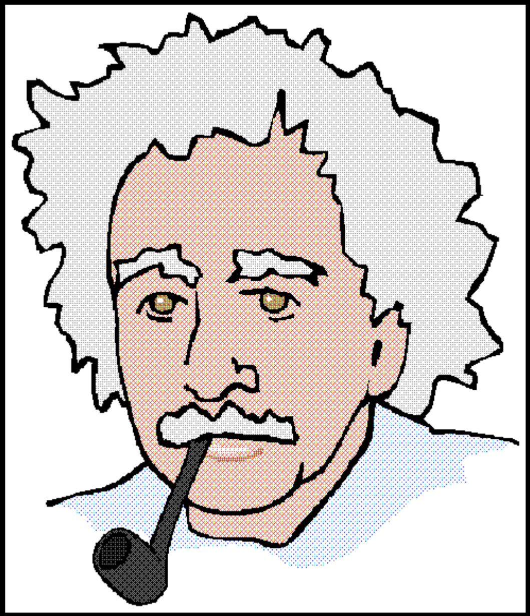


3-Colorability
Search Oracle

Better 3-CRAYOLA Oracle





3-Colorability
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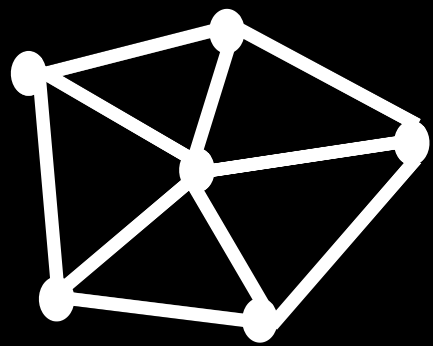
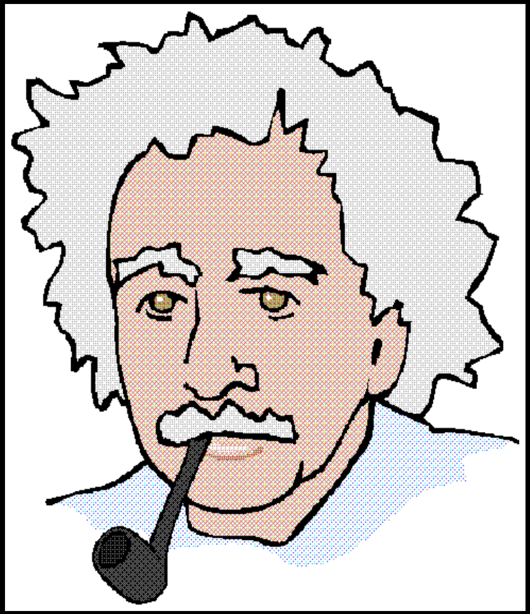
Better 3-CRAYOLA Oracle



NO, or

YES here is how:
gives 3-coloring
of the nodes

3-Colorability
Search Oracle



3-Colorability
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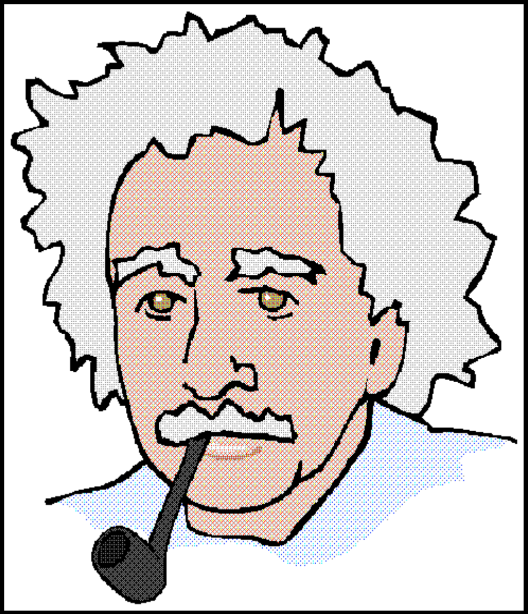
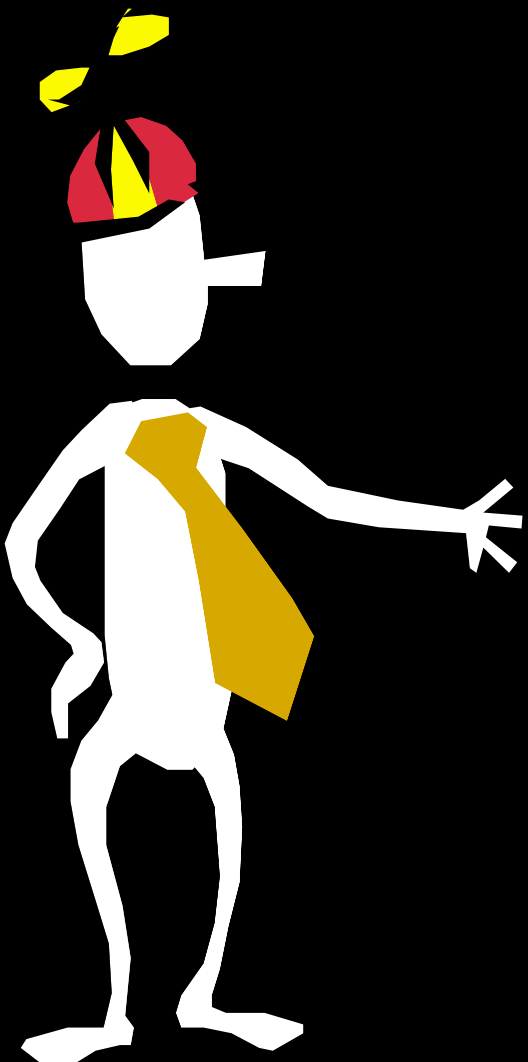
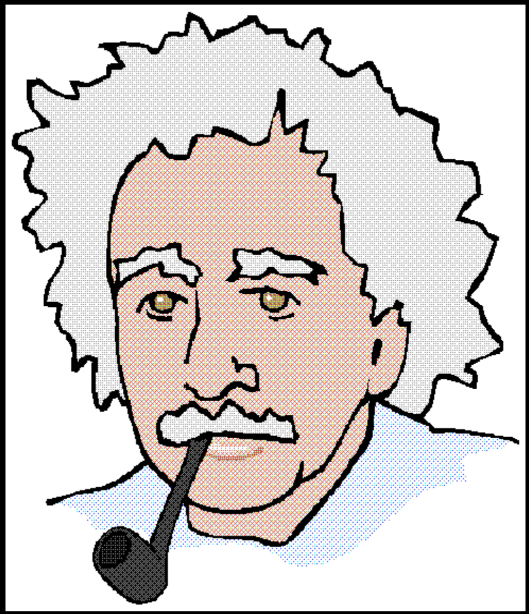


3-Colorability
Decision Oracle

Christmas Present





GIVEN:
3-Colorability
Decision Oracle

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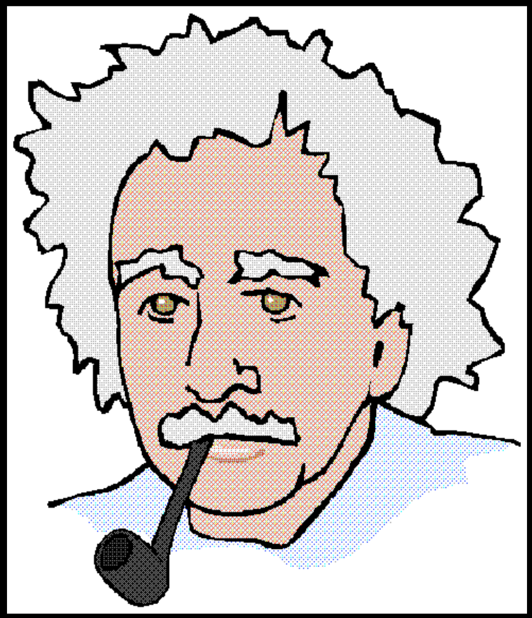

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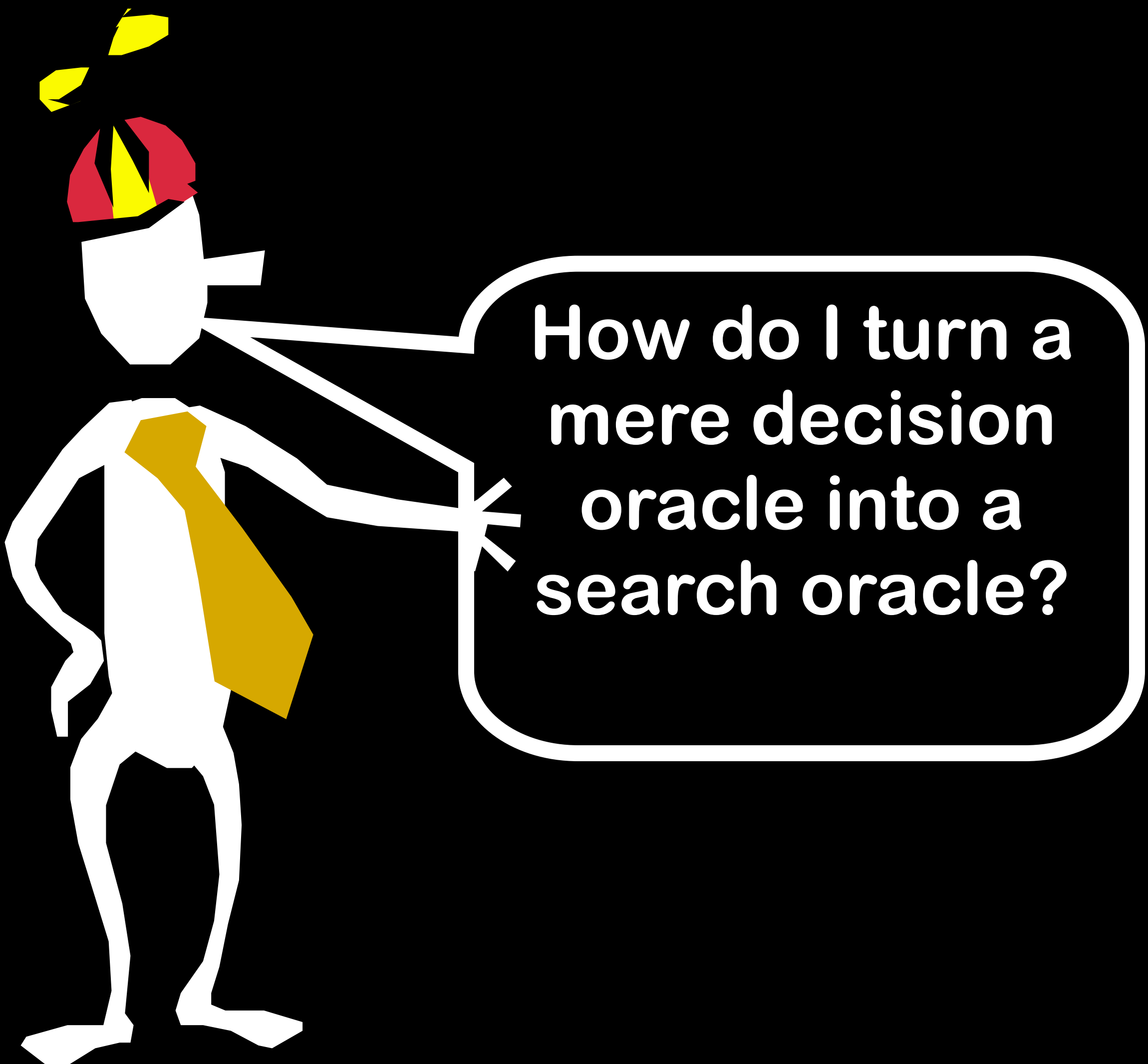
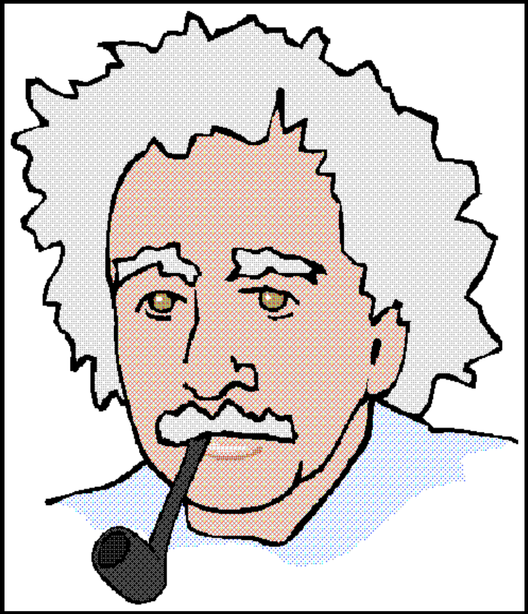
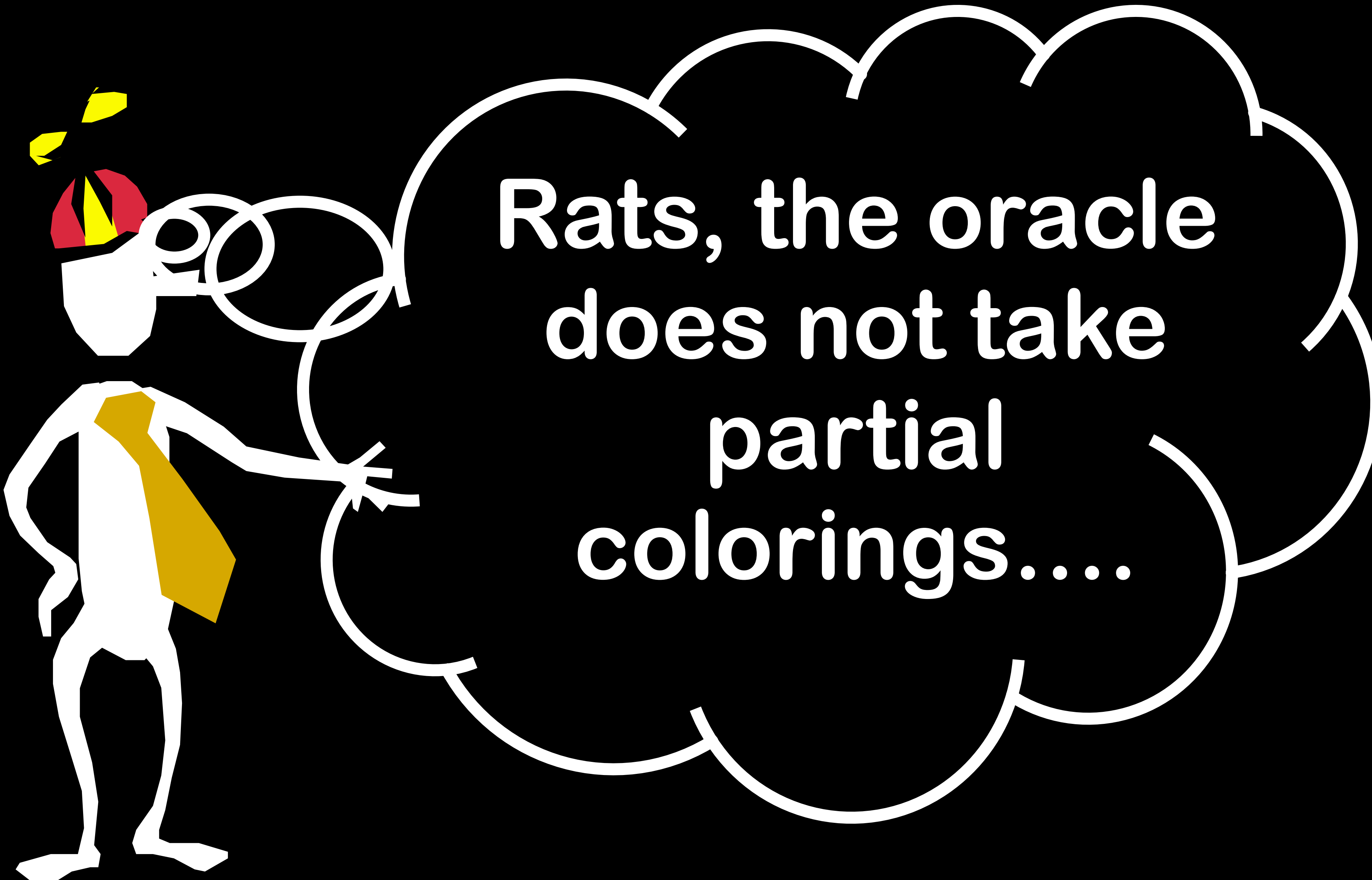




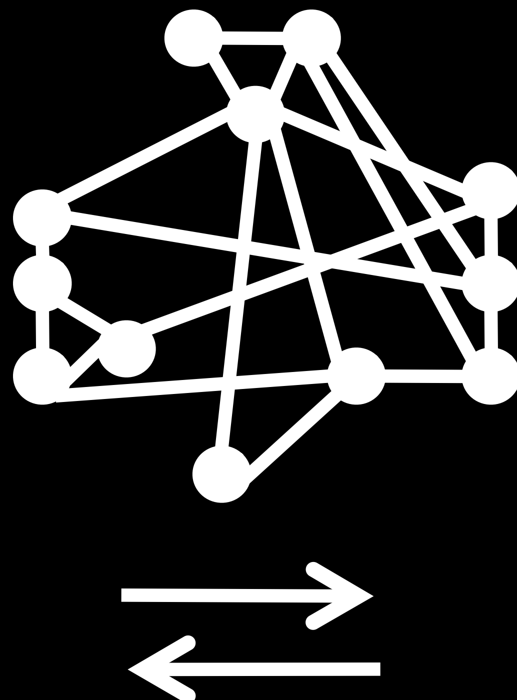
GIVEN:
3-Colorability
Decision Oracle

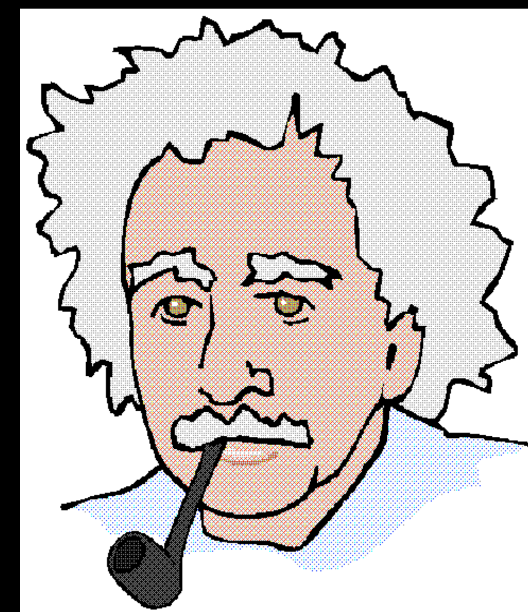
What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says “YES”

Beanie’s Flawed Idea



Beanie's Fix





GIVEN:
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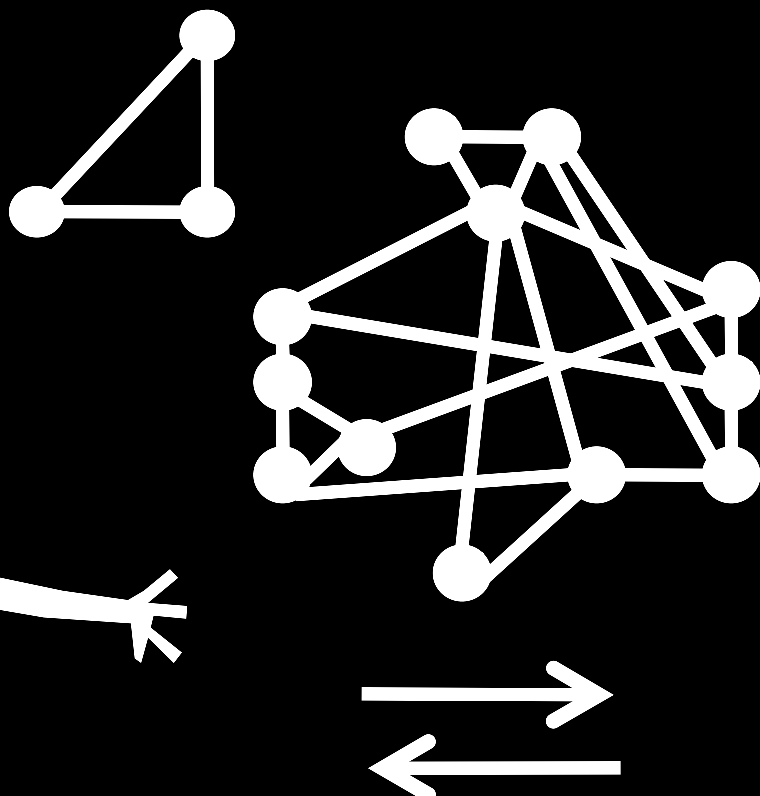

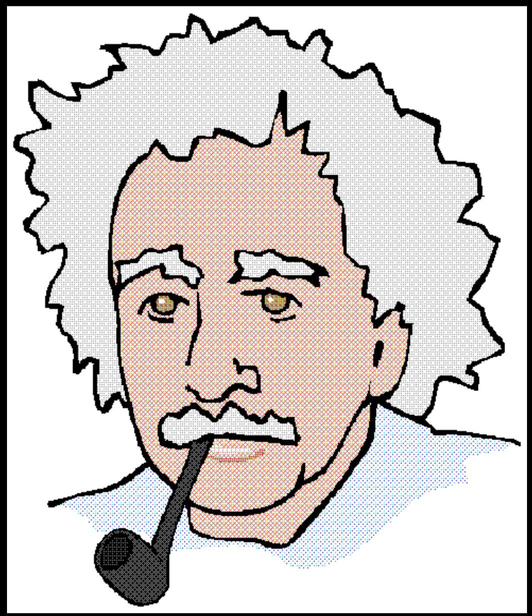

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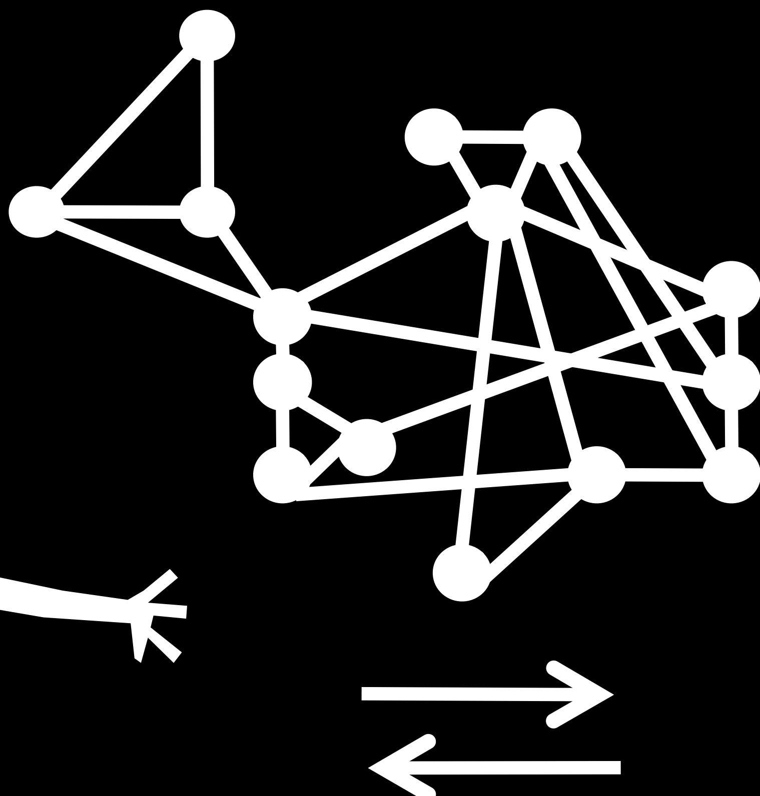

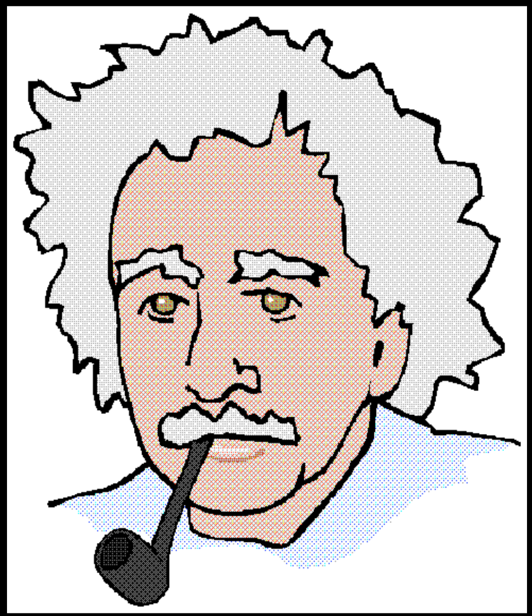

GIVEN:
3-Colorability
Decision Oracle

Let’s now look at two other problems:

1. K-Clique
2. K-Independent Set

K-Cliques

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A K-clique is a set of K nodes with all K(K-1)/2 possible edges between them

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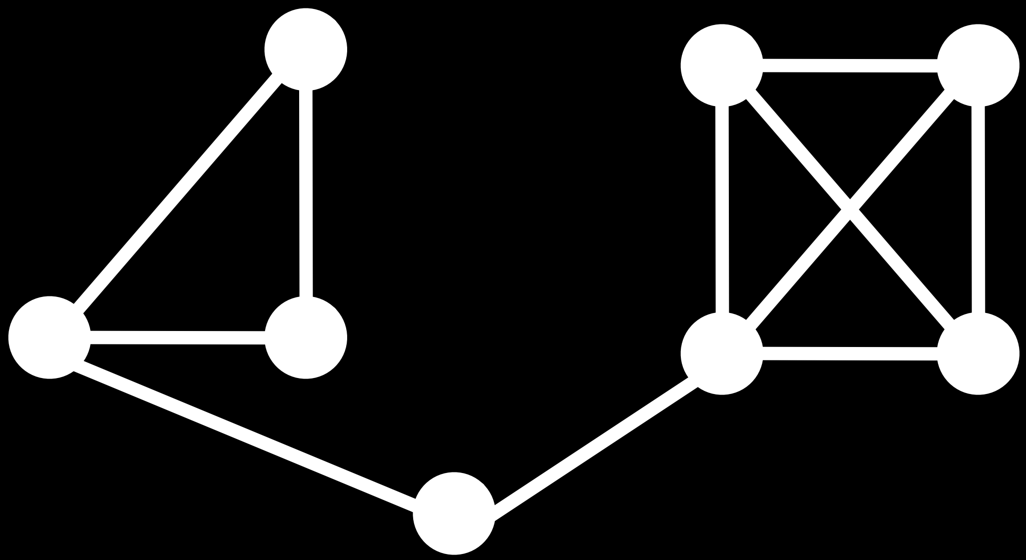
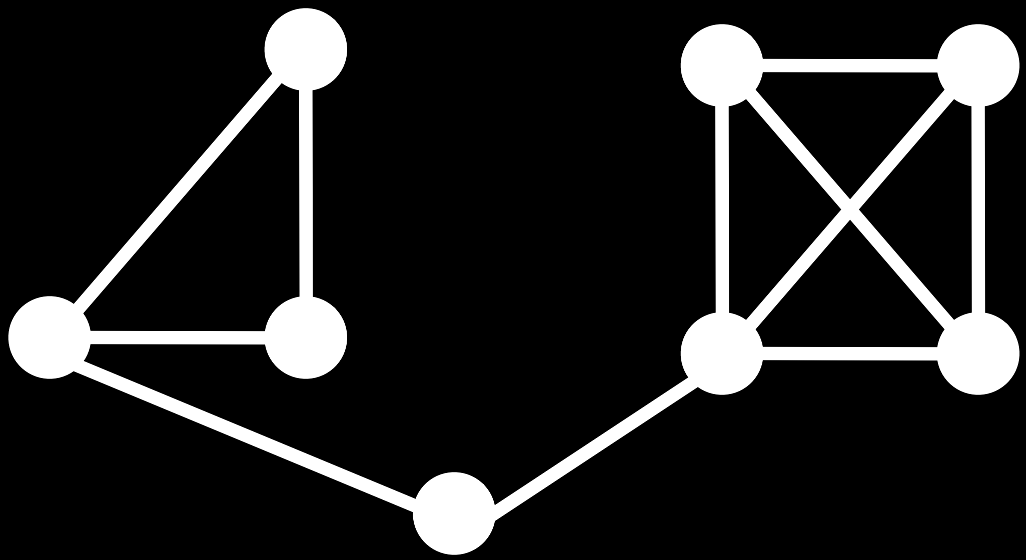
A K-clique is a set of K nodes with all $K(K-1)/2$ possible edges between them



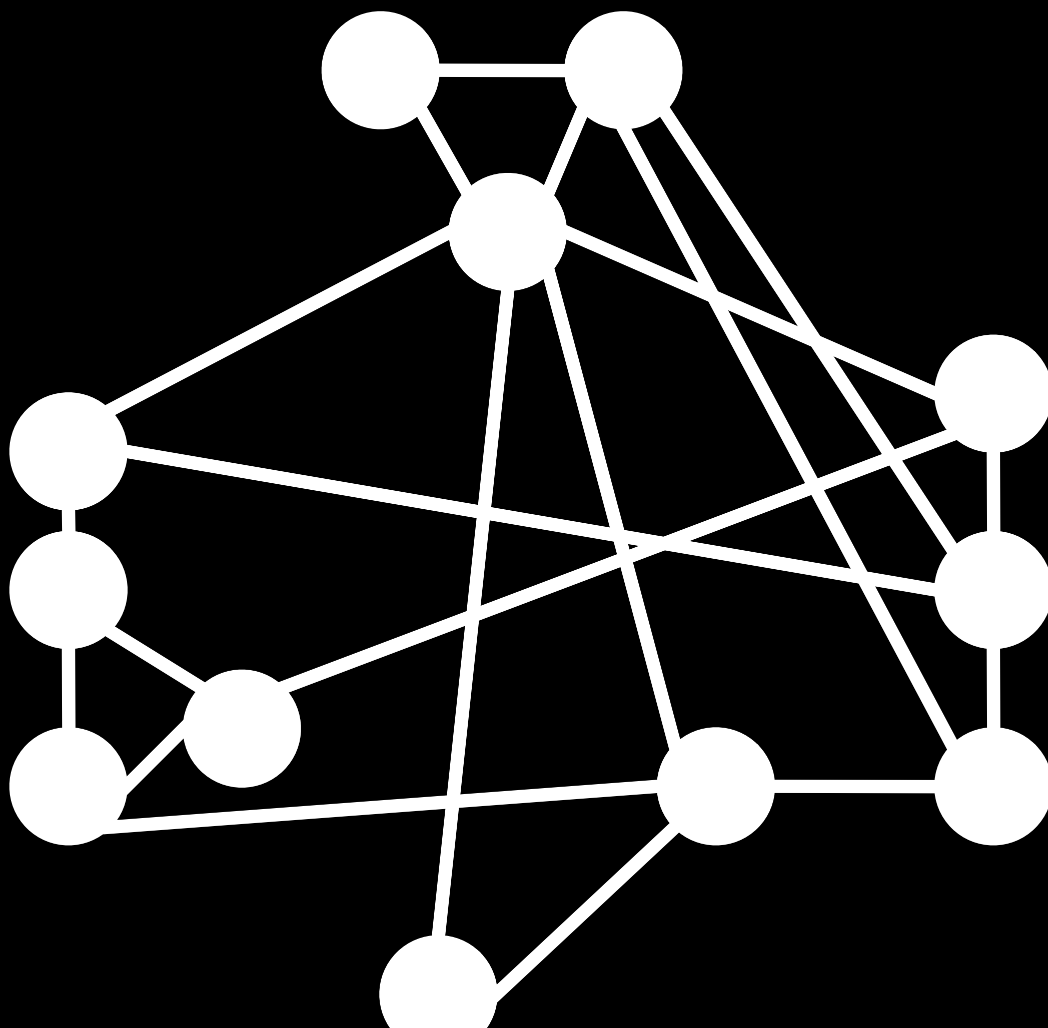
K-Cliques

A K-clique is a set of K nodes with all K(K-1)/2 possible edges between them



This graph contains a 4-clique

A Graph Named “Gadget”



Given: (G, k)
Question: Does G contain a k-clique?

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BRUTE FORCE: Try out all n choose k possible locations for the k clique

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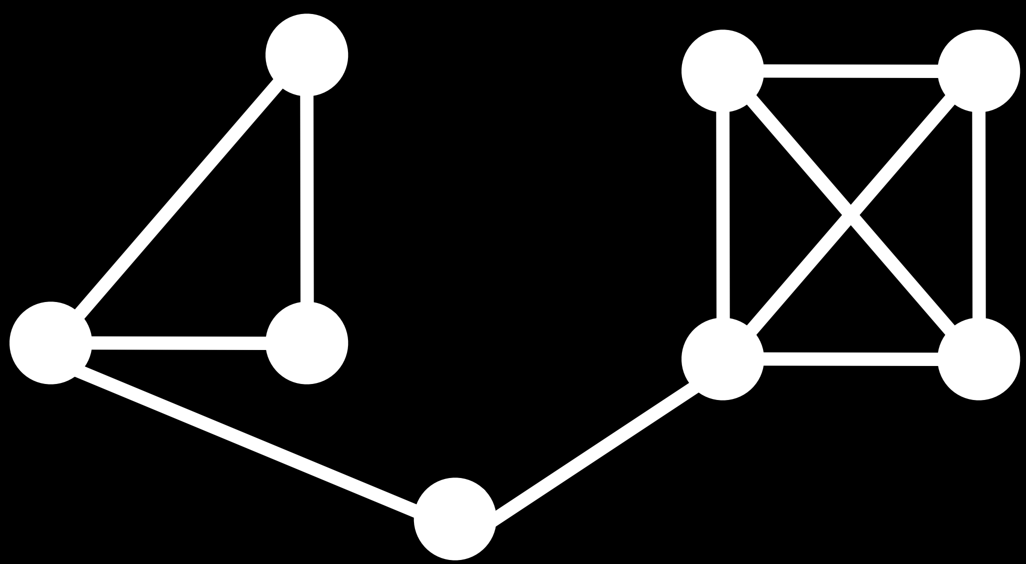

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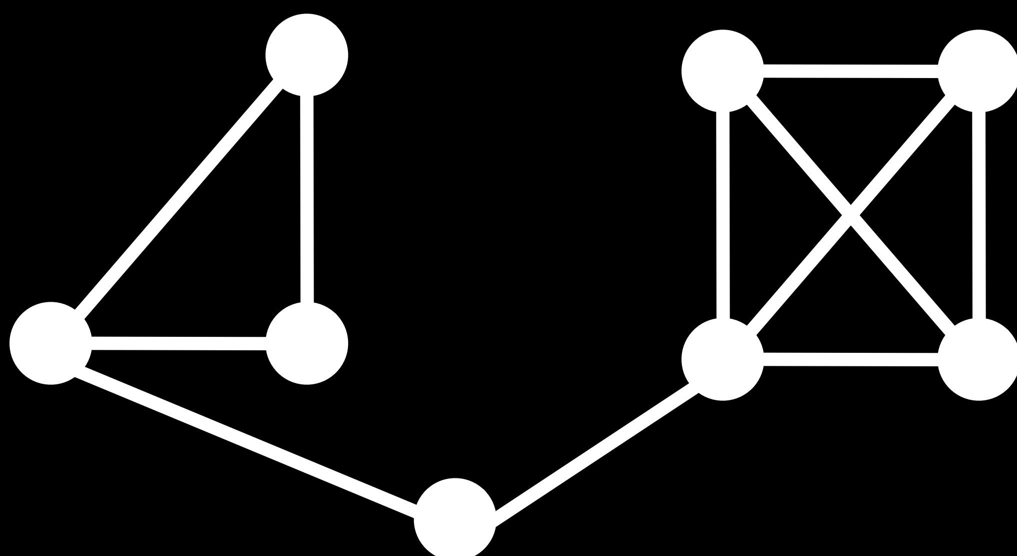

This graph contains an independent set of size 3

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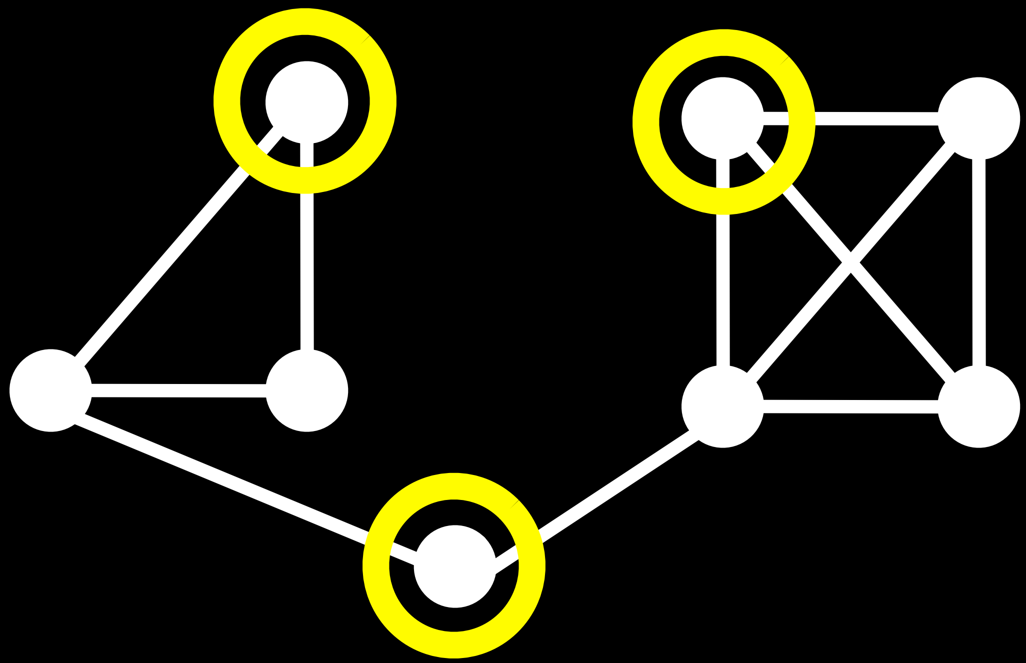

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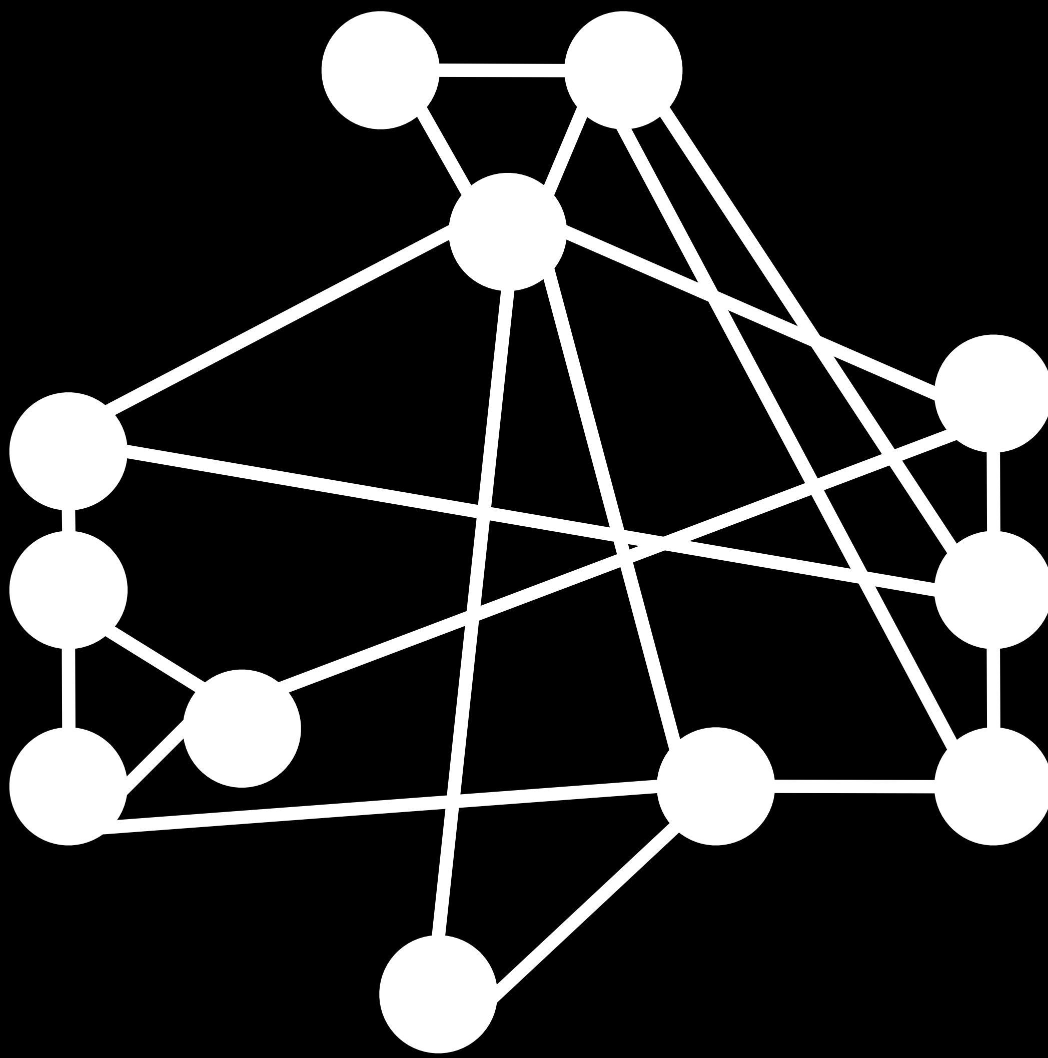
A Graph Named “Gadget”



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Clique / Independent Set

Two problems that are cosmetically different, but substantially the same

Complement of G

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Given a graph G, let G*, the complement of G, be the graph obtained by the rule that two nodes in G* are connected if and only if the corresponding nodes of G are not connected

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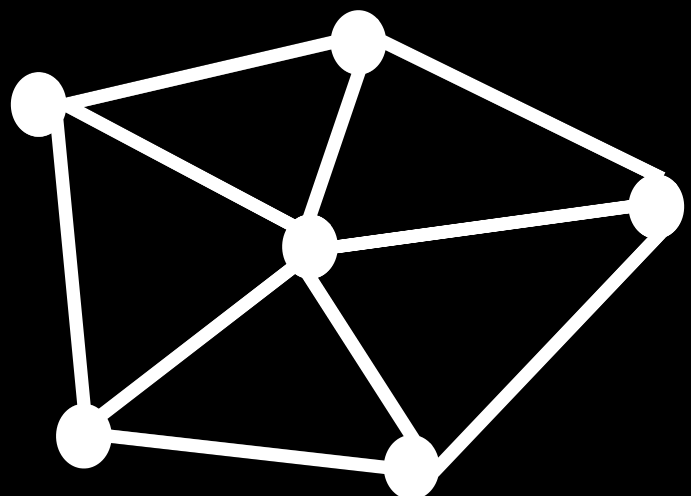

G

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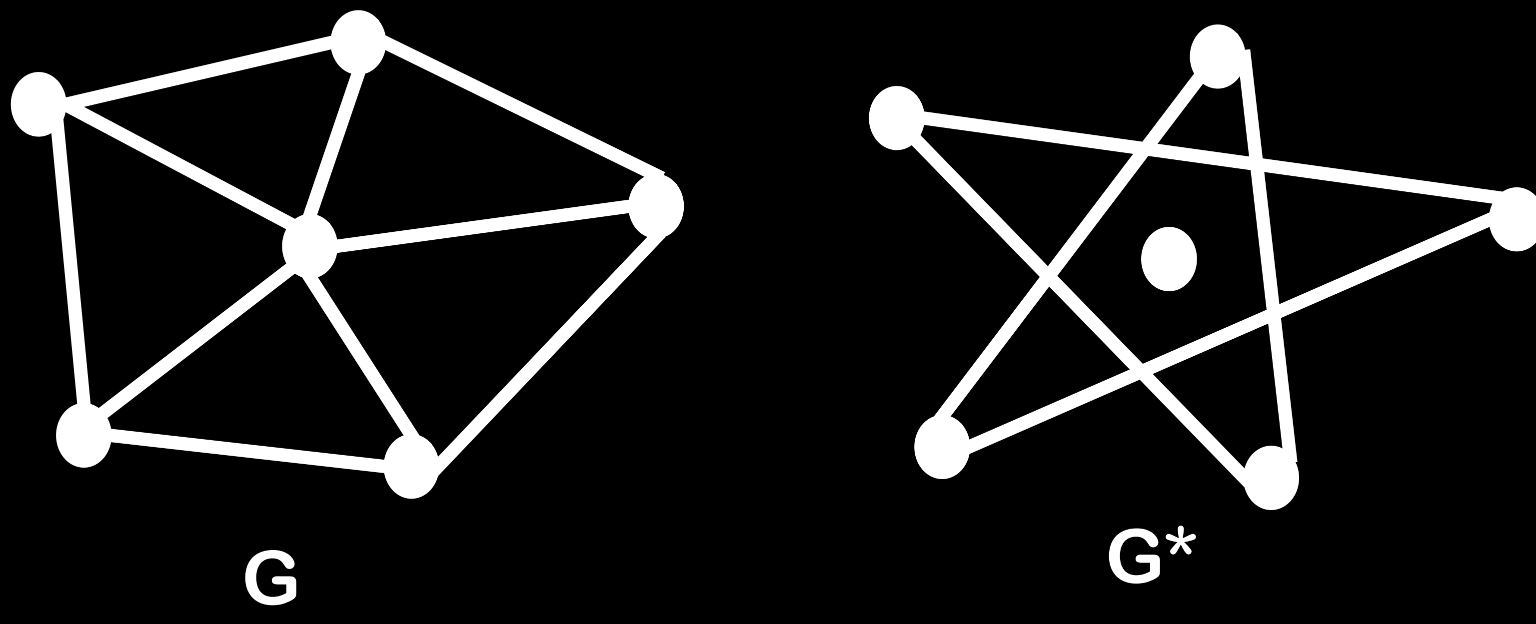

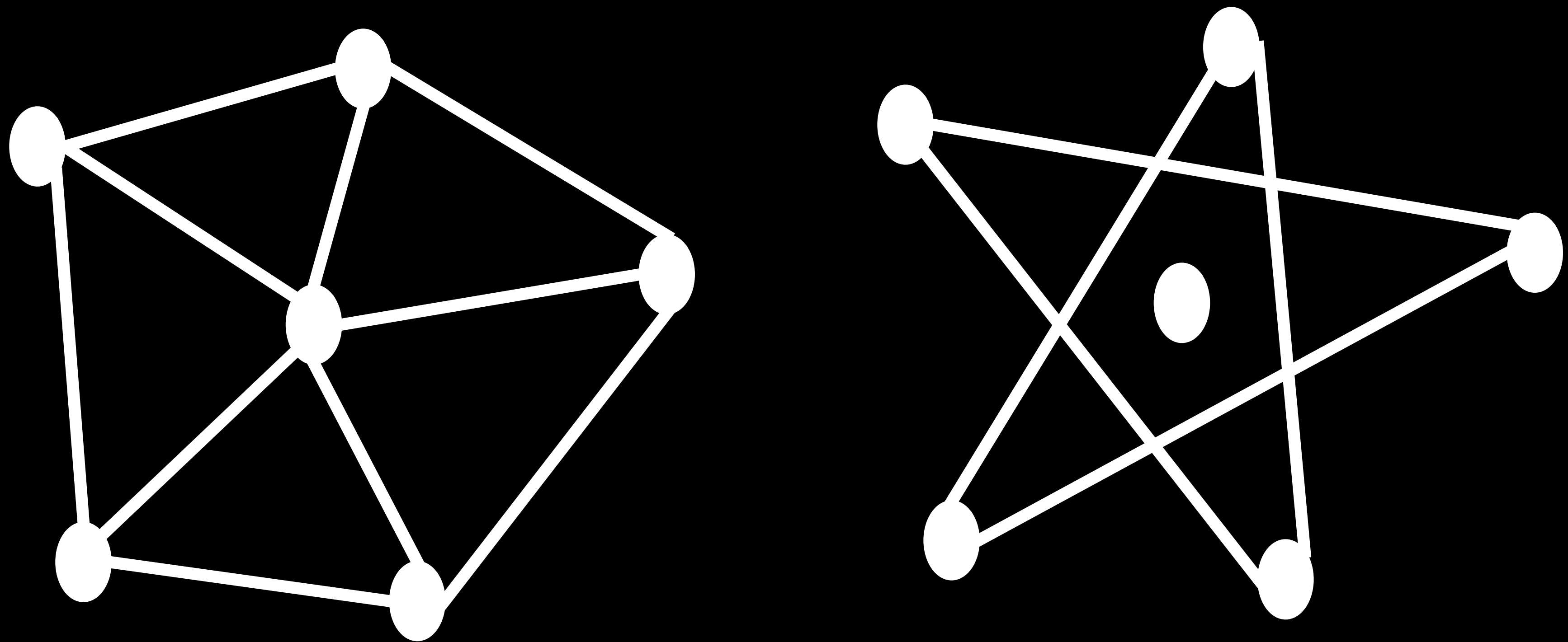

G has a k-clique $\Leftrightarrow$ G* has an independent set of size k

Let G be an n-node graph

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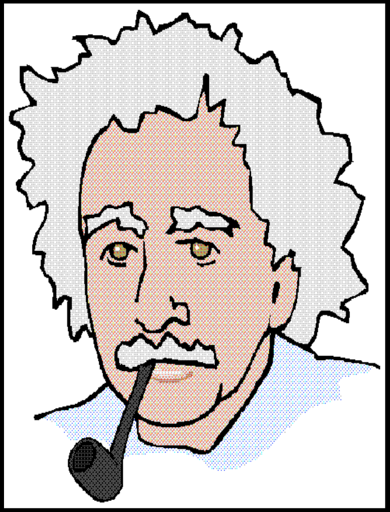


GIVEN: Clique Oracle

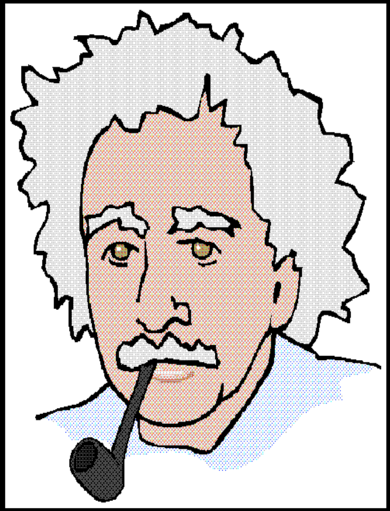
Let G be an n-node graph

BUILD:
Independent
Set Oracle

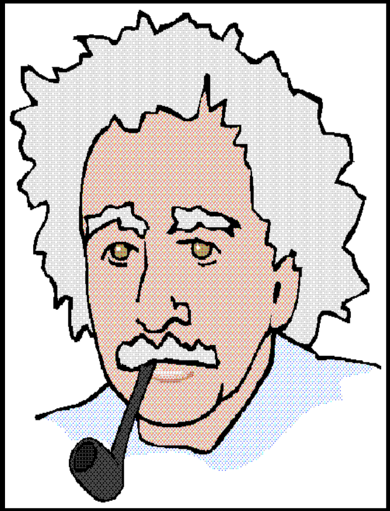
GIVEN:
Clique
Oracle

Let G be an n-node graph

(G,k)



BUILD:
Independent Set Oracle



GIVEN:
Clique Oracle

Let G be an n-node graph

(G,k)

(G*, k)

BUILD:
Independent Set Oracle

GIVEN:
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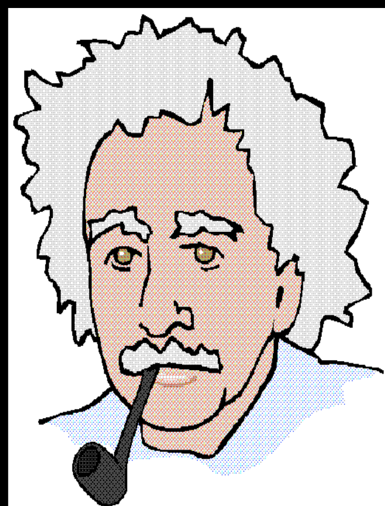

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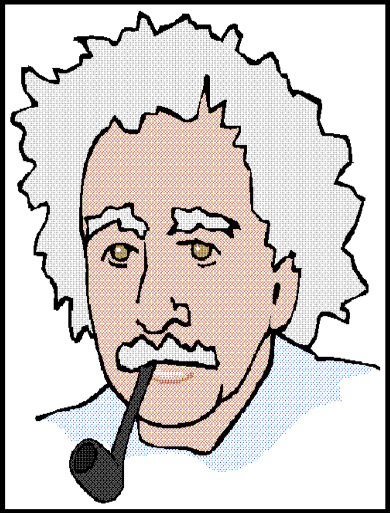
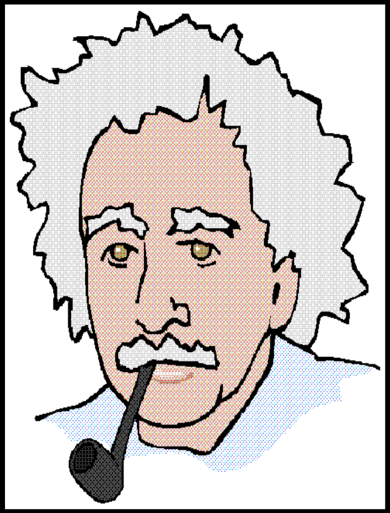


(G*, k)



BUILD:
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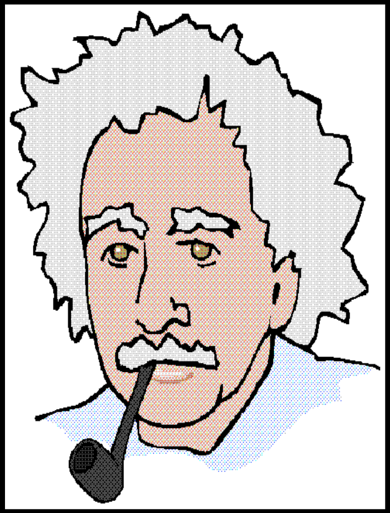

GIVEN:
Independent
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Let G be an n-node graph

BUILD:
Clique
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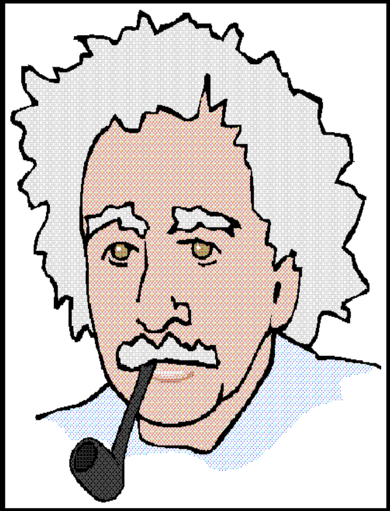

BUILD:
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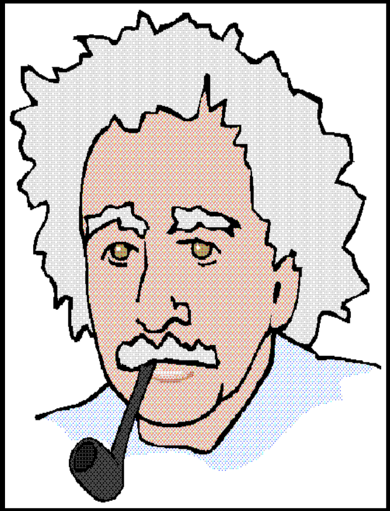

GIVEN:
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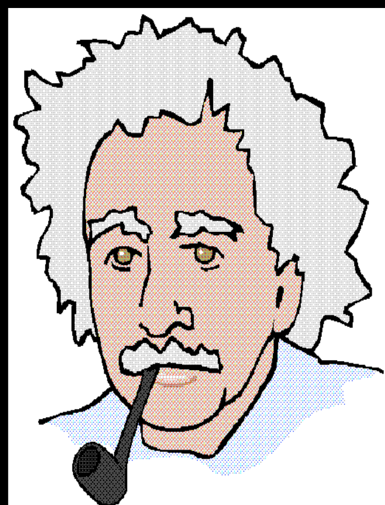
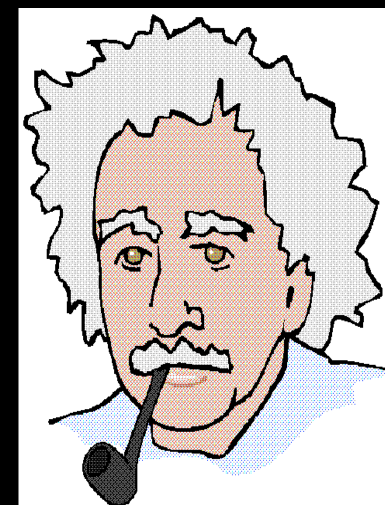


(G*, k)



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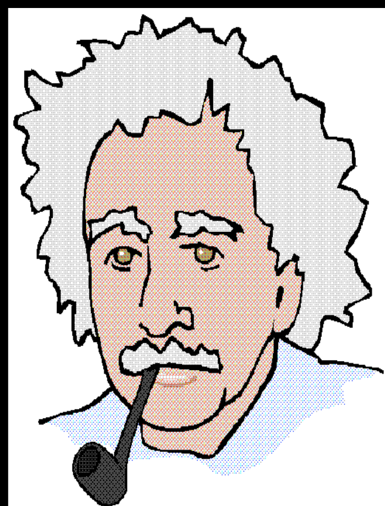

(G*, k)

BUILD:
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Let G be an n-node graph

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(G*, k)

BUILD:
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Clique / Independent Set

Two problems that are cosmetically different, but substantially the same

Thus, we can quickly reduce a clique problem to an independent set problem and vice versa

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There is a fast method for one if and only if there is a fast method for the other

Let's now look at two other problems:

1. Circuit Satisfiability
2. Graph 3-Colorability

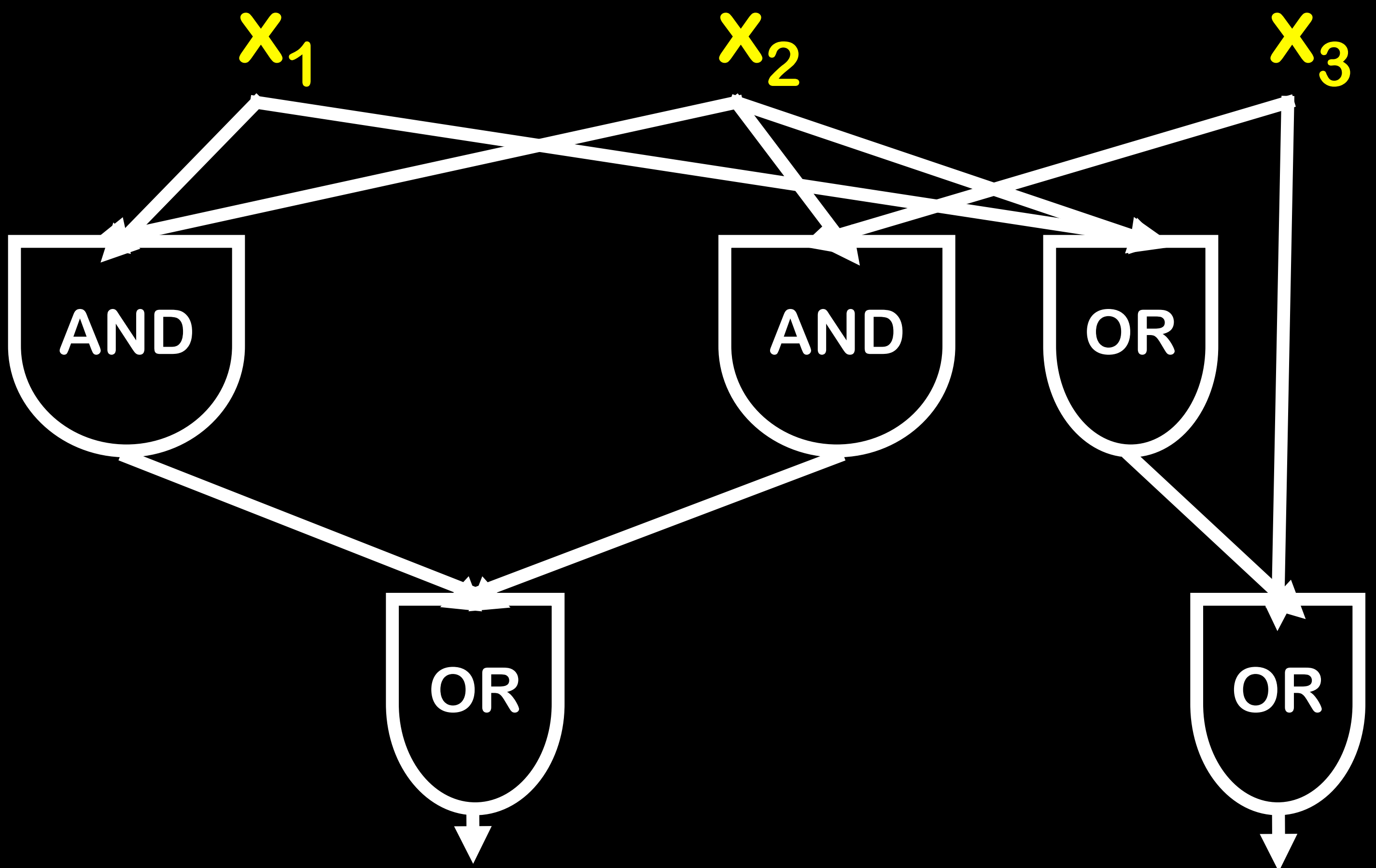
Combinatorial Circuits

Combinatorial Circuits

AND, OR, NOT, 0, 1 gates wired together with no feedback allowed

Combinatorial Circuits

AND, OR, NOT, 0, 1 gates wired together with no feedback allowed



Circuit-Satisfiability

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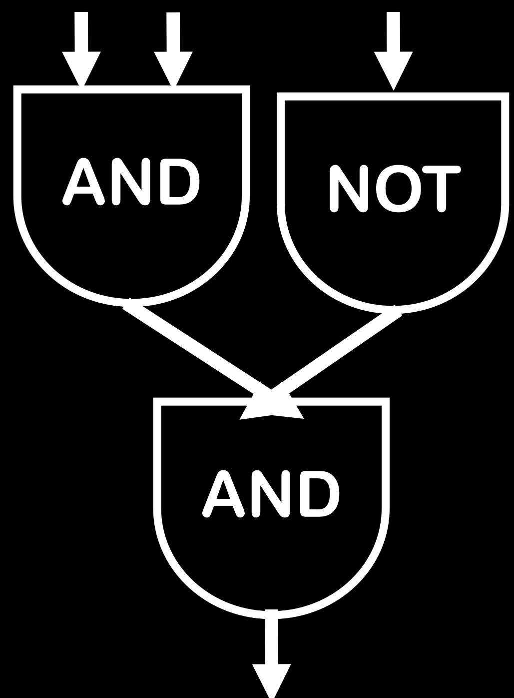

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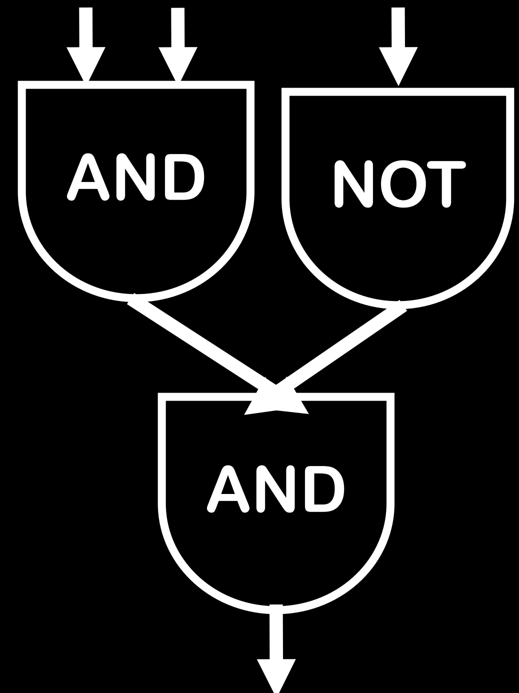
Yes, this circuit is satisfiable: 110



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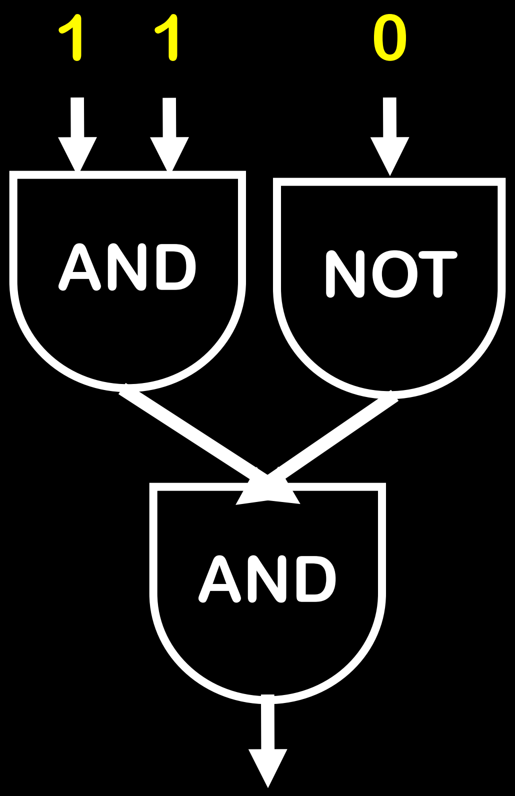

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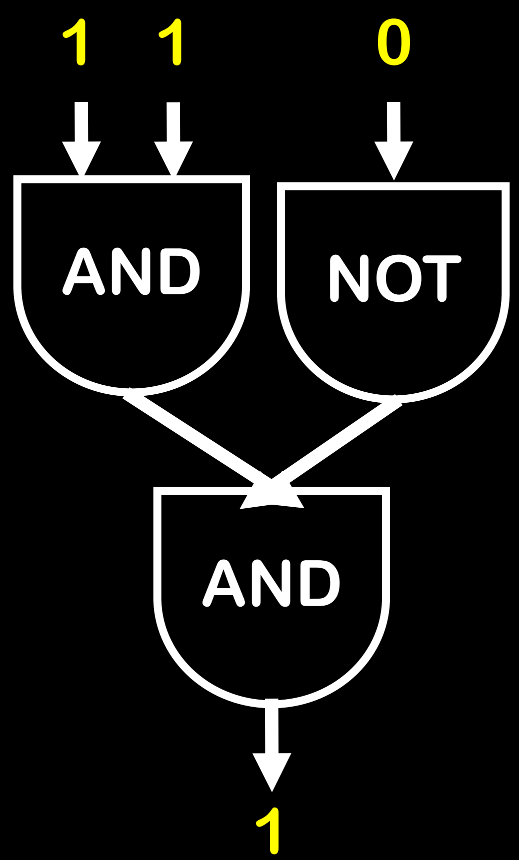

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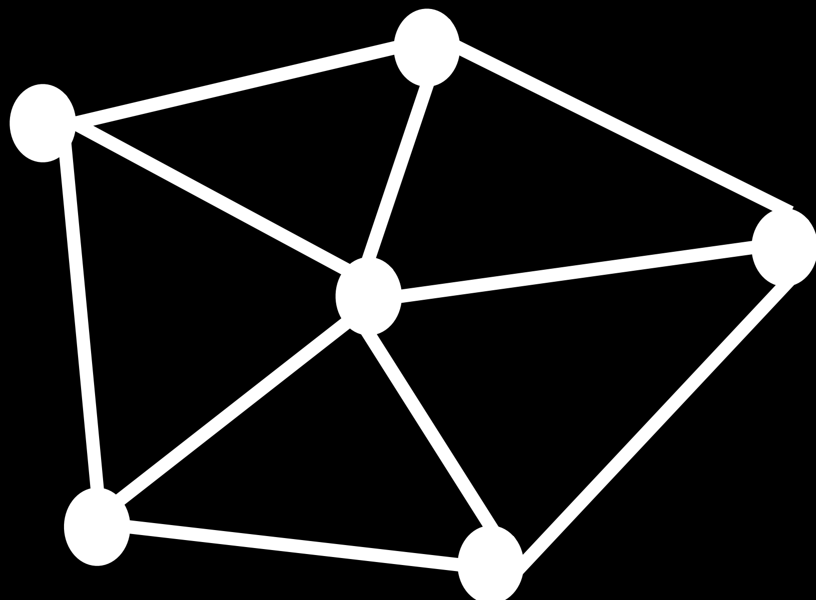
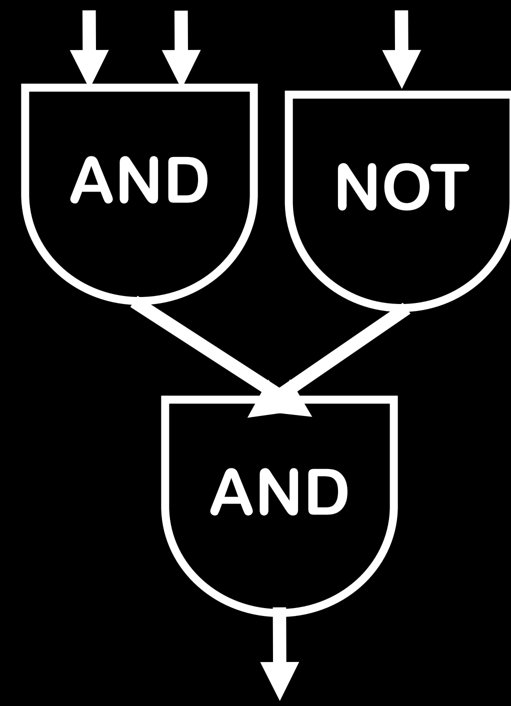
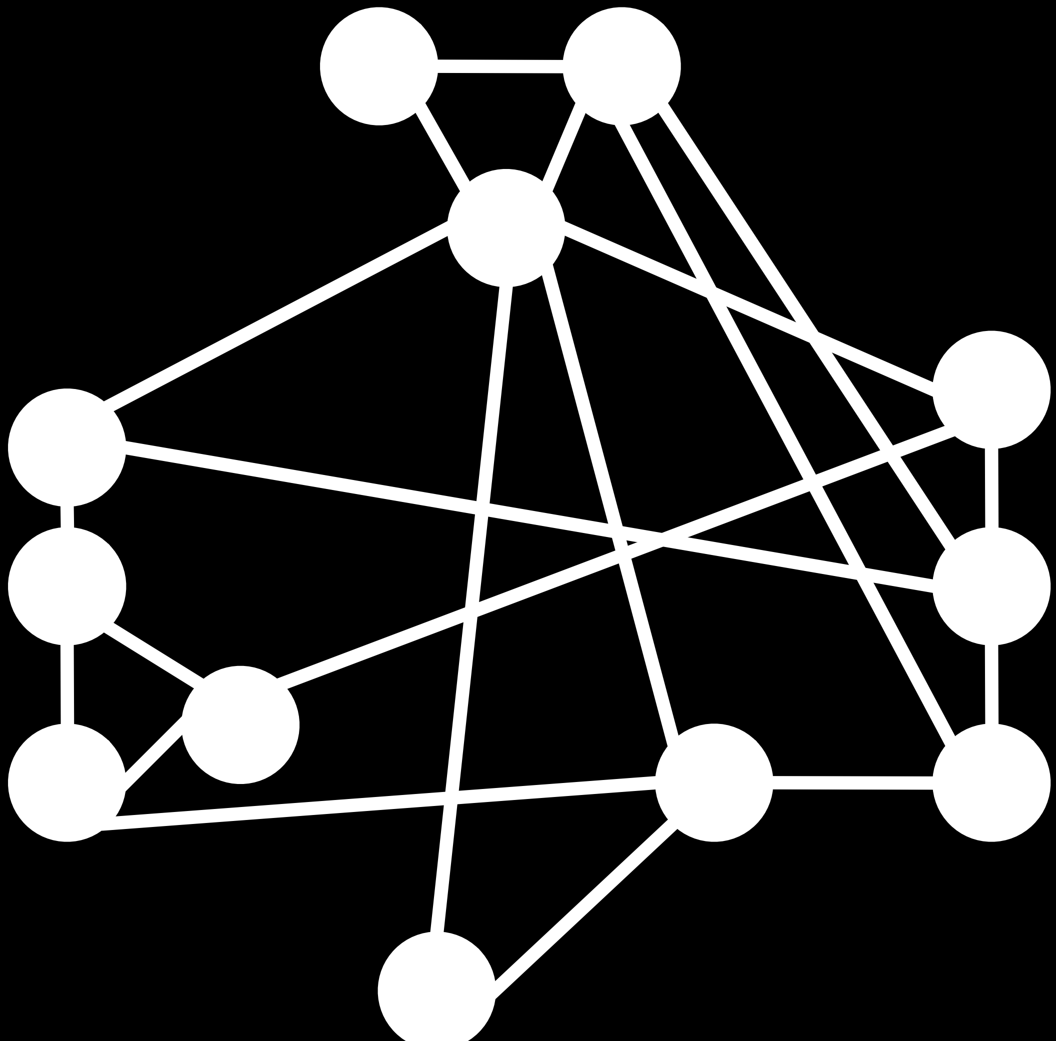
BRUTE FORCE: Try out all 2^n assignments

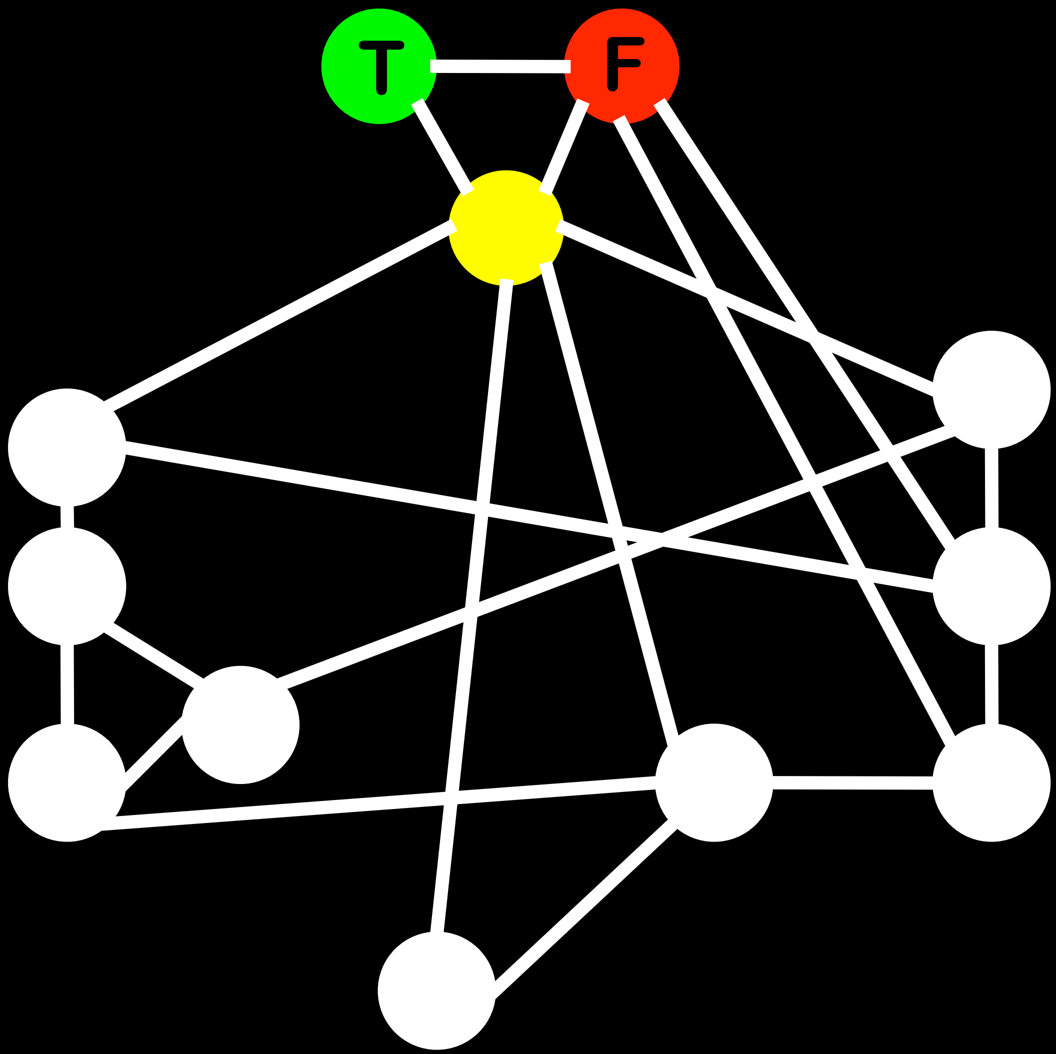
3-Colorability

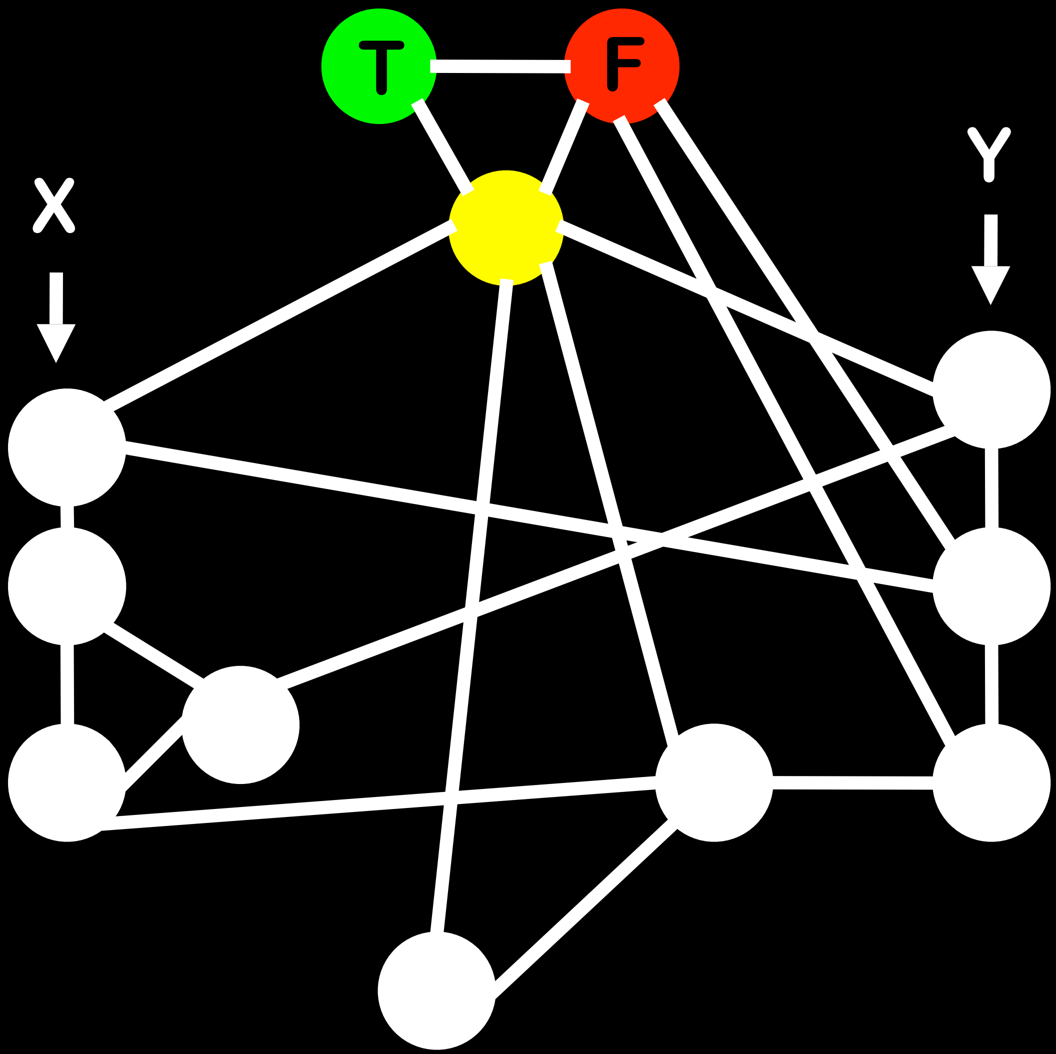


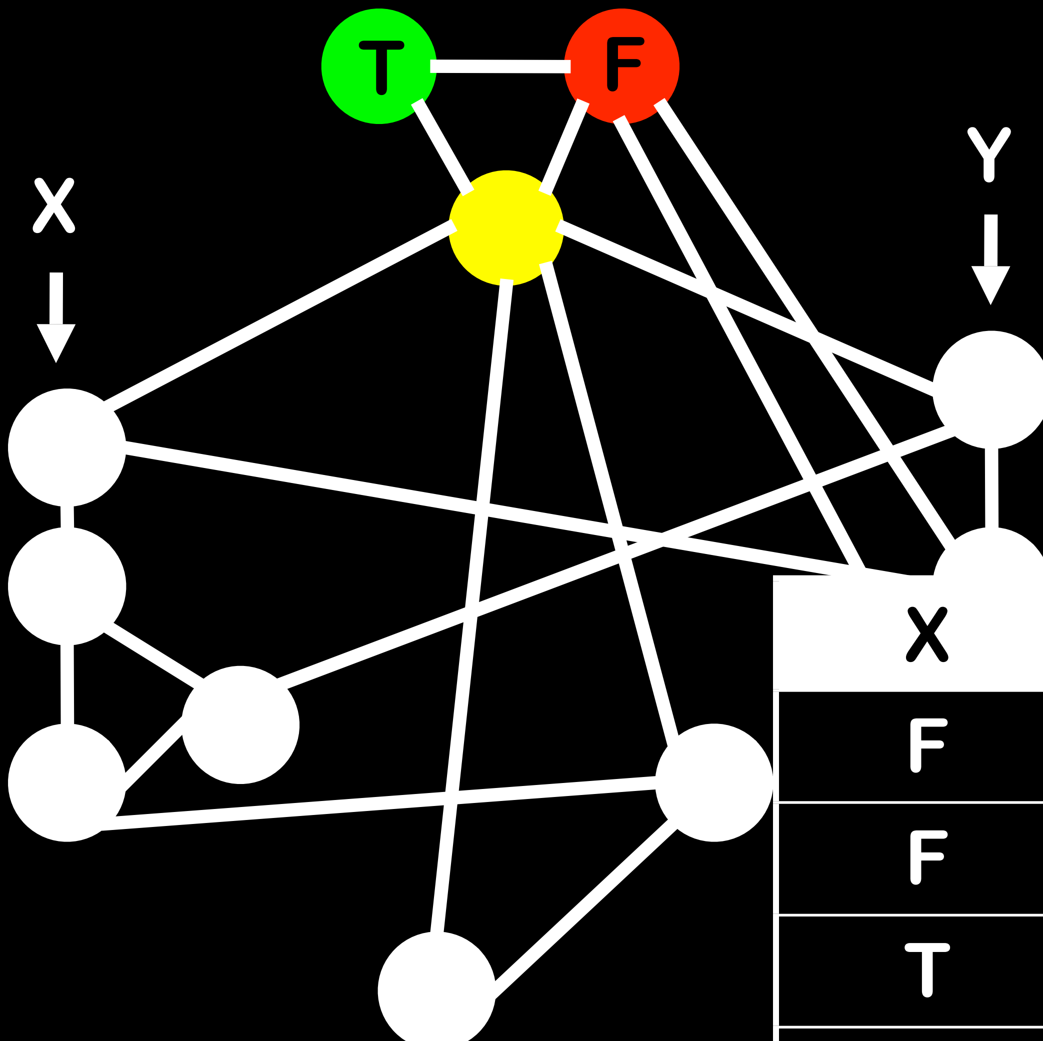
Circuit Satisfiability



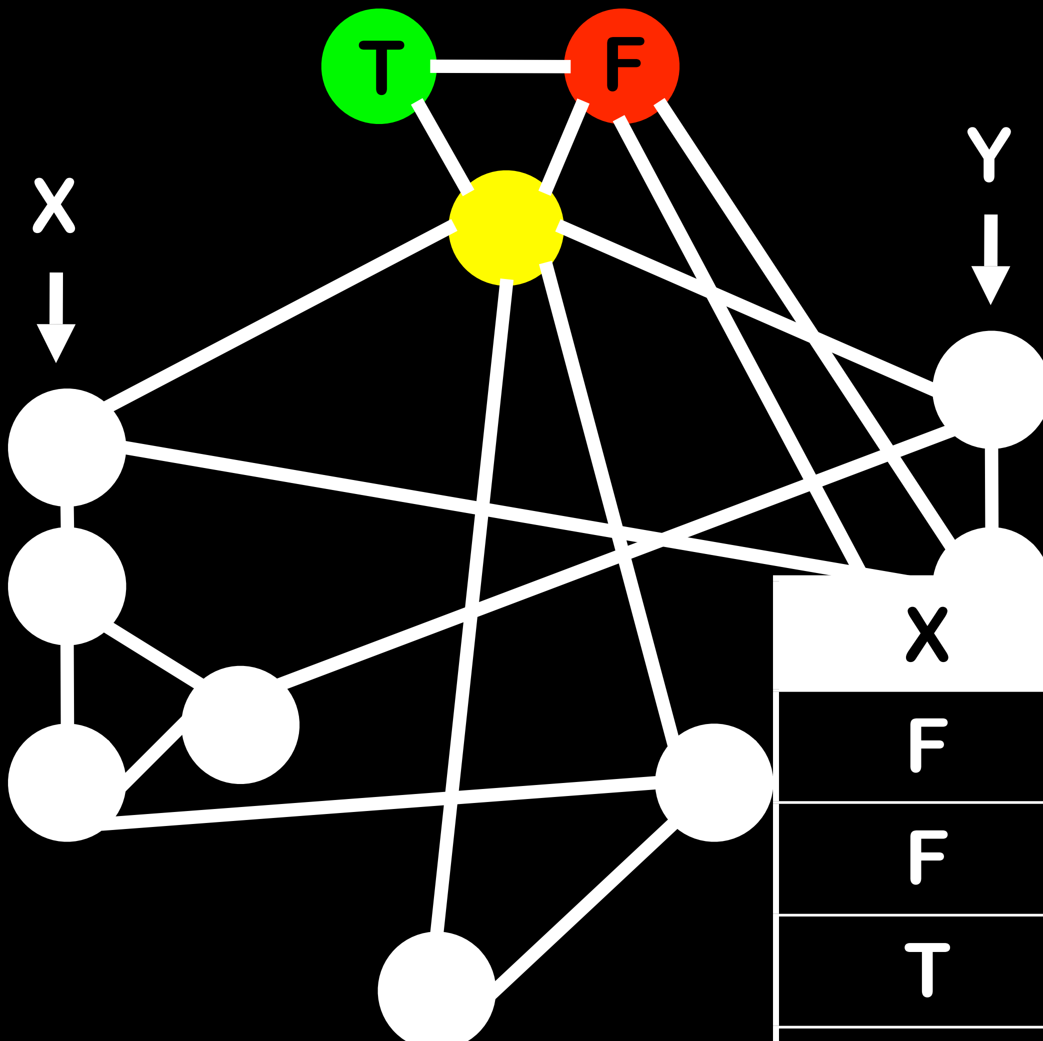



T
F

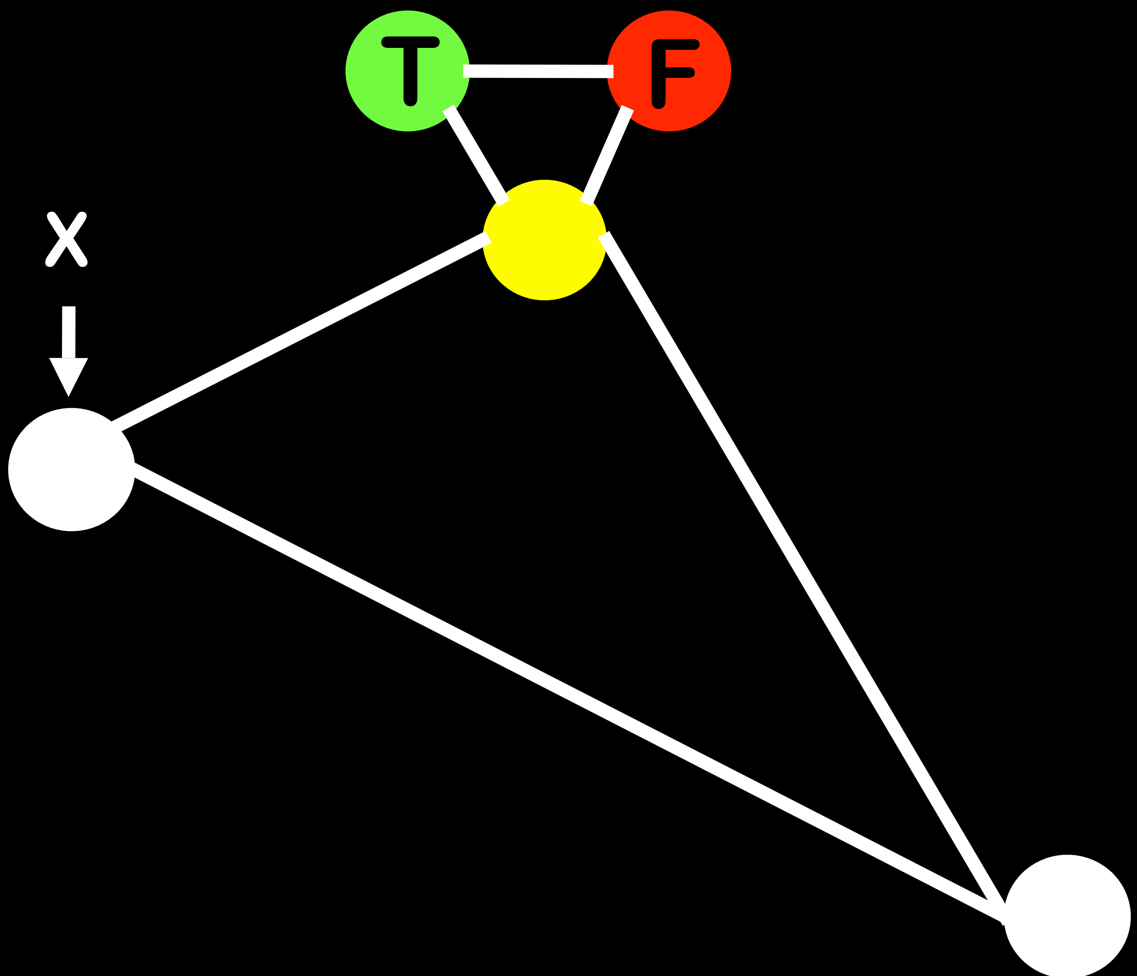

T
F
X
Y

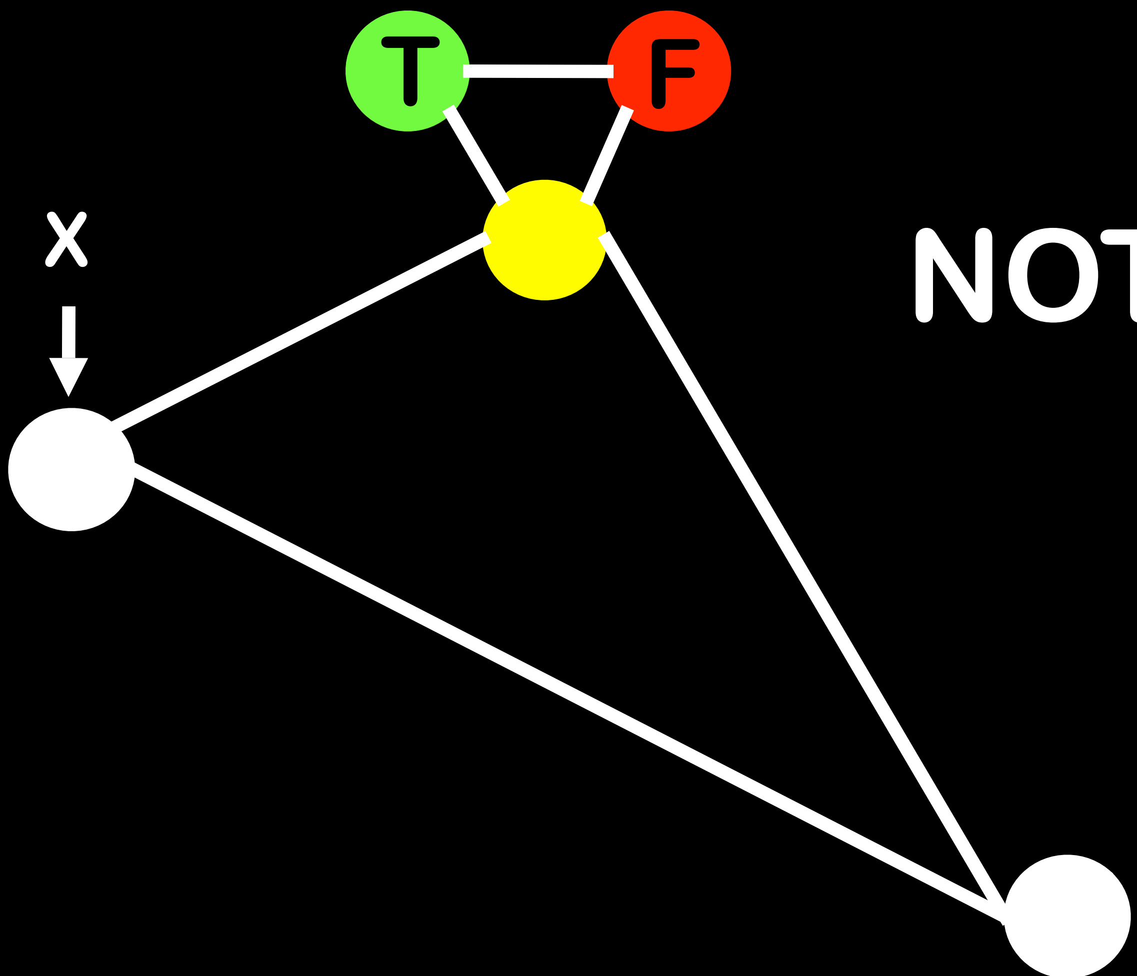


X	Y	
F	F	F
F	T	T
T	F	T
T	T	T

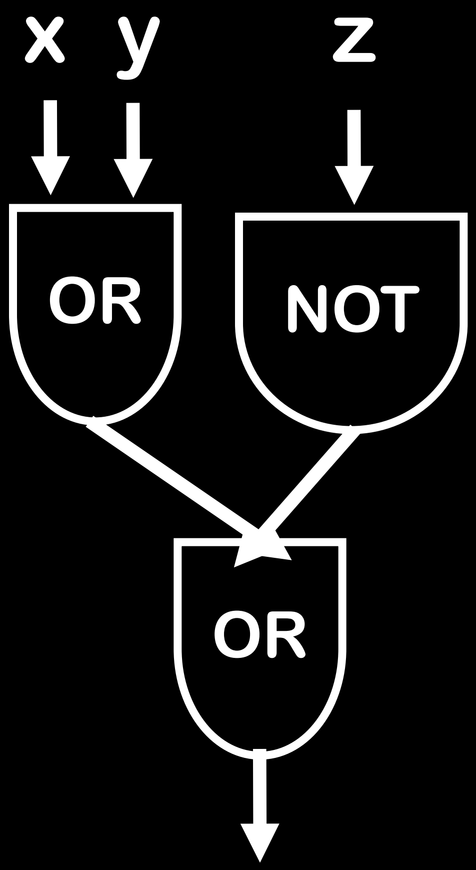


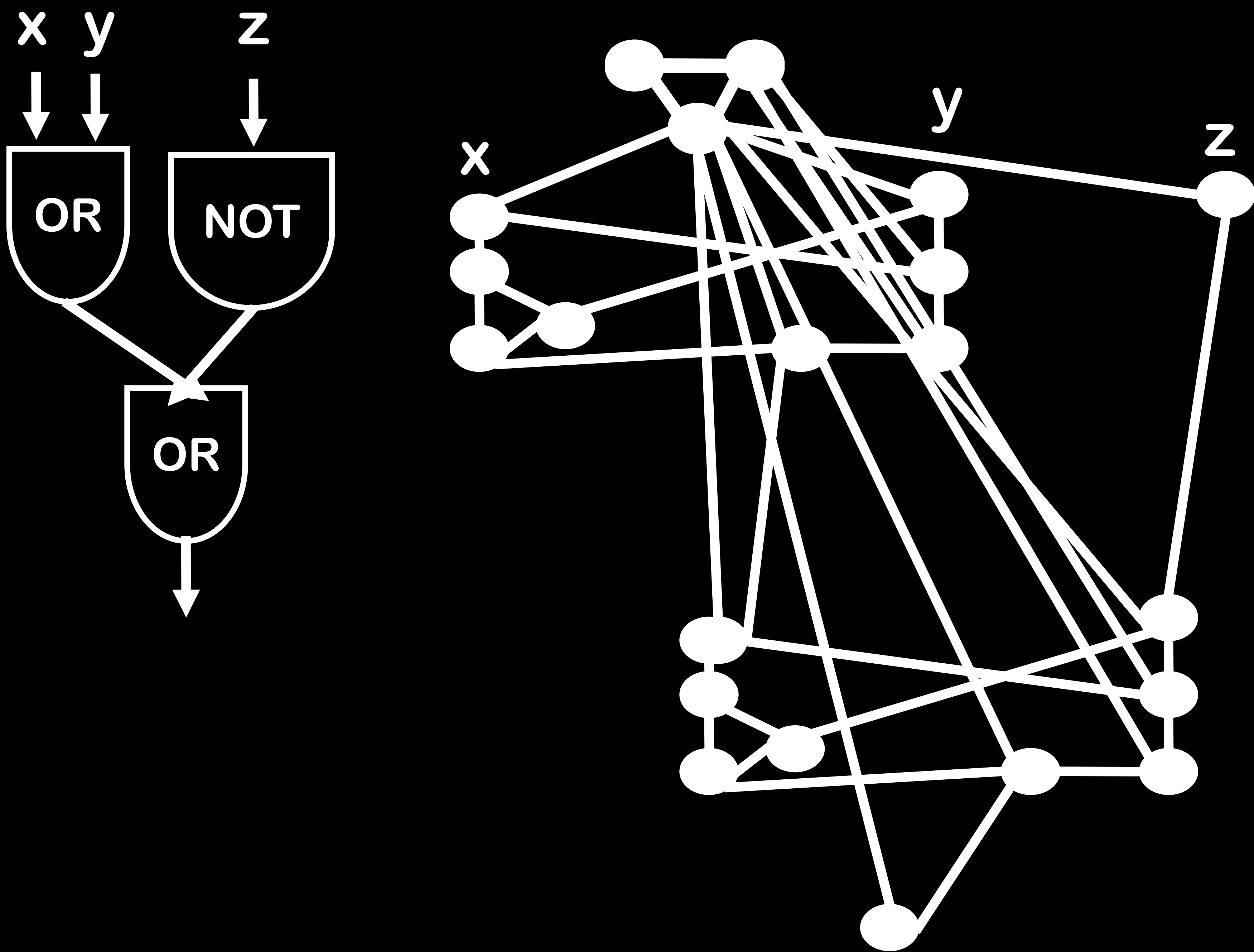
X	Y	OR
F	F	F
F	T	T
T	F	T
T	T	T

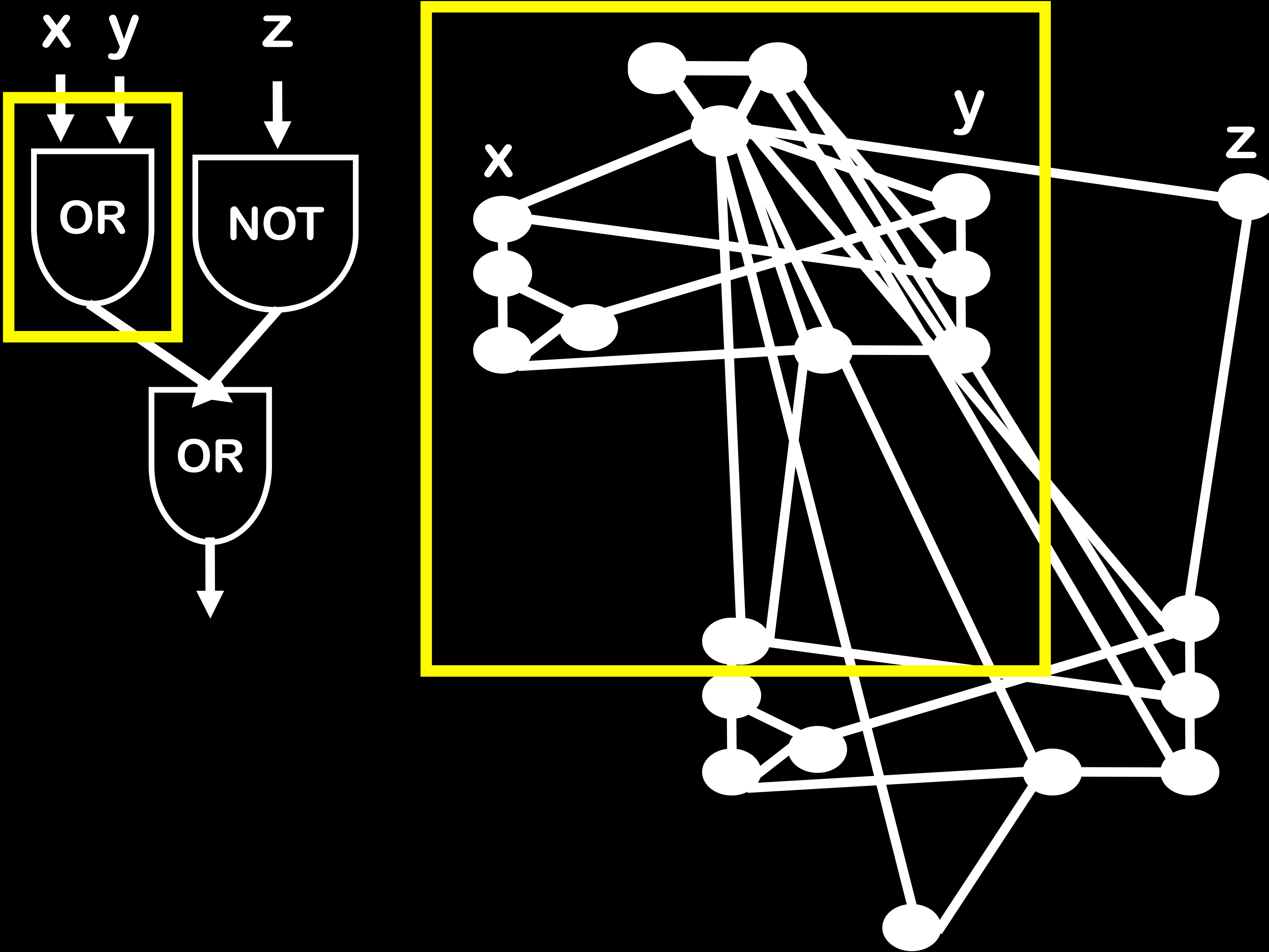

T
F
x

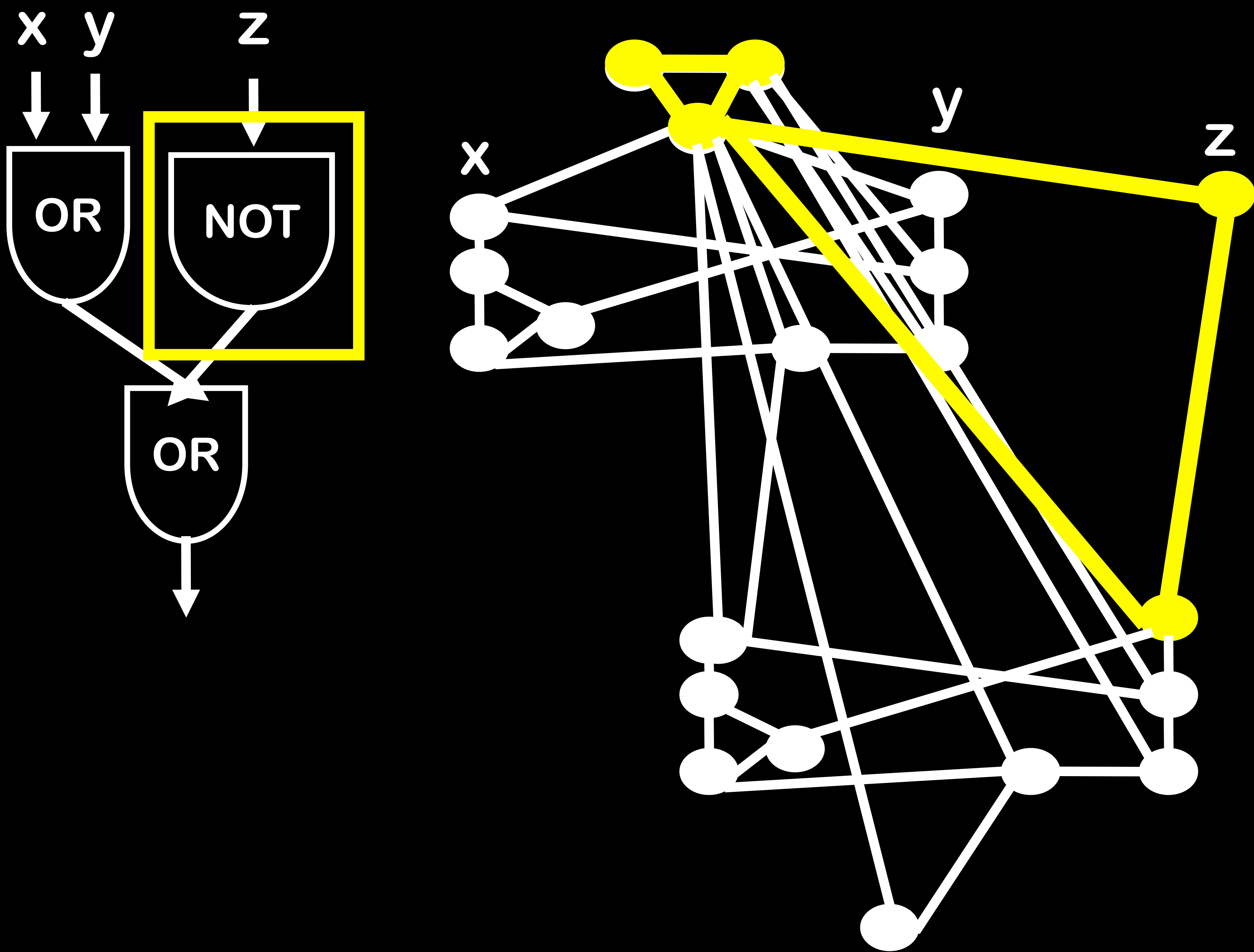


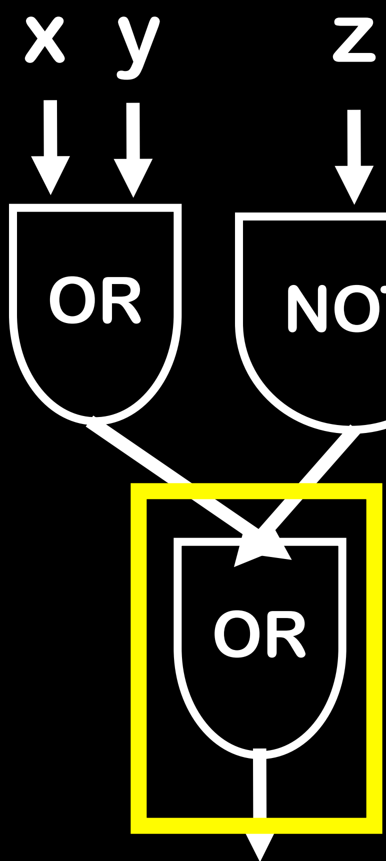
NOT gate!

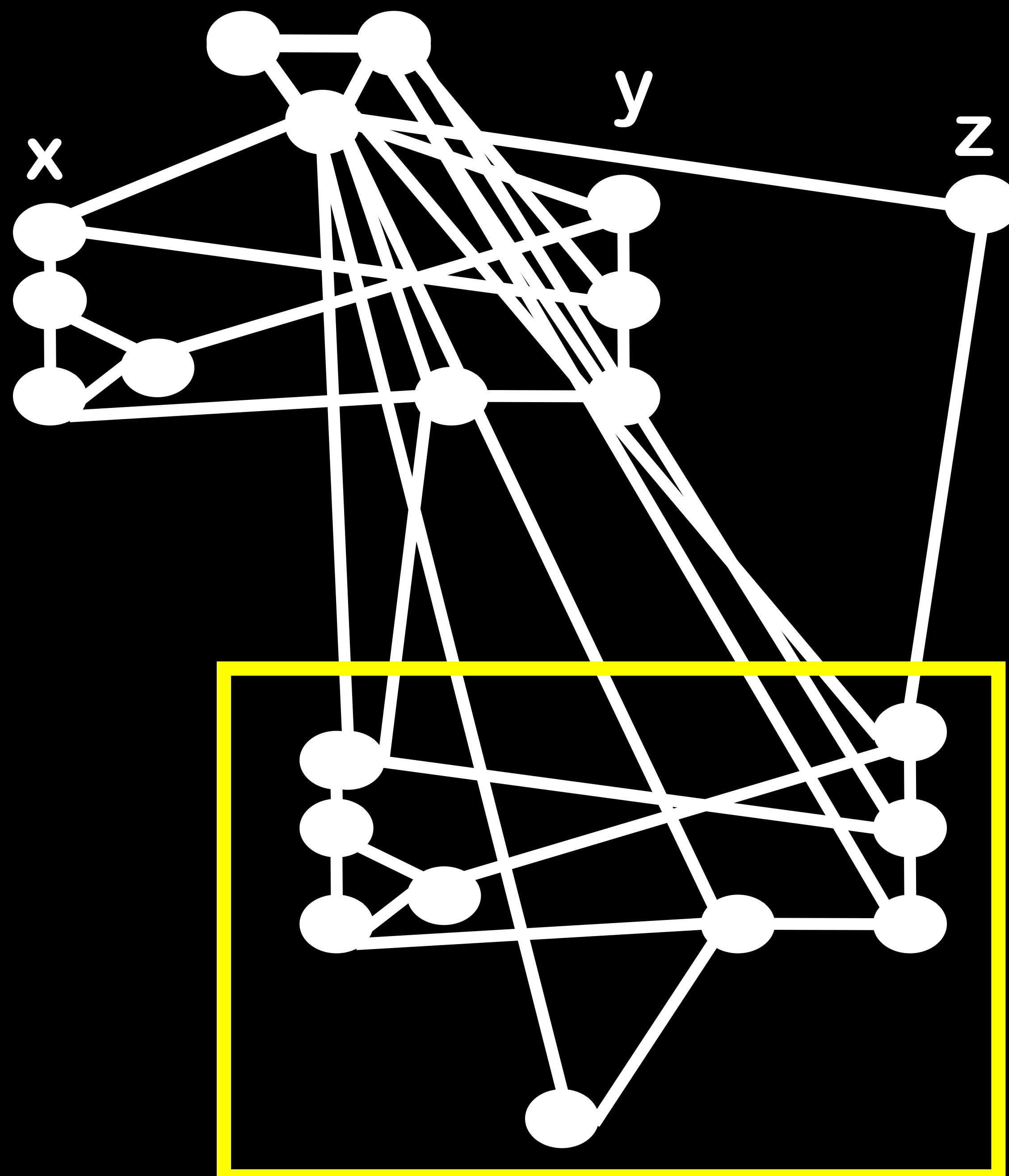

x
y
z
OR
NOT
OR

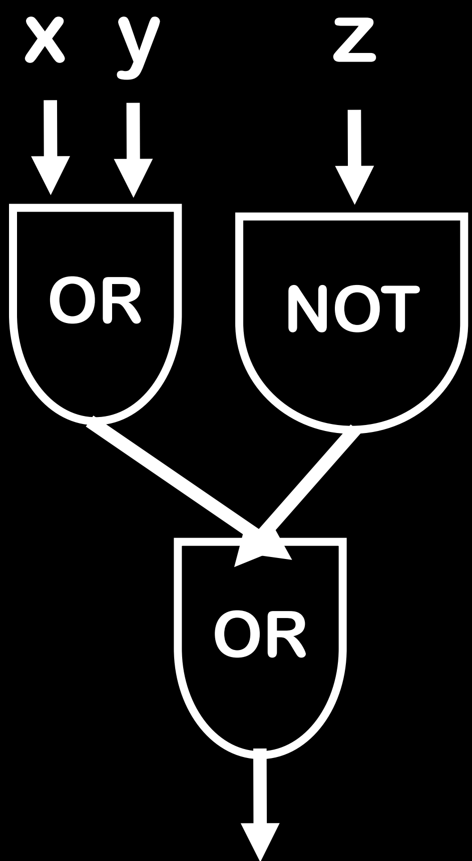

x
y
z
OR
NOT
OR
x
y
z

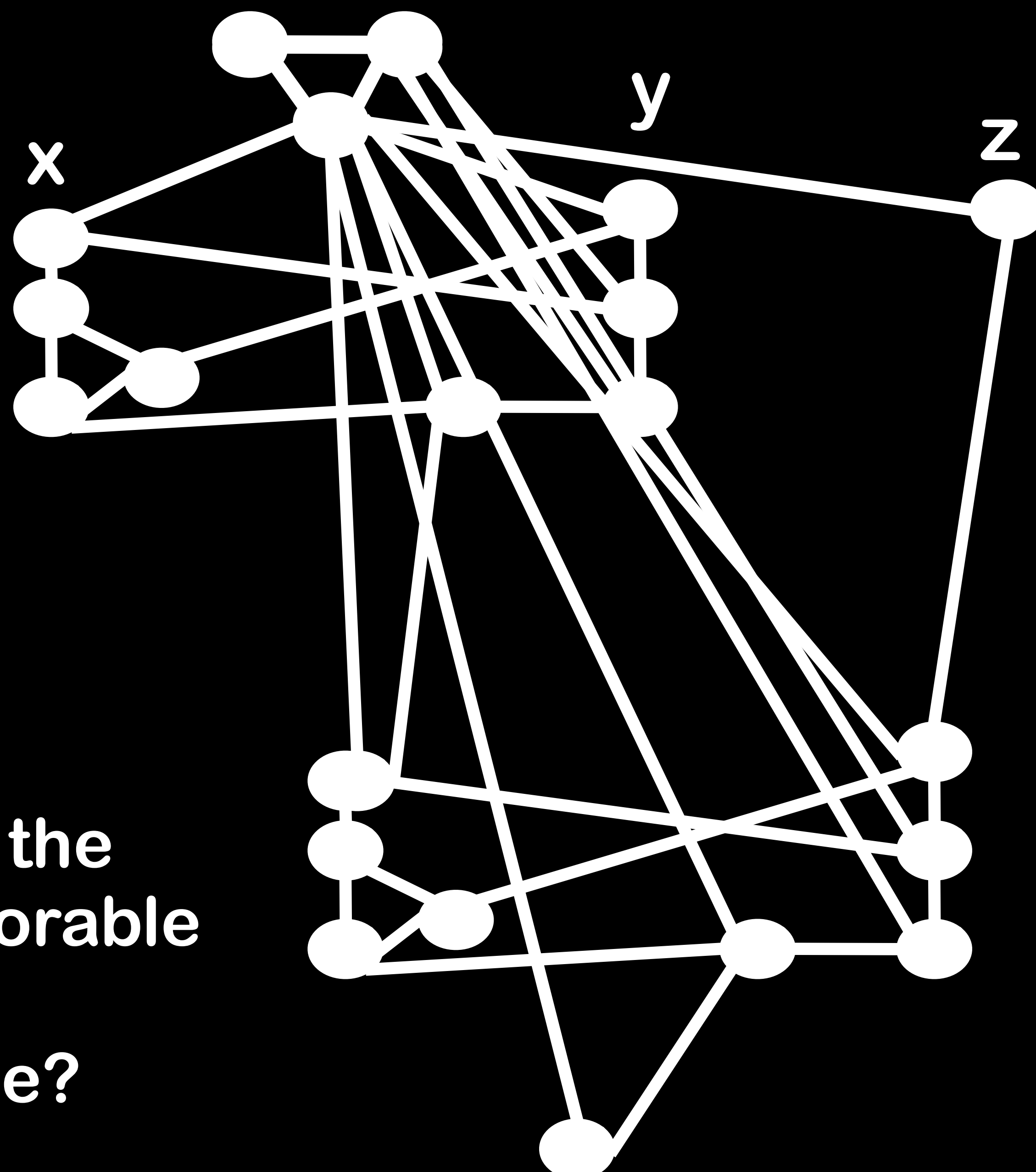

x
y
z
OR
NOT
OR
y
x
z


x
y
z
OR
NOT
OR
y
x
z

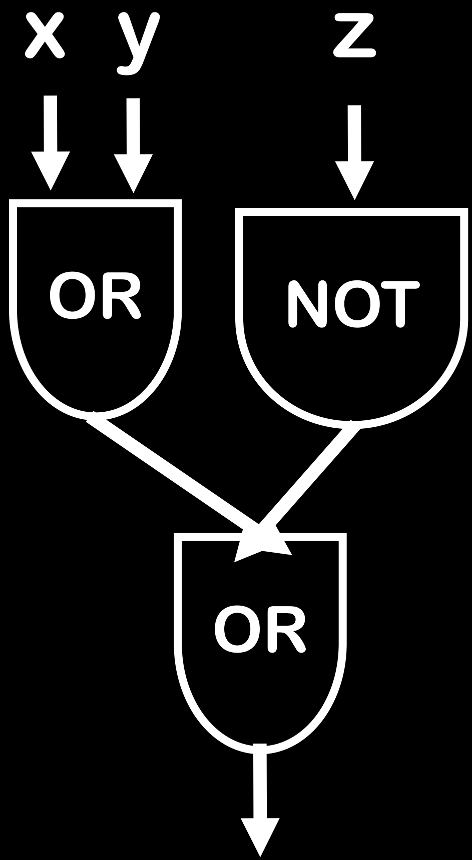

x
y
z
OR
NOT
OR

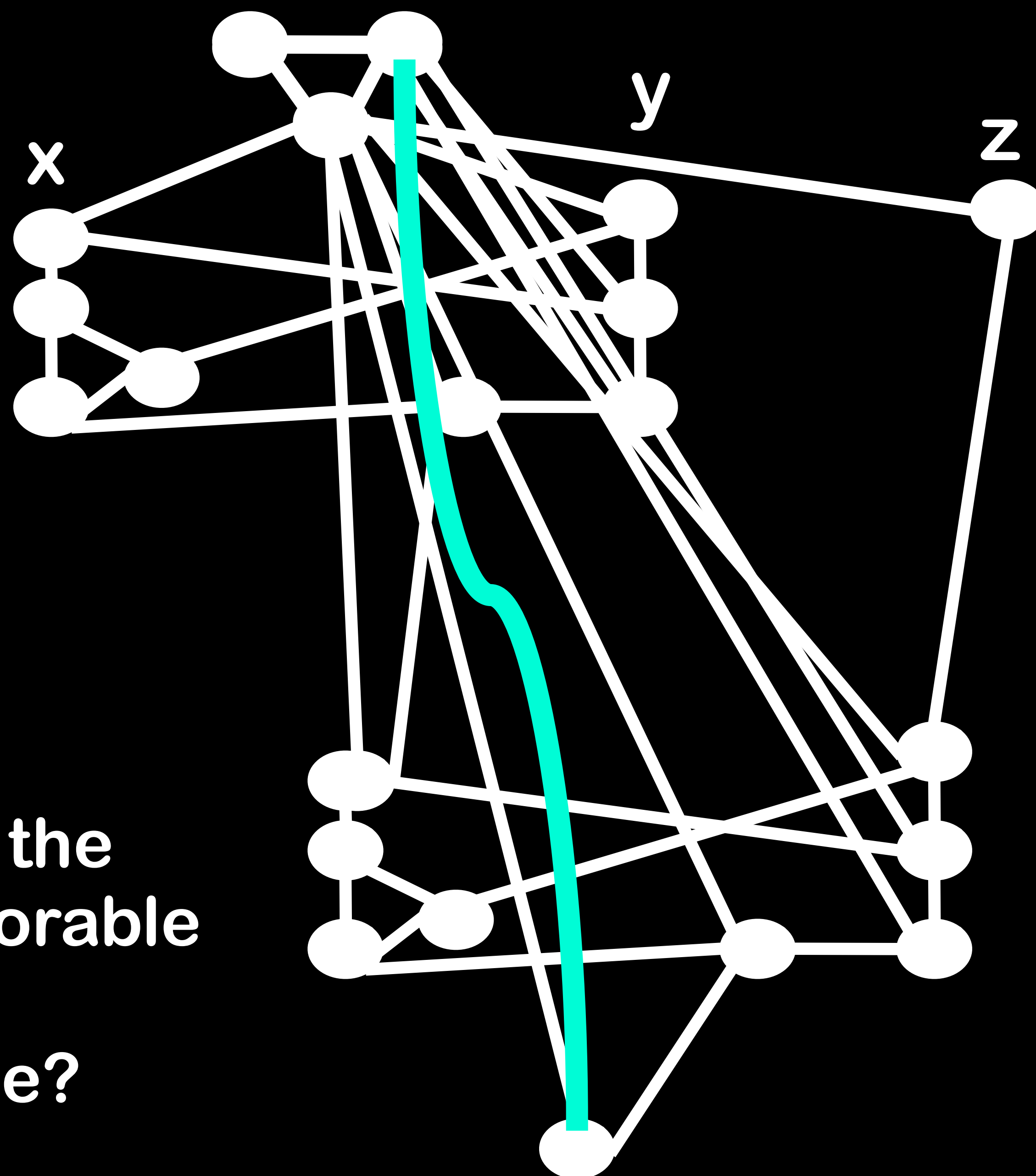

x
y
z





How do we force the graph to be 3 colorable exactly when the circuit is satifiable?





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Let C be an n-input circuit

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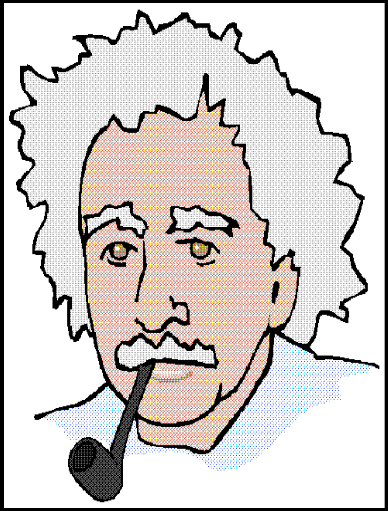

GIVEN:
3-color
Oracle

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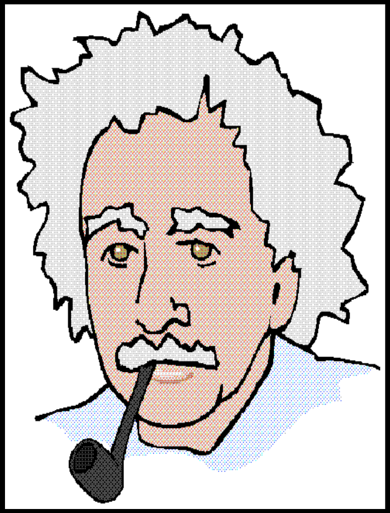
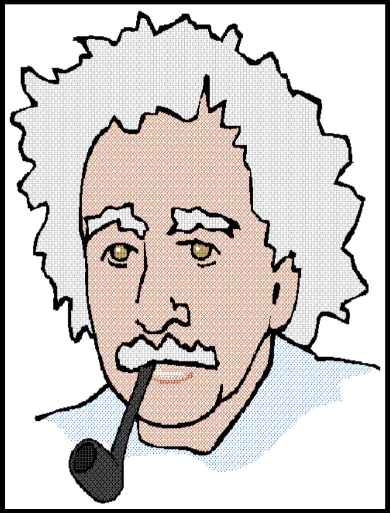


BUILD:
SAT
Oracle

GIVEN:
3-color
Oracle

Let C be an n-input circuit

C



BUILD:
SAT
Oracle

GIVEN:
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Let C be an n-input circuit

C

Graph composed of gadgets that mimic the gates in C

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SAT
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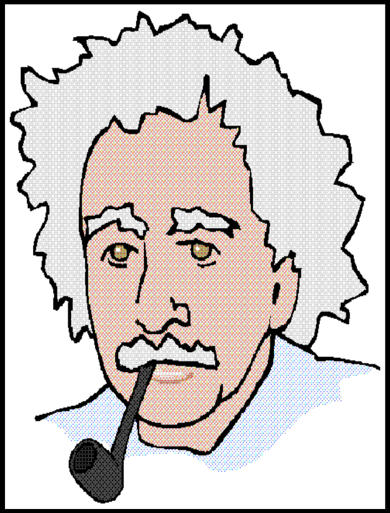
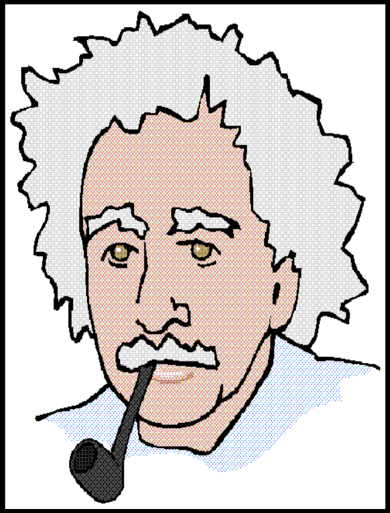


BUILD:
SAT
Oracle



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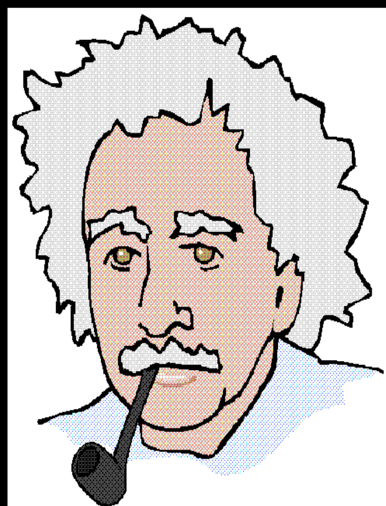
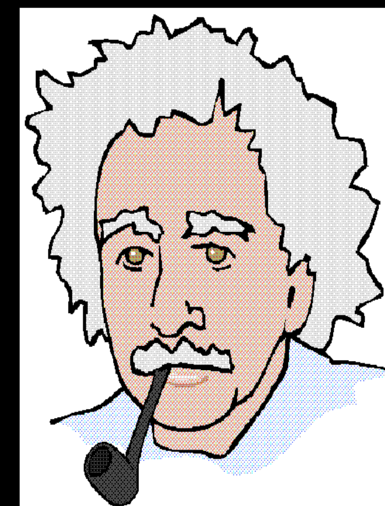
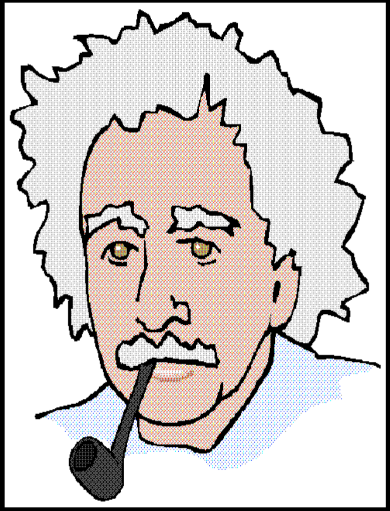


BUILD:
SAT
Oracle

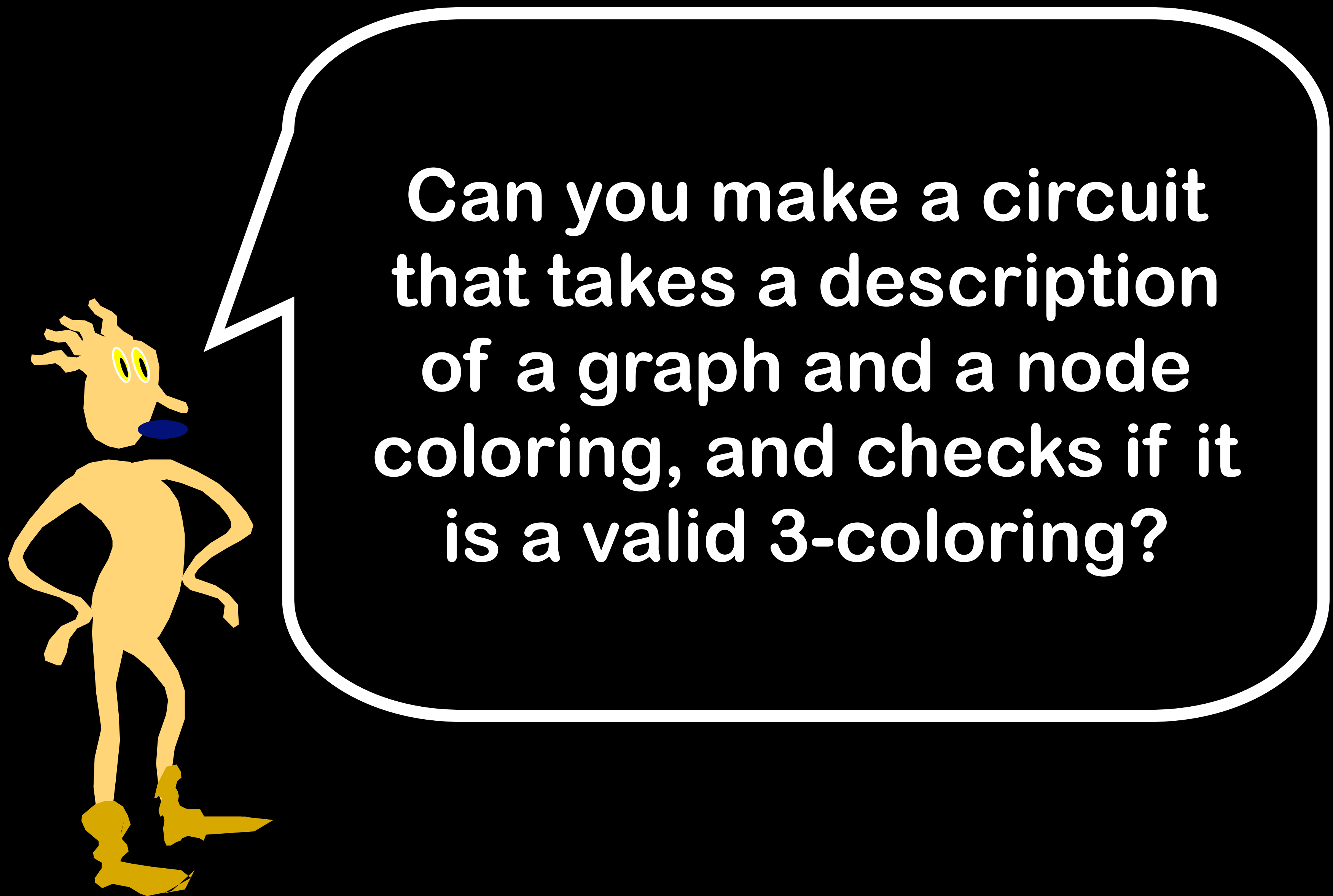


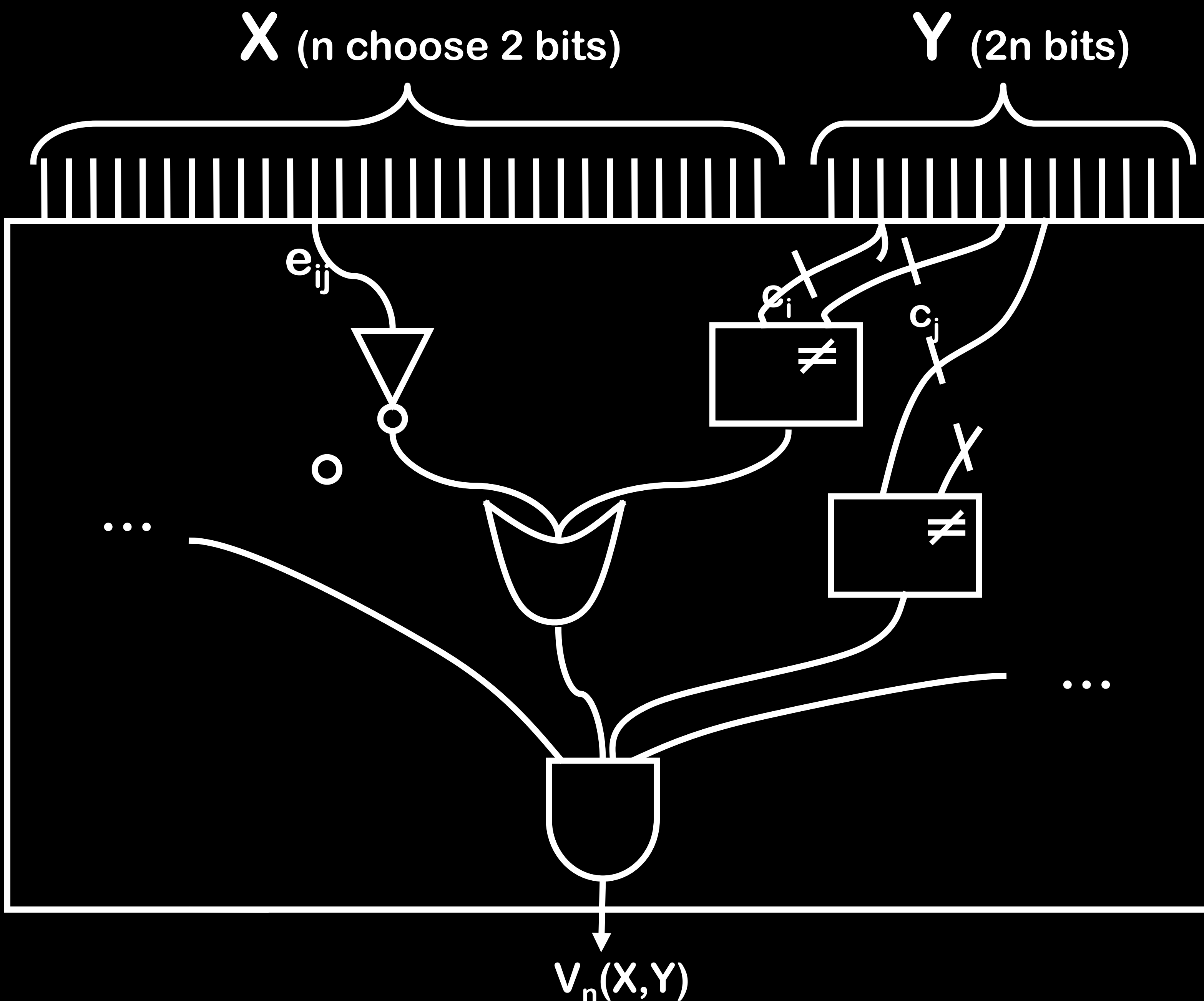
GIVEN:
3-color
Oracle

You can quickly transform a method to decide 3-coloring into a method to decide circuit satifiability!



Given an oracle for circuit SAT, how can you quickly solve 3-colorability?


Can you make a circuit that takes a description of a graph and a node coloring, and checks if it is a valid 3-coloring?



X (n choose 2 bits)
Y (2n bits)
e_{ij}
c_i
c_j
≠
≠
…
…
$V_n(X,Y)$

$V_n(X,Y)$

$V_n(X,Y)$

Let V_n be a circuit that takes an n-node graph X and an assignment of colors to nodes Y, and verifies that Y is a valid 3 coloring of X. I.e., $V_n(X,Y) = 1$ iff Y is a 3 coloring of X

$V_n(X,Y)$

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Given n, we can construct V_n in time $O(n^2)$

Let G be an n-node graph

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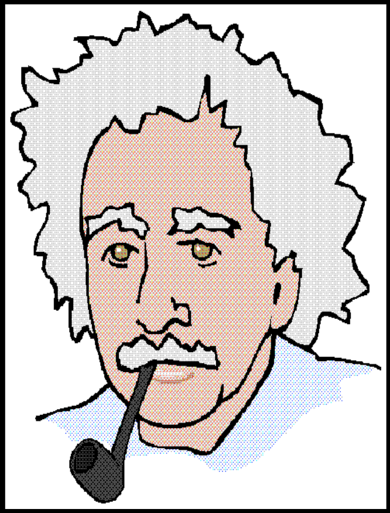

GIVEN:
SAT
Oracle

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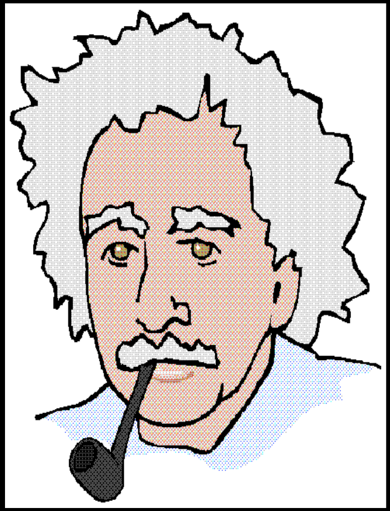
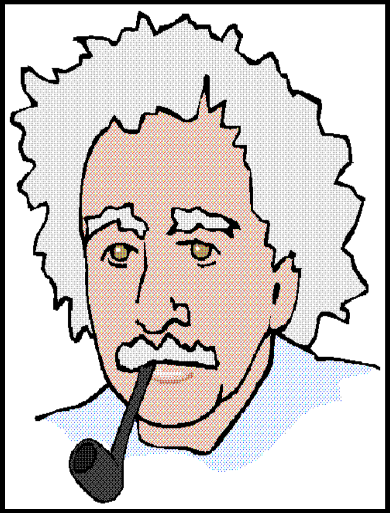


BUILD:
3-color
Oracle

GIVEN:
SAT
Oracle

Let G be an n-node graph

G

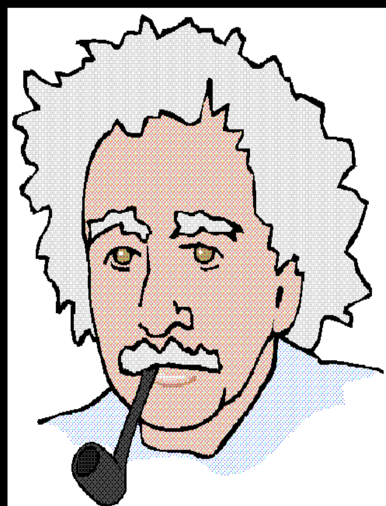


BUILD: 3-color Oracle

GIVEN: SAT Oracle

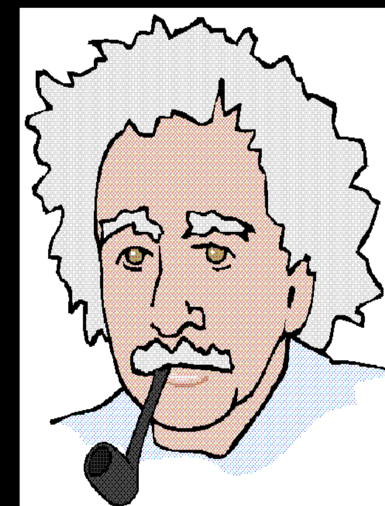
Let G be an n-node graph

G



BUILD:
3-color
Oracle

$V_n(G,Y)$



GIVEN:
SAT
Oracle

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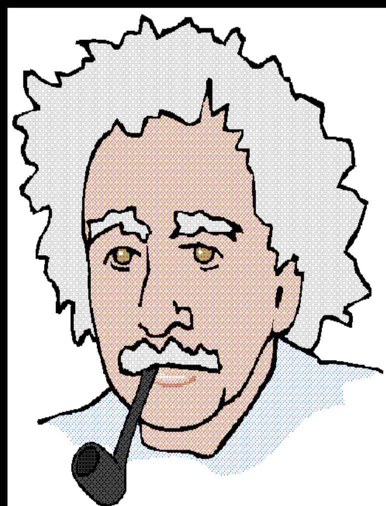
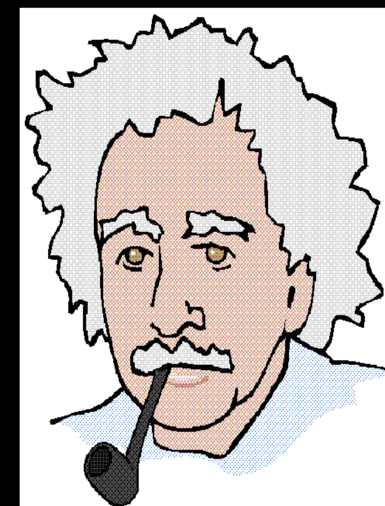


$V_n(G,Y)$



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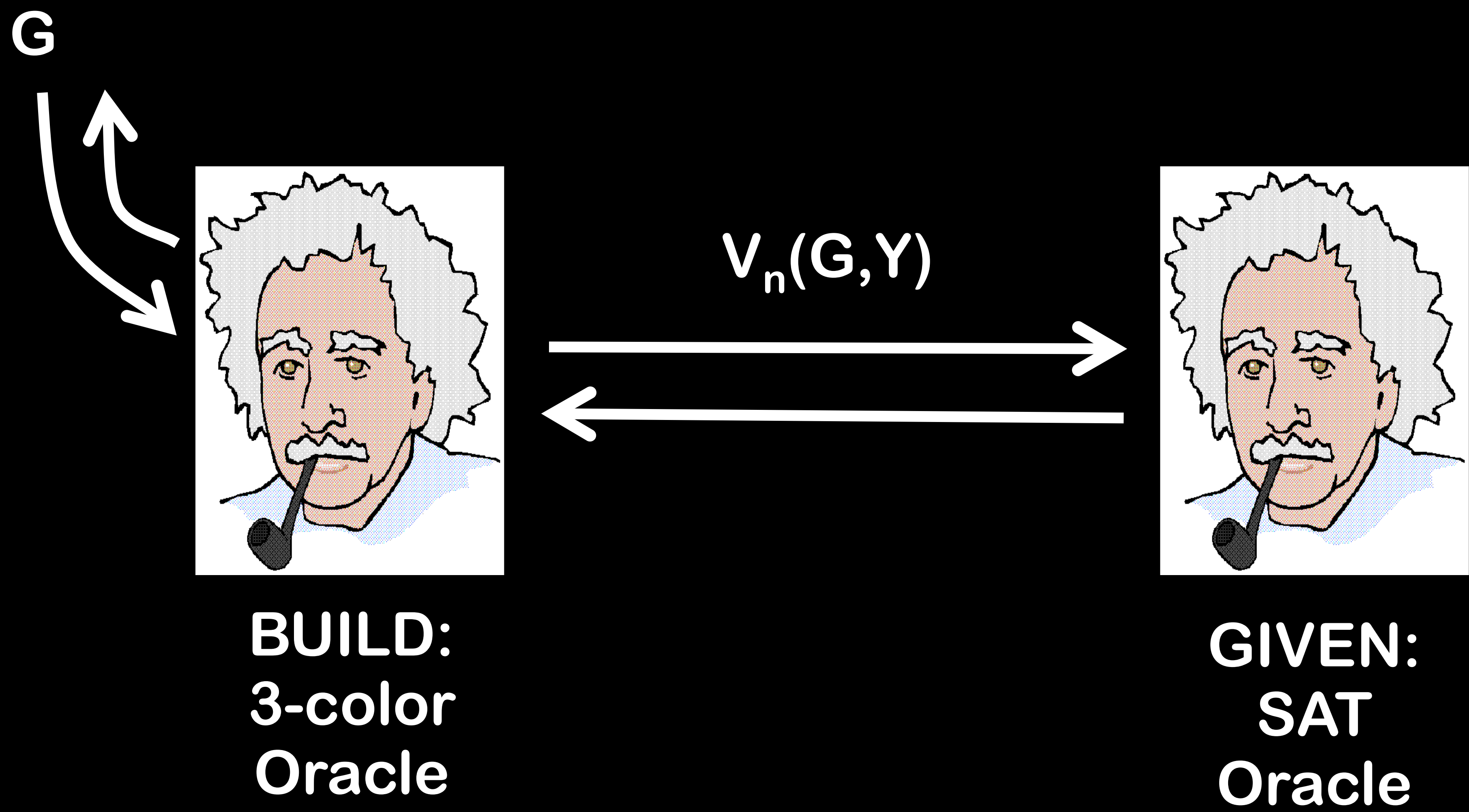

Circuit-SAT / 3-Colorability

Two problems that are cosmetically different, but substantially the same

Circuit-SAT / 3-Colorability

Clique / Independent Set

Circuit-SAT / 3-Colorability

Clique / Independent Set

Given an oracle for circuit SAT, how can you quickly solve k-clique?

Circuit-SAT / 3-Colorability

Clique / Independent Set

Four problems that are cosmetically different, but substantially the same

FACT: No one knows a way to solve any of the 4 problems that is fast on all instances

Summary

Many problems that appear different on the surface can be efficiently reduced to each other, revealing a deeper similarity